Canad. Math. Bull. Vol. **22** (4), 1979

MODULES BEHAVING LIKE TORSION ABELIAN GROUPS

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Introduction. Recently H. Marubayashi [1, 2] and S. Singh [10, 11, 12] generalized some results of torsion abelian groups for modules over some restricted rings, like bounded Dedekind prime rings, bounded hereditary Noetherian prime rings. Singh [12] introduced the concept of h-purity for a module M_R satisfying the following conditions:

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any nonzero homomorphism $f: W \to V$ can be extended to a homomorphism $g: U \to V$ provided the composition length $d(U/W) \leq d(V/f(W))$.

We call a module M_R satisfying (I) and (II) an S_2-module. Singh [12] generalized some of the results of [10] and has shown that a large number of decomposition theorems true for torsion abelian groups also hold for S_2-modules. The purpose of this paper is to generalize some more results of torsion abelian groups for S_2-modules. The results in Section (I) are all analogous to known results in abelian torsion group theory. Analogous to high subgroups as defined by Irwin [3], we define high submodules of S_2-modules. Irwin [3, Theorem 7] proved that high submodules of primary modules over principal ideal domain are pure submodules. We prove here that high submodules are h-pure (Theorem 7). We have also obtained a necessary and sufficient condition for a submodule of an S_2-module to be embeddable in a bounded direct summand (Theorem 9). Section (II) deals with the applications to torsion modules over bounded (hnp)-rings. It is proved that if N is any high submodule of a torsion module M over a bounded (hmp)-ring then M/N is divisible and if N is a direct sum of cyclic modules then all high submodules are isomorphic (Proposition 17, Th. 18). Lastly the concept of Σ-modules analogous to Σ-groups is introduced and the results of [4] have been generalized.

Some of the results have been announced by the author in [8].

Preliminaries. All rings considered here are associative, contain unity and modules are unital S_2-modules. An element $x \in M$, $x \neq 0$ and M is an S_2-

Received by the editors August 2, 1977 and, in revised form, September 12, 1978.

https://doi.org/10.4153/CMB-1979-058-0 Published online by Cambridge University Press

module, is called uniform if xR is a uniform module. For any uniform element $x \in M$, $d(xR)$ the composition length of xR is called exponent of x and is denoted as $e(x)$; $\sup\{d(U/xR)\}$ where U runs over all uniserial submodules of M, containing x is called height of x and is denoted as $H_M(x)$ (or simply $H(x)$). For any $k \geq 0$, $H_k(M)$ denotes the submodule generated by uniform elements of M of height at least k. A submodule N of M is called h-pure if $H_k(M) \cap N = H_k(N)$ for all $k \geq 0$. M is said to be bounded if $H(x) \leq k$ for every uniform element $x \in M$. We denote by M^1 the submodule generated by all the uniform elements of M of height infinity. Analogous to neat subgroups a submodule N of M is called h-neat if $N \cap H_1(M) = H_1(N)$. This concept is introduced in [8].

Section I. Results on S_2-module. If M is an S_2-module and N is a submodule of M then a submodule K of N is called complement of N if K is maximal with respect to $K \cap N = 0$. A complement of M^1 is called high submodule.

Now we prove the following:

LEMMA 1. *If M is an S_2-module and N is h-neat submodule of M With same socle then $N = M$.*

Proof. We do this by induction. Let every uniform element of M of exponent n belong to N. Suppose x is a uniform element in M with $e(x) = n+1$ then we can get a submodule $zR \subseteq xR$ such that $d(xR/zR) = 1$. By induction $z \in N$, hence by h-neatness of N there exists a uniform element $u \in N$ such that $z \in uR$ and $d(uR/zR) = 1$. Hence by (II) we get an isomorphism $f: xR \to uR$ which is identity on zR and can be chosen as $xr \leftrightarrow ur$. Define $g: xR \to (x - f(x))R$ given as $xr \to (x - f(x))r$. Then g is an R-epimorphism with $zR \subseteq \ker g$. Hence $e(x - f(x)) \leq d(xR/zR) = 1$ so $x - f(x) \in N$ and by induction we get $N = M$.

