A NOTE CONCERNING THE IDEAL OF NUCLEAR OPERATORS

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(Received 11 January, 1995)

Abstract. Let $\mathscr{A}$ be a Banach algebra of bounded linear operators such that $\mathscr{A}$ contains every operator with finite dimensional range. Then $\mathscr{A}$ contains every nuclear operator.

1. Introduction. Let X be an infinite dimensional Banach space. We adopt the following notation for the various spaces of operators involved here:

$$\mathscr{B}(X) \equiv \text{the algebra of all bounded linear operators on } X;$$

$$\mathscr{F}(X) \equiv \text{the ideal of finite rank operators in } \mathscr{B}(X);$$

$$\mathscr{N}(X) \equiv \text{the ideal of nuclear operators in } \mathscr{B}(X).$$

Also, let $\|\cdot\|_{op}$ denote the usual operator norm, and let $\|\cdot\|_1$ denote the usual natural complete algebra norm on $\mathscr{N}(X)$. A useful discussion of algebras of operators and of $\mathscr{N}(X)$ is given in [**2**, §1.7].

The main purpose of this note is to prove the following result.

THEOREM 1. *Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a Banach algebra of operators with*

$$\mathscr{F}(X) \subseteq \mathscr{A} \subseteq \mathscr{B}(X).$$

Then $\mathscr{N}(X) \subseteq \mathscr{A}$, and furthermore, there exists $M > 0$ such that for all $T \in \mathscr{N}(X)$, $\|T\|_{\mathscr{A}} \leq M\,\|T\|_1$.

As a consequence of the theorem, $\mathscr{N}(X)$ is the smallest Banach algebra of operators that contains $\mathscr{F}$(X). It has long been noted that $\mathscr{N}$(X) is the smallest non-zero Banach ideal of operators in $\mathscr{B}$(X).

A natural example of an algebra of operators to which Theorem 1 applies occurs in the theory of linear operators on a real or complex Banach lattice X. A good brief introduction to the algebras of operators involved can be found in W. Arendt's paper [**1**]. We use the terminology from this paper. Let $\mathscr{B}^r(X)$ be the algebra of all regular operators on the Banach lattice X. $\mathscr{B}^r(X)$ is a Banach algebra in the r-norm, and it is known that $\mathscr{F}(X) \subseteq \mathscr{B}^r(X)$. The closure of $\mathscr{F}(X)$ in the r-norm is the Banach algebra of all r-compact operators, denoted by $\mathscr{K}^r(X)$. Applying Theorem 1, we have $\mathscr{N}(X) \subseteq \mathscr{K}^r(X)$.

2. Results. Before proving Theorem 1, we deal with some preliminary results. For $x \in X$ and $\alpha \in X'$ (the dual space of X), let $\alpha \otimes x$ be the operators in $\mathscr{F}(X)$ given by

$$(\alpha \otimes x)(y) = \alpha(y)x \qquad (y \in X);$$

for $\alpha \in X' \backslash \{0\}$, let $\alpha \otimes X$ denote the space

$$\alpha \otimes X = \{\alpha \otimes x : x \in X\}.$$

Then $\alpha \otimes X$ is a minimal left ideal of $\mathscr{F}(X)$. The minimal right ideals of $\mathscr{F}(X)$ have the form

$$X' \otimes x = \{\alpha \otimes x : \alpha \in X'\},$$

where $x \in X \backslash \{0\}$. An algebra norm on $\mathscr{F}(X)$ is complete on minimal left and right ideals when the spaces $\alpha \otimes X$ and $X' \otimes x$ (as above) are all complete with respect to the norm.

In what follows we use the notation $\|\cdot\|$ for both the norm on X and the norm on X'.

Proposition 2. *Let $|\cdot|$ be an algebra norm on $\mathscr{F}(X)$ that is complete on minimal left and right ideals. Then there exist constants $m > 0$ and $M > 0$ such that*

$$m \|\alpha\| \|x\| \leq |\alpha \otimes x| \leq M \|\alpha\| \|x\|,$$

for all $x \in X$ and $\alpha \in X'$.

