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A COSINE FUNCTIONAL EQUATION WITH RESTRICTED ARGUMENT

BY
L. B. ETIGSON[1]

We name a *functional equation with restricted argument* one in which at least one of the variables is restricted to a certain discrete subset of the domain of the other variable(s). In particular, the subset may consist of a single element.

The purpose of this paper is to present a functional equation satisfied only by cosine functions.

We begin by quoting a theorem of Wilson (Theorem 1, Section 3 of [7]) for D'Alembert's functional equation

$$f(x+y)+f(x-y) = 2f(x)f(y). \tag{1}$$

Since Wilson did not state specific domain and range, we use general domain and range for which his result holds.

THEOREM 1. *If G is an additive abelian group in which it is possible to divide by* 2, *and F a field of characteristic not equal to* 2, *then for $f: G \to F$, all solutions of*

$$f(x+y)f(x-y) = f(x)^2+f(y)^2-1, \qquad f(0) = 1, \tag{2}$$

and of (1) *are common, except the trivial solution $f(x)\equiv 0$ of* (1).

Equation (2) is a special case of

$$g(x+y)g(x-y) = g(x)^2+g(y)^2-C^2, \qquad g(0) = C \neq 0, \tag{C}$$

C a constant in the range of g. Dividing (C) by C^2, we have (2) for $f(x)=g(x)/C$.

In the case of $f: R \to C$, it is known ([4], [7]) that (except for the trivial solution $f(x)\equiv 0$) the solutions of (1) are of the form

$$f(x) = \frac{h(x)+h(x)^{-1}}{2}, \tag{3}$$

where

$$h(x+y) = h(x)h(y). \tag{4}$$

If f is a measurable solution of (1), then f is continuous ([2]), and the corresponding

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h in (3) is continuous ([4]). Then every non-trivial h satisfying (4) is of the form ([1], page 216)

$$h(x) = \exp(cx) \tag{5}$$

where c is an arbitrary complex constant, so that by (3) and (5) f is of the form $f(x) = \cosh cx$, or, since $\cosh ix = \cos x$,

$$f(x) = \cos bx, \tag{6}$$

where b is an arbitrary complex constant.

In Theorem 2 we shall make use of the functional equation

$$f(x+y)f(x-y) = f(x)^2 + f(y)^2 - 1. \tag{7}$$

Replacing x and y by 0 in (7) we see that $f(0)^2 = 1$, so that (7) is equivalent to either

$$f(x+y)f(x-y) = f(x)^2 + f(y)^2 - 1, \qquad f(0) = 1 \tag{2}$$

or

$$f(x+y)f(x-y) = f(x)^2 + f(y)^2 - 1, \qquad f(0) = -1. \tag{8}$$

For $f: R \to C$, Theorem 1 and the preceding remarks show that all measurable solutions of (2) are given by (6). A proof similar to that in [7] for Theorem 1 shows that all solutions of (8) and of

$$f(x+y) + f(x-y) = -2f(x)f(y) \tag{9}$$

are common, except the trivial solution $f(x) \equiv 0$ of (9). But (9) results from (1) by replacing the function f by $-f$, so that all previous remarks about f satisfying (1) also hold for $-f$ satisfying (9). In particular, by (6), all non-trivial, measurable solutions of (9), and hence all measurable solutions of (8), for $f: R \to C$, are given by

$$f(x) = -\cos bx. \tag{10}$$

Hence all measurable solutions $f: R \to C$ of (7) are given by (6) and (10).

Further, if f is measurable and satisfies (2) or (8), or (7), then f is continuous.

In this paper we shall consider the functional equation with restricted argument

$$f(x+y+A)f(x-y+A) = f(x)^2 + f(y)^2 - 1, \tag{11}$$

where $f: R \to C$ is measurable and $A \neq 0$ is a fixed real constant. It will be shown that f is continuous, and that (besides the trivial solution $f(x) \equiv 1$), the only functions which satisfy (11) are the cosine functions $\cos ax$ and $-\cos ax$, where for the constant a, only a countable set of numbers is admissible.

Other functional equations with restricted argument solved in the literature, but not given that name, are

$$f(x-y+A) - f(x+y+A) = 2f(x)f(y), \tag{12}$$

$$f(x+y+A)f(x-y+A) = f(x)^2 - f(y)^2 \tag{13}$$

and

$$f(x+y+2A)+f(x-y+2A) = 2f(x)f(y), \tag{14}$$

considered respectively in [6], [5], and [3], where it is shown that (12) and (13) have only sine functions as solutions, and (14) has only cosine functions as solutions (in addition to their trivial solutions).