The following lemma generalizes [1, Lemma 3.22]:

LEMMA 2. *If N is a submodule of an S_2-module M and for every uniform element $x \in \mathrm{soc}(N)$, $H_N(x) = H_M(x)$. Then N is h-pure submodule of M.*

Proof. Let y be a uniform element of N such that if $e(y) = n$ then $H_N(y) = H_M(y)$. Let x be a uniform element of N with $e(x) = n+1$ and $H_M(x) = k$. There is a uniform element $t \in M$ such that $x \in tR$ and $d(tR/xR) = k$. Choose $z \in xR$ such that $d(xR/zR) = 1$ then $d(tR/zR) = k+1$. Hence by supposition there exists a uniform element $u \in N$ such that $z \in uR$ and $d(uR/zR) = k+1$. Let $u'R/zR = \mathrm{soc}(uR/zR)$, then there exists an isomorphism $f: xR \to u'R$ which is identity on zR. Now the map $g: xR \to (x - f(x))R$ given as $xr \to (x - f(x))r$ is an R-epimorphism with $zR \subseteq \ker g$; so $e(x - f(x)) \leq d(xR/zR) = 1$ and we get $x - f(x) = u_1$ with $u_1 \in \mathrm{soc}(N)$. Trivially $u_1 \in H_k(M)$. Hence by hypothesis $u_1 \in H_k(N)$, consequently $x \in H_k(N)$. Therefore by induction the lemma follows.

PROPOSITION 3. *If M is an S_2-module and $M = M_1 \oplus M_2$ and N is a submodule such that $\operatorname{soc}(N) \subseteq \operatorname{soc}(M_1)$ then any projection π of M onto M_1 restricted on N is an isomorphism and $\pi(N)$ is h-pure submodule of M provided N is h-pure in M.*

Proof. Let x be a uniform element of N and $zR = \operatorname{soc}(xR)$ then $z = xr$ for some $r \in R$ and $\pi(z) = \pi(x)r$ but $\pi(z) = z$, hence $z = \pi(x)r$ consequently $\pi(x) \neq 0$ and therefore π, restricted on N, is an isomorphism. Now for h-purity of $\pi(N)$, let x be a uniform element in $\operatorname{soc}(\pi(N))$ such that $H_M(x) = n$ then by h-purity of N there exists a uniform element $y \in N$ such that $x \in yR$ and $d(yR/xR) = n$. Now as π, restricted on N, is an isomorphism, $d(\pi(y)R/\pi(x)R) = d(\pi(y)R/xR) = n$. Hence by lemma 2, $\pi(N)$ is h-pure submodule of M.

Now we prove the complement submodules are h-neat.

PROPOSITION 4. *If M is an S_2-module and N is a submodule of M then any complement T of N is h-neat.*

Proof. Let x be a uniform element in $T \cap H_1(M)$, then there exists a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR) = 1$. If $y \in T$ then we are done. Let $y \notin T$ and $K = (T + yR) \cap N$. Then $K \neq 0$ and so for some non-zero uniform element $u \in N$ we have $u = t + yr$, $t \in T$, $r \in R$. As yR is totally ordered either $yrR \subseteq xR$ or $xR \subseteq yrR$. If $yrR \subseteq xR$ then $yr \in T$ and so $u \in T \cap N$, a contradiction, therefore $xR \subseteq yrR$. Now as yR/xR is a simple module, $yrR = yR$ and hence without any loss we can assume $u = t + y$. Since $x \in yR$, $x = yr_0$ for some $r_0 \in R$, hence using $T \cap N = 0$ we get $x = yr_0 = -tr_0$. Consider the map $\mathfrak{R}: yR \to tR$ given as $yr \to tr$ then as $N \cap T = 0$, $\mathfrak{R}$ is a well defined onto homomorphism. Hence tR is uniform. Now we assert that $tr_0R \neq tr$. Suppose $tr_0R = tR$ then $t \in yR$ and hence $u = yc$ for some $c \in R$. Now it is easy to see that $uR < yR$. Again as yR is totally ordered either $xR \subseteq uR$ or $uR \subseteq xR$, but $d(yR/xR) = 1$ and $N \cap T = 0$, therefore none is possible. Consequently $tR > tr_0R$ and $d(tR/xR) > 1$ which gives $x \in H_1(T)$. So T is h-neat in M.