Proof. By [**3**, Theorem (2.4.17), p. 69], there exists $m > 0$ such that for all $x \in X$ and $\alpha \in X'$ we have

$$m \|\alpha\| \|x\| = m \|\alpha \otimes x\|_{op} \leq |\alpha \otimes x|. \tag{1}$$

Fix $\alpha \in X' \backslash \{0\}$, and consider the minimal left ideal $\alpha \otimes X$. By hypothesis, $|\cdot|$ is complete on $\alpha \otimes X$. Also, clearly $\|\cdot\|_{op}$ is complete on $\alpha \otimes X$ and so, by the open mapping theorem and (1), there exists $J_\alpha > 0$ such that

$$J_\alpha \|\alpha\| \|x\| = J_\alpha \|\alpha \otimes x\|_{op} \geq |\alpha \otimes x|, \quad \text{for all} \quad x \in X. \tag{2}$$

The same argument applied to minimal right ideals implies that for each $x \in X \backslash \{0\}$ there exists $K_x > 0$ such that

$$K_x \|\alpha\| \|x\| = K_x \|\alpha \otimes x\|_{op} \geq |\alpha \otimes x|, \quad \text{for all} \quad \alpha \in X'. \tag{3}$$

For any $\alpha \in X'$ define $\varphi_\alpha : X \to (\mathscr{F}(X), |\cdot|)$ by $\varphi_\alpha(x) = \alpha \otimes x$. By (2) φ_α is continuous for each α. Let $\mathscr{C}$ be the collection

$$\mathscr{C} \equiv \{\varphi_\alpha : \alpha \in X', \|\alpha\| = 1\}.$$

By (3) this collection of operators is pointwise bounded on X. Thus, by the uniform boundedness theorem, there exists $M > 0$ such that $\|\varphi_\alpha\|_{op} \leq M$, for all $\alpha \in X'$ with $\|\alpha\| = 1$. Therefore

$$|\alpha \otimes x| = |\varphi_\alpha(x)| \leq M \|x\| \|\alpha\|, \quad \text{for all} \quad x \in X \quad \text{and} \quad \alpha \in X'.$$

Any operator $F \in \mathscr{F}(X)$ can be written in the form $F = \sum_{k=1}^{n} \alpha_k \otimes x_k$, where $\{x_k\} \subseteq X$ and $\{\alpha_k\} \subseteq X'$. The projective tensor norm on $\mathscr{F}(X)$, denoted by $\|\cdot\|_p$, is defined by

$$\|F\|_p \equiv \inf\left\{\sum_{k=1}^{n} \|\alpha_k\| \|x_k\| : F = \sum_{k=1}^{n} \alpha_k \otimes x_k\right\};$$

see [**2**, p. 99].

Corollary 3. *Let $|\cdot|$ be an algebra norm on $\mathscr{F}(X)$ that is complete on minimal left and right ideals. Then there exist $m > 0$ and $M > 0$ such that for all $F \in \mathscr{F}(X)$, we have*

$$m \|F\|_{op} \leq |F| \leq M \|F\|_p.$$

https://doi.org/10.1017/S0017089500031487 Published online by Cambridge University Press

Proof. Again, the existence of $m>0$ for which $m\|F\|_{op}\le|F|$, for all $F\in\mathscr{F}(X)$, follows from [**3**, Theorem (2.4.17)]. Now let $M>0$ be as in the statement of Proposition 2. If $F\in\mathscr{F}(X)$ with $F=\sum_{k=1}^{n}\alpha_k\otimes x_k$, then

$$|F|\le\sum_{k=1}^{n}|\alpha_k\otimes x_k|\le M\left(\sum_{k=1}^{n}\|\alpha_k\|\,\|x_k\|\right).$$

It follows that $|F|\le M\,\|F\|_p$.

At this point we remind the reader of the definition of nuclear operator (following [**2**, p. 98]). An operator $T\in\mathscr{B}(X)$ is *nuclear* if there exist sequences $\{x_k\}\subseteq X$ and $\{\alpha_k\}\subseteq X'$ satisfying

$$\sum_{k=1}^{\infty}\|\alpha_k\|\,\|x_k\|<\infty\quad\text{and}\quad T(x)=\sum_{k=1}^{\infty}\alpha_k(x)x_k,$$

for all $x\in X$. The natural norm on $\mathcal{N}(X)$ is

$$\|T\|_1=\inf\left\{\sum_{k=1}^{\infty}\|\alpha_k\|\,\|x_k\|:T\text{ is represented by }\sum_{k=1}^{\infty}\alpha\otimes x_k\text{ (as above)}\right\}.$$

The proof of Theorem 1. Let $(\mathscr{A},\|\cdot\|_{\mathscr{A}})$ be as in the statement of Theorem 1. Assume that $\alpha\in X'\backslash\{0\}$. Choose $y\in X$ with $\alpha(y)=1$. Then $E=\alpha\otimes y$ satisfies $E^2=E$, and $\mathscr{A}E=\alpha\otimes X$. Since $\mathscr{A}E$ is a closed, and hence complete, subspace of $\mathscr{A}$, we have $\|\cdot\|_{\mathscr{A}}$ is complete on minimal left ideals of $\mathscr{F}(X)$. A similar argument shows that $\|\cdot\|_{\mathscr{A}}$ is complete on minimal right ideals of $\mathscr{F}(X)$. Therefore Proposition 2 applies.