THEOREM 2. *Let $f: R \to C$ satisfy* (11), *where A is a non-zero real constant. Then the most general solution of* (11) *is given by*

$$f(x) = g(x-A), \tag{15}$$

where g is an arbitrary solution of (7), *periodic with period $2A$.*

Proof. First we show (15) is a solution of (11), provided g is a solution of (7) with period $2A$.

Defining f by (15), where g is a solution of (7) with period $2A$, we have by (15), (7), and the periodicity of g

$$\begin{aligned} f(x+y+A)f(x-y+A) &= g(x+y)g(x-y) \\ &= g(x+y-2A)g(x-y) \\ &= g(x-A)^2+g(y-A)^2-1 \\ &= f(x)^2+f(y)^2-1, \end{aligned}$$

so that (11) holds.

Conversely, we show every solution of (11) is of the form (15), where g of (15) has period $2A$ and satisfies (7).

Replacing y by $-y$ in (11) and comparing the result with (11), we get

$$f(y)^2 = f(-y)^2. \tag{16}$$

Replacing x by $-A$ and y by 0 in (11), and using (16), we obtain

$$f(A)^2 = 1. \tag{17}$$

Replacing y by A in (11), we get, using (17),

$$f(x+2A)f(x) = f(x)^2. \tag{18}$$

For all those x_1 such that $f(x_1) \neq 0$, we have by (18)

$$f(x_1+2A) = f(x_1). \tag{19}$$

For all those x_2 such that $f(x_2)=0$, we have by (16)

$$f(-x_2) = 0. \tag{20}$$

Replacing x by A and y by x_2+2A in (11), we have

$$f(x_2+4A)f(-x_2) = f(A)^2+f(x_2+2A)^2-1. \tag{21}$$

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Thus by (21), (20), and (17) we have

$$f(x_2+2A) = 0 = f(x_2). \tag{22}$$

By (19) and (22) we get

$$f(x+2A) = f(x) \tag{23}$$

for all real x, so that f is periodic with period $2A$.

If g is defined by (15), then by (23) and (15) we have

$$g(x+2A) = f(x+3A) = f(x+A) = g(x),$$

so that g is periodic with period $2A$.

Replacing x by $x+A$ and y by $y+A$ in (11) and using the periodicity of f we have

$$f(x+y+A)f(x-y+A) = f(x+A)^2+f(y+A)^2-1. \tag{24}$$

Finally by (24) and (15) we obtain

$$g(x+y)g(x-y) = g(x)^2+g(y)^2-1.$$

Thus g is a solution of (7), and the proof of the theorem is complete.

THEOREM 3. *The only measurable solutions of* (11), *where* $A\neq 0$ *is a fixed real constant, are*

$$f(x) = \cos ax \quad \text{and} \quad f(x) = -\cos ax, \tag{25}$$

where

$$a = \frac{n\pi}{A}, \qquad n = 0, 1, 2, \ldots .$$

Proof. We use Theorem 2. Since f is measurable, then the function g defined by (15) is also measurable. Since g satisfies (7), which is equivalent to (2) or (8), then by the introductory remarks it is seen that g is continuous. Thus f is continuous, by (15). Since the continuous solutions for g satisfying (7) are $g(x)=\cos bx$ and $g(x)=-\cos bx$, where b is an arbitrary complex constant, and since g is periodic with period $2A$, we have $\cos b(x+2A)=\cos bx$, which is possible only if $b2A=2n\pi$, $n=0, \pm 1, \pm 2, \ldots$, in which case we have

$$g(x) = \cos \frac{n\pi}{A} x, \qquad n = 0, 1, 2, \ldots, \tag{26}$$

and

$$g(x) = -\cos \frac{n\pi}{A} x, \qquad n = 0, 1, 2, \ldots . \tag{27}$$

Thus by (15) we obtain as measurable solutions of (11), in the cases of (26) and (27) respectively,

$$f(x) = \cos\left(\frac{n\pi}{A} x - n\pi\right) \quad \text{and} \quad f(x) = -\cos\left(\frac{n\pi}{A} x - n\pi\right), \qquad n = 0, 1, 2, \ldots .$$

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When n is even, $n=2k$, $k=0, 1, 2, \ldots$, we have, respectively, $f(x)=\cos px$ and $f(x)=-\cos px$, where $p=2k\pi/A$, $k=0, 1, 2, \ldots$. When n is odd, $n=2k+1$, $k=0, 1, 2, \ldots$, we have, respectively, $f(x)=-\cos qx$ and $f(x)=\cos qx$, where $q=(2k+1)\pi/A$, $k=0, 1, 2, \ldots$. Grouping the solutions $\cos px$, $\cos qx$, respectively, $-\cos px$, $-\cos qx$, completes the proof of the theorem.

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Department of Mathematics
Atkinson College, York University
Downsview, Ontario.

Environmental Health Studies,
Ontario Ministry of Health
Toronto, Ontario.

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