Now the following proposition answers about the converse part of the Proposition 4.

PROPOSITION 5. *If M is an S_2-module and N is a h-neat submodule of M such that $\operatorname{soc}(N) \oplus \operatorname{soc}(T) = \operatorname{soc}(M)$ then N is a complement of T.*

Proof. Trivially $N \cap T = 0$. We embed N into a complement K of T. Then $K \oplus T$ is an essential submodule of M and hence $\operatorname{soc}(K) \oplus \operatorname{soc}(T) = \operatorname{soc}(M)$.

Therefore $\operatorname{soc}(K)=\operatorname{soc}(N)$ and by Lemma 1, $N=K$. Hence N is a complement of T.

Now the following theorem gives a characterization of complement submodules.

THEOREM 6. *If M is an S_2-module such that M is essential in a module E. If N is a submodule of M such that $N\subseteq D\subseteq E$ and N is essential in D, then the set of complements of N in M is the set of intersections of M with complements of D in E.*

Proof. Let A be a complement of D in E then $A\oplus D$ is essential in E and hence $\operatorname{soc}(A)\oplus\operatorname{soc}(D)=\operatorname{soc}(E)$. Let $T=A\cap M$, then trivially $\operatorname{soc}(M)=\operatorname{soc}(T)\oplus\operatorname{soc}(N)$. We prove that T is h-neat in M. Let x be a uniform element in $T\cap H_1(M)$, then there exists a uniform element $y\in M$ such that $x\in yR$ and $d(yR/xR)=1$. Since by Proposition 4 A is h-neat, there exists a uniform element $z\in A$ such that $d(zR/xR)=1$. Hence by (II) there exists an isomorphism $f\colon yR\to zR$, such that f is the identity on xR. The map $g\colon yR\to(y-f(y))R$ is trivially an R-epimorphism with $xR\subseteq\ker g$. Hence $e(y-f(y))\le d(yR/xR)=1$ and $y-f(y)\in\operatorname{soc}(M)$. Consequently $f(y)\in M$ and we get T to be h-neat in M. Now by Proposition 5, T is a complement of N. Conversely let B be a complement of N in M then $(B\cap D)\cap N=0$, hence $B\cap D=0$. We embed B into a complement C of D in E. Then $B\subseteq C\cap M$ but $(C\cap M)\cap N=C\cap N=0$ and we get $B=C\cap M$.

We have proved that the complement submodules are h-neat. Now we prove that the complement submodules under some condition are h-pure. The following theorem generalizes [3, Theorem 7].

THEOREM 7. *If M is an S_2-module and N is a submodule of M such that $N\subseteq M^1$. Then any complement T of N is h-pure submodule of M.*

Proof. Appealing to Proposition 4, we get $T\cap H_1(M)=H_1(T)$. Now suppose $H_n(T)=T\cap H_n(M)$. Then we prove $H_{n+1}(T)=T\cap H_{n+1}(M)$. Let x be a uniform element in $T\cap H_{n+1}(M)$ then there exists a uniform element $y\in M$ such that $x\in yR$ and $d(yR/xR)=n+1$. Let $zR/xR=\operatorname{soc}(yR/xR)$ then $d(yR/zR)=n$. If $z\in T$, then by induction $x\in H_{n+1}(T)$ and we get T to be h-pure. Now let $z\notin T$ then $(T+zR)\cap N\neq 0$ and so we can find a non-zero uniform element $w=t+zr$ for some $0\neq r\in R$ and $t\in T$. As T is a complement of N and zR/xR is simple, $zrR=zR$ and hence without loss of generality we can suppose $w=t+z$. As $x\in zR$, $x=zr'$ for some $r'\in R$, so $wr'=tr'+zr'$, but $T\cap N=0$, $x=zr'=-tr'$. As in Proposition 4 it is easy to see that t is uniform.