Now let $T\in\mathcal{N}(X)$, so that there exist $\{x_k\}\subseteq X$ and $\{\alpha_k\}\subseteq X'$ with $\sum_{k=1}^{\infty}\|\alpha_k\|\,\|x_k\|<\infty$, and $T(x)=\sum_{k=1}^{\infty}\alpha_k(x)x_k$, for all $x\in X$. Let M be as in Proposition 2. Set $S_n=\sum_{k=1}^{n}\alpha_k\otimes x_k$. For $m>n$, we have

$$\begin{aligned}\|S_m-S_n\|_{\mathscr{A}}&\le\sum_{k=n+1}^{m}\|\alpha_k\otimes x_k\|_{\mathscr{A}}\\&\le M\left(\sum_{k=n+1}^{m}\|\alpha_k\|\,\|x_k\|\right)\to0\quad\text{as}\quad m>n\to\infty.\end{aligned}$$

It follows that there exists $S\in\mathscr{A}$ with $\|S-S_n\|_{\mathscr{A}}\to0$. Also, by [**3**, Theorem (2.4.17)], the $\mathscr{A}$-norm dominates the operator norm, and so $\|S-S_n\|_{op}\to0$. This implies that $T=S\in\mathscr{A}$. Also,

$$\begin{aligned}\|T\|_{\mathscr{A}}=\lim_{n\to\infty}\|S_n\|_{\mathscr{A}}&\le\limsup_{n\to\infty}\left(\sum_{k=1}^{n}\|\alpha_k\otimes x_k\|_{\mathscr{A}}\right)\\&\le\limsup_{n\to\infty}M\left(\sum_{k=1}^{n}\|\alpha_k\|\,\|x_k\|\right)=M\sum_{k=1}^{\infty}\|\alpha_k\|\,\|x_k\|.\end{aligned}$$

Thus, $\|T\|_{\mathscr{A}}\le M\,\|T\|_1$.

Finally, we present two more applications of Theorem 1. The following result is

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essentially Theorem (2.8.21) in C. Rickart's book [**3**]. The conclusion of the result has been strengthened by a direct (and obvious) application of Theorem 1.

COROLLARY 4. *Let* $|\cdot|$ *be an algebra norm on* $\mathcal{F}(X)$ *and let* $\mathcal{A}$ *be the completion of* $\mathcal{F}(X)$ *with respect to this norm. Then there exists a representation* $a \to T_a$ *of* $\mathcal{A}$ *on* X *whose kernel is the radical of* $\mathcal{A}$ *and such that each element of* $\mathcal{F}(X)$ *maps into itself. Furthermore, the image* $\{T_a : a \in \mathcal{A}\}$ *contains* $\mathcal{N}(X)$.

When H is a HIlbert space, two of the most important ideals in $\mathcal{B}(H)$ are $\mathcal{C}_2(H)$, the Hilbert-Schmidt operators on H, and $\mathcal{C}_1$(H), the trace class operators on H; see R. Schatten [**4**]. Of course $\mathcal{C}_1(H) = \mathcal{N}(H)$. As is well-known (sometimes by definition), if $T, S \in \mathcal{C}_2(H)$, then $TS \in \mathcal{C}_1(H)$. Theorem 1 has the following amusing corollary in this context.

COROLLARY 5. *Let* $\mathcal{A}$ *be a Banach algebra of operators on* H *with*

$$\mathcal{F}(H) \subseteq \mathcal{A} \subseteq \mathcal{C}_2(H).$$

Then $\mathcal{A}$ *is an ideal in* $\mathcal{C}_2(H)$.

REFERENCES

1. W. Arendt, On the o-spectrum of regular operators and the spectrum of measures, *Math. Z.* **178** (1981), 271–278.

2. T. Palmer, Banach algebras and the general theory of *-algebras in *Vol. I*, *Encyclopedia of Math. and its Applications* (Cambridge University Press, 1994) 49.

3. C. Rickart, *Banach Algebras* (D. Van Nostrand Co., 1960).

4. R. Schatten, *Norm ideals of completely continuous operators* (Springer-Verlag, 1960).

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