Let O be the centre of the given circle and Q that of the circle $HKFA$.

$$\text{Join } QK, OK. \quad O\widehat{K}B = O\widehat{B}K = K\widehat{H}Q = H\widehat{K}Q.$$

$\therefore$ QK and KO are in the same straight line.

Hence by Prop. 1 the circles whose centres are O and Q touch externally at C.

Another solution is got by taking EH in the opposite direction. Two more solutions, with internal contact, are got by interchanging B and C in the figure.

If A lies within the given circle, or if the given circle and A are on opposite sides of XY, there is no solution.

If XY cuts the given circle there are in general two solutions, both internal, or both external.

R. F. MUIRHEAD.

On the Treatment in School Texts of the Sphere, Cone and Pyramid.

An important note in the *Proc. Edin. Math. Soc.* of May 1904 pointed out that the then prevalent School method of tangents made it difficult for the pupil to gain sound knowledge in his later studies. The same criticism may, I submit, be directed against the treatment, now prevalent in school texts, of (1) surface of sphere, (2) volume of sphere, and (3) volume of cone and pyramid.

"Area of sphere......Two planes cut the diameter AOB perpendicularly at M and N, and intercept the arc PQ. Revolving about the axis AOB, the arc PQ generates a belt of a sphere, while pq generates a corresponding belt of the circumscribing cylinder. The planes are supposed to be near together. As PQ is short we may suppose it to be a straight line; and the surface generated by PQ is the curved surface of a frustum of a cone, and hence, from a previous formula, equals $2\pi \times PQ \times$ half-way perpendicular; hence etc. in the usual way, surface of the PQ-belt of the sphere equals surface of corresponding pq-belt of cylinder."

From this position advance is made to total surface of sphere.

The above, quoted almost verbatim from a 1926 textbook by a first-rate publisher, is a fair sample of the now prevailing treatment

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in school texts; and to me it seems that, *at a stage before the calculus*, anything more damaging to the pupil, in the sense of the 1904 note, can hardly be conceived. The treatment is of course suitable, with certain necessary adjustments, *after some knowledge of the calculus has been gained*: its proper form is $dS/dx = 2\pi a$, whence the rest follows by integration; but coming before the calculus such treatment can only be regarded as pernicious.

In his *Newton*, I., II., III., Frost gives a solution very like the one I am criticising, under Lemma 4; but Frost's *Newton* is a definite study of Newton's method of limits; and even if it be granted—an unlikely concession—that Newton's method of ultimate ratios in Lemma 4 is suitable for pupils at the stage under consideration, Frost's presentation of the solution is not sufficiently careful for school use.

The fact however that Frost's proof depends on Lemma 4 makes a big difference between his method and the method of our schoolbooks of to-day. For at an even later stage, in a note on Lemma 5, Frost definitely states that no assumption is made that each arc is ultimately equal to the corresponding chord. That equality which our schoolbooks assume at the very start is not established in Newton till Lemma 7; and throughout the proof referred to in Frost, *PQ* means the *chord PQ* and *not the arc PQ*. Further, Frost goes on to speak of the inscribed polygon, a reference which implies, though the emphasis is not clear enough for school work, that the sum of the surfaces of the inscribed frusta is under consideration. Though Frost is not sufficiently explicit for school teaching, it is to be remembered that he is applying a carefully stated and carefully proved theorem on ultimate ratios, namely, Lemma 4.

It so happens however—and this is my main point—that in the proof of the three school theorems mentioned neither Lemma 7 is required, with its equality of arc and chord, nor Lemma 4. Our school books begin by assuming what Newton did not prove till Lemma 7, an equality of vanishing arc and chord which can only have the vaguest of meanings to the pupil; indeed it is doubtful whether in the mind of the pupil the equality is not really due to the fact that both arc and chord vanish. Our school books begin with Lemma 7; Frost uses Lemma 4; and all we need in school can be proved without the use of either lemma.

Not a single word is needed in school about infinitesimals or vanishing quantities or their ultimate ratios; all we need is given at

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pages 23 and 19 of Frost. At page 23, "*state*, as in Whewell's Doctrine of Limits, that, if a *finite* portion of a curve be taken, and many successive points in the curve be joined so as to form a polygon, the sides of which, taken in order, are chords of portions of the curve, when the number of these points is increased indefinitely, the curve will be the limit of the polygon." At page 19, "The strength of the proofs lies in the examination of the quantities while the hypothesis is in a finite state, before arrival at the ultimate form, and the deduction of properties by which the relations of the quantities can be pursued accurately to the ultimate state."

Area of sphere....to prove spherical belt generated by finite arc AB equal in area to cylindrical belt generated by ab. Inscribe polygon of n sides from A to B, with equal sides all at distance d from centre. The side PQ generates a frustum of area $2\pi . pq . d$, proved in the usual way. Hence by summation, surface generated by the polygon is $2\pi . ab . d$. Apply the Whewell doctrine, and obtain surface of spherical belt AB as $2\pi . ab . r$. Corollary; area of sphere is $2n . 2r . r$ or $4nr^2$.

Volume of sphere....Do not "imagine the surface divided into an infinite number of infinitely small polygons"; to do so is to call in Lemma 7 and the equality of infinitesimals. Begin with the circumscribing cube; keep cutting off corners by tangent planes. At any finite stage of the process, volume of circumscribing solid is surface $\times$ radius $\div$ 3. Pursue this property to the ultimate state.

Equal volumes of tetrahedra with equal bases and equal heights ....To take the planes close is to call in Lemma 4 and ratios of vanishing quantities unnecessarily. Begin with a triangle, divided by parallels to the base into a finite number of trapezia of equal heights; at the end of each add a triangle to complete an external parallelogram, and substract a triangle to leave an internal parallelogram; and show that the triangle ABC lies in area between the sum of the internal and the sum of the external parallelograms. Similarly the volume of a tetrahedron lies between the sum of the internal and the sum of the external prisms; and consequently the difference between the two tetrahedra is less than the lowest prism. Pursue this property accurately to the ultimate state.

None of these proofs makes appeal to Lemma 4 or to Lemma 7; they are of a type suited to schoolwork; their method is a useful, even a necessary, preparation for the calculus.

G. Lawson.