Now we have the following two cases:

CASE 1. Let $tr'R<tR$, then $d(tR/xR)\ge 1$. As $N\subseteq M^1$ and $z\in H_n(M)$ we get $t\in H_n(M)\cap T=H_n(T)$, so $x\in H_{n+1}(T)$ and T is h-pure by induction.

https://doi.org/10.4153/CMB-1979-058-0 Published online by Cambridge University Press

CASE 2. Let $tr'R = tR$, then we get $t = xa = zr'a$ for some $a \in R$ and $w = za'$, $a' \in R$. Trivially $wR \subseteq zR$, as zR is totally ordered either $wR \subseteq xR$ or $wR = zR$, but on account of $N \cap T = 0$ none is possible. Therefore $tr'R \neq tR$.

Hence T is h-pure submodule of M.

COROLLARY 8. *If M is an S_2-module and T is a high submodule of M then T is h-pure in M.*

The following theorem generalizes Erdeyli's theorem for primary groups [5, Corollary 27.8]. The proof of the theorem is very much similar to that of Theorem 7.

THEOREM 9. *If M is an S_2-module and N is a submodule of M, then N can be embedded in a bounded summand of M if and only if the heights of the uniform elements of N in M are bounded.*

Proof. If N is embeddable in a bounded summand of M then trivially for every uniform element $x \in N$, $H_M(x) \leq k$. For the converse let $m = \sup\{H(x)\}$ for every uniform element $x \in N$. Then trivially $N \cap H_{m+1}(M) = 0$. Now we embed N into a complement K of $H_{m+1}(M)$. Obviously K is bounded. Now we prove that $K \cap H_n(M) = H_n(K)$ for all $n \geq 0$. The result is trivially true for $n = 0$ and $n \geq m+1$. Now let $0 < n < m$ and $K \cap H_n(M) = H_n(K)$. Now we prove that $K \cap H_{n+1}(M) = H_{n+1}(K)$. Suppose x is a uniform element in $K \cap H_{n+1}(M)$, then there exists a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR) = n+1$. Let $zR/xR = \mathrm{soc}(yR/xR)$ then $d(yR/zR) = n$. If $z \in K$ then by supposition $z \in H_n(K)$ and hence $x \in H_{n+1}(K)$ and we get the assertion. Let $z \notin K$ then $(K + zR) \cap H_{m+1}(M) \neq 0$. Hence for some uniform element $u \in H_{m+1}(M)$ we have $u = t + zr$ where $t \in K$, $r \in R$. As zR is totally ordered either $zrR \subseteq xR$ or $xR < zrR$, but due to the simplicity of zR/xR and $K \cap H_{m+1}(M) = 0$, none is possible, consequently $zrR = zR$. Hence without any loss of generality we can assume $u = t + z$. As in Theorem 7, it is easy to see that t is uniform. Now $x = zr'$ for some $r' \in R$ and $x = zr' = -tr'$. Now we have the following two cases:

CASE 1. If $tr'R < tR$ then $d(tR/zR) \geqq 1$. As $n < m$ and $z \in H_n(M)$ we get $t \in K \cap H_n(M) = H_n(K)$. Hence $x \in H_{n+1}(K)$. Consequently by induction K is h-pure submodule of M.

CASE 2. If $tr'R = tR$ then $tR = xR$ and $t = xa$ for some $a \in R$. Now $u = t + z = xa + z = zr'a + z = zb$ for some $b \in R$. As done earlier it is easy to see that $uR = zbR = zR$. Hence $z \in H_{m+1}(M)$ and we get $t = 0$ which is a contradiction. Therefore this case is not possible.

Consequently K is a bounded h-pure submodule of M. Appealing to [12, Theorem 3] we get K to be a direct summand of M. Hence the theorem follows:

The following corollary generalizes a theorem of Khabbaz [5, Theorem 27.7]:

COROLLARY 10. *If M is an S_2-module then every complement of $H_k(M)$ is a direct summand of M.*

P. Hill [9] has shown that two pure subgroups with the same socle are not necessarily isomorphic. The following theorem which generalizes a result of E. Enoch [6, Lemma 66.1]. shows that under some conditions two h-pure submodules with same socle are isomorphic.

THEOREM 11. *If M is an S_2-module and N, K are h-pure submodules of M with* $\operatorname{soc}(N)=\operatorname{soc}(K)$. *If $M=K\oplus T$ then $M=N\oplus T$ and $N\cong K$.*

Proof. It is trivial to see that $N\cap T=0$ and $\operatorname{soc}(M)=\operatorname{soc}K\oplus\operatorname{soc}T=\operatorname{soc}(N)\oplus\operatorname{soc}(T)$. Now we apply induction. Suppose every uniform element $y\in M$ with $e(y)=n$ belongs to $N\oplus T$. Since for every projection π of M onto K and uniform element $x\in M$, $x=\pi(x)+(I-\pi)(x)$ and $e(\pi(x))\le n$, hence for completing the proof it is sufficient to show that if u is any uniform element of K with $e(u)=n+1$ then $u\in N\oplus T$. Let $zR=\operatorname{soc}(uR)$ then $d(uR/zR)=n$. As $\operatorname{soc}(N)=\operatorname{soc}(K)$, $z\in N$ and by h-purity of N, there exists a uniform element $v\in N$ such that $z\in vR$ and $d(vR/zR)=n$. Appealing to (II) there exists an isomorphism $\sigma: uR\to vR$ such that σ is the identity on zR. The map $\mathfrak{N}$: $uR\to(u-\sigma(u))R$ is trivially an R-epimorphism with $zR\subseteq\ker\mathfrak{N}$. Therefore $e(u-\sigma(u))\le d(uR/zR)=n$ which yields by induction $u-\sigma(u)\in N\oplus T$ and so $u\in N\oplus T$. Therefore $M=N\oplus T$ and we have $N\cong K$.

COROLLARY 12. *If M is an S_2-module and $M=N_1\oplus N_2=T_1\oplus T_2$ with* $\operatorname{soc}(N_1)=\operatorname{soc}(T_1)$ *then $M=N_1\oplus T_2=T_1\oplus N_2$.*

In Section II we shall prove that high submodules under some conditions are isomorphic. For proving this fact we need the following:

PROPOSITION 13. *If M is an S_2-module and $N\subseteq M^1\neq 0$ then for any complement T of N in M, M/T is direct sum of infinite length uniform submodules.*

Proof. If every uniform element of $\operatorname{soc}(M/T)$ is of infinite height then by [12, Theorem 4] the assertion follows. If it is not so, then by [12, Theorem 5] $M/T=L/T\oplus K/T$ where L/T is of finite length. Hence $(M/K)^1=0$ and we get $M^1\subseteq K$, so $0=T\cap N=(L\cap K)\cap N=L\cap N$ which is a contradiction. Hence the assertion follows

LEMMA 14. *If N is a submodule of an S_2-module M and T_1, T_2 are complements of N then* $\operatorname{soc}((T_1\oplus N)/N)=\operatorname{soc}((N_2\oplus T)/N)$.

Proof. For any uniform element $x\in T_1$ we assert that $e(x)=1$ if and only if $e(\bar{x})=1$ where $\bar{x}=x+N$. If $e(x)=1$, then trivially $e(\bar{x})=1$. Let $e(\bar{x})=1$ and $e(x)>1$ then there exists a submodule $yR\subseteq xR$ such that $d(xR/yR)=1$.

Trivially $y \notin N$ and so $d(\bar{x}R/\bar{y}R)=1$ which gives $e(\bar{x})>1$, a contradiction and hence the assertion follows. Let $\bar{x}$ be a uniform element in $\mathrm{soc}((T_1 \oplus N)/N)$ then by above argument x can be taken to be a uniform element in $\mathrm{soc}(T_1)$. As $\mathrm{soc}(T_2)\oplus \mathrm{soc}(N)=\mathrm{soc}(M)$, we get $x \in \mathrm{soc}(T_2)\oplus\mathrm{soc}(N)$ and consequently $\bar{x} \in \mathrm{soc}((T_2\oplus N)/N)$. Similarly the other inclusion follows.

LEMMA 15. *If N is a submodule of an S_2-module M with $N \subseteq M^1 \neq 0$ then for any complement T of N, $(T\oplus N)/N$ is h-pure in M/N.*

Proof. Let $\bar{x}$ be a uniform element in $((T\oplus N)/N)\cap H_k(M/N)$ then x can be chosen to be uniform in T. As $N \subseteq M^1$, $x \in T \cap H_k(M)$, hence by Theorem 7, $x \in H_k(T)$. Therefore there exists a uniform element $y \in T$ such that $x \in yR$ and $d(yR/xR)=k$. As $x \notin N$, $d(\bar{y}R/\bar{x}R)=k$ and so $\bar{x} \in H_k((T\oplus N)/N)$. Hence $(T\oplus N)/N$ is h-pure in M/N.

Section II. Applications to torsion modules over bounded (hnp)**-rings.** Proceeding on similar lines as in Theorem 6, we have the following proposition which generalizes [4, Theorem 3].

PROPOSITION 16. *Let N be a submodule of a torsion module M over a bounded (hnp)-ring R and E be divisible hull of M and D be divisible hull of N in E, then the set of complements of N in M is the set of intersections of M with complementary summands of D in E.*

PROPOSITION 17. *If M is a torsion module over a bounded (hnp)-ring R and $N \subseteq M^1 \neq 0$ then for any complement T of N, M/T is divisible.*

Proof. It runs on similar lines as in Proposition 13.

It is proved in [7] that torsion modules over bounded (hnp)-ring contain basic submodules and any two basic submodules are isomorphic. Now appealing to Theorem 7, Proposition 17, Lemma's 14, 15 and the fact mentioned above and [10, Corollary 1] we have the following theorem which generalizes [4, Theorem 7]. Since the proof is on similar lines, it is omitted.

THEOREM 18. *If M is a torsion module over a bounded (hnp)-ring R and N, T are high submodules of M. If N is direct sum of cyclic modules then T is also direct sum of cyclic modules and $T \cong N$.*

Analogous to Σ-groups [4], now we define Σ-module.

DEFINITION. If M is a torsion module over a bounded (hnp)-ring R then M is called Σ-module if all of its high submodules are direct sum of cyclic modules.

Now in view of the results proved we have the following theorem which generalizes [4, Theorems 10, 11]. Since the proof is almost similar it is omitted.

THEOREM 19. *If M is a torsion module over a bounded (hmp)-ring R then the following hold:*

(a) *M contains a Σ-submodule T such that T is h-pure in M and $T^1=M^1$.*

(b) *If M is a Σ-module then any submodule T with $T^1=T\cap M^1$ is also a Σ-module.*

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(c) *If M is a Σ-module and T is h-pure submodule of M then T is a Σ-module.*

PROPOSITION 20. *If M is a torsion module over a bounded (hnp)-ring R and $M^1 \neq 0$. If N and T are high submodules of M then the following hold:*

(a) *$H_k(T)$ is high submodule of $H_k(M)$ for all $k \geq 0$.*

(b) $\mathrm{soc}(H_k((T \oplus M^1)/M^1)) = \mathrm{soc}(H_k((N \oplus M^1)/M^1))$, $k \geq 0$.

(c) *M/N is divisible hull of $(M^1 \oplus N)/N$.*

(d) *$M/N \cong M/T$.*

(e) *$M = N + H_k(M)$ for all $k \geq 0$.*

(f) *$H_k(M)/H_{k+n}(M) \cong H_k(N)/H_{k+n}(N)$ for all n, $k \geq 0$.*

(g) *$M/H_k(N) = N/H_k(N) \oplus H_k(M)/H_k(N)$, $k \geq 0$.*

(h) *M is minimal h-pure module containing $T \oplus M^1$.*

Proof. In view of the results proved, the proofs of the above can be well adopted from that of groups, hence we prove only (e) by Proposition 17, M/N is divisible hence for every positive integer k, and uniform element $\bar{x} \in M/N$ we can find a uniform element $\bar{y}_k \in M/N$ such that $d(\bar{y}_k R/\bar{x}R) = k$ then $d(y_k R/t_k R) = k$ where $t_k = y_k r$, $r \in R$ and $\bar{x} = \bar{t}_k$. Consequently $x - t_k \in N$ but $t_k \in H_k(M)$, so $x \in N + H_k(M)$ and we get (e).

COROLLARY 21. *If M is a torsion module over a bounded (hnp)-ring R then M is a Σ-module if and only if $H_k(M)$ is a Σ-module for $k \geq 0$.*

Proof. If M is a Σ-module then by Theorem 19, $H_k(M)$ is a Σ-module. Conversely suppose $H_k(M)$ is a Σ-module. Let N be any high submodule of M then $H_k(N)$ is a high submodule of $H_k(M)$ (Proposition 20). Hence $H_k(N)$ is a direct sum of cyclic modules. Appealing to [7, Theorem 2.2.13] we get N to be direct sum of cyclic modules and so by Theorem 18, the assertion follows.

COROLLARY 22. *If M is a torsion module over a bounded (hnp)-ring R and N is a submodule such that $N \supseteq H_k(M)$ then N is a Σ-module provided M is a Σ-module.*

PROPOSITION 23. *If M is a reduced torsion module over a bounded (hnp)-ring R and $M^1 \neq 0$. If N is a high submodule of M then the following hold:*

(a) *$N \oplus M^1 < M$*

(b) *N cannot be bounded.*

Proof. It can be well adopted from that of groups and the results proved so far.

ACKNOWLEDGEMENT. The author is extremely thankful to Professor Surjeet Singh of Guru Nanak Dev University, Amritsar for his valuable suggestions and help in the preparation of this paper. The author also wishes to thank the referee for his suggestions about the presentation of this paper and for pointing out some errors.

REFERENCES

1, 2. H. Marubayashi, *Modules over bounded dedekind prime rings, I, II*, Osaka J. Math. **9**(1872), 95–110, 427–445.

3. J. M. Irwin, *High subgroups of abelian torsion groups*, Pacific J. Math. **11** (1961).

4. J. M. Irwin and E. A. Walker, *On N-high subgroups of abelian groups*, Pacific J. Math. **11** (1961), 1363–1374.

5. L. Fuchs, *Infinite abelian groups, Vol. I*, Academic press, New York (1970).

6. L. Fuchs, *Infinite abelian groups, Vol. II*, Academic Press, New York (1973).

7. Musharafuddin Khan, *Torsion modules over bounded (hnp)rings* Ph-D. Dissertation (1976).

8. M. Zubair Khan, *Modules behaving like torsion abelian groups (Abstract)*, Notices Amer. Math. Soc. June (1977).

9. P. Hill, *Certain pure subgroups of primary groups; Topics in abelian groups*, Scott Foresman and Co. (1963).

10, 11. S. Singh, *Modules over (hnp)-rings, I, II*; Canadian J. Math. **27** (1975) 867–883, Canadian J. Math **28** (1976), 73–82.

12. S. Singh, *Some decomposition theorems in abelian groups and their generalizations; Ring theory*, Proc. of Ohio Univ. Conference Marcel Dekker, N.Y. (1976), 183–189.

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