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The enumeration and bifurcations of ranking functions

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Suppose n competitors each compete in r races and a ranking function F assigns a score $F(j)$ to the competitor finishing in the jth position in each race. The sum of the scores over r races gives each competitor a final ranking. If n is fixed, the ranking function F bifurcates as r increases. The complete bifurcation behaviour is determined for $n = 3$ and some information obtained for $n > 3$.

1. Introduction

A ranking function is used to give an overall ranking to n competitors who compete in a sequence of r races. We define a "ranking function F " to be a nonnegative function defined on the first n positive integers and satisfying the condition $F(j) > F(j+1)$, for $1 \leq j \leq n-1$. In each race the competitor finishing in the jth position is awarded a score $F(j)$. The sum of the scores over the r races gives each competitor a final score and the competitors are ranked by these final scores.

A "result" will be simply a finite set of positive integers $\{\alpha_k\}_{1 \leq k \leq r}$, where for each k , $1 \leq \alpha_k \leq n$. That is, a result represents the placings of a single competitor over the r races. (We do not allow the possibility that two competitors be placed equal in a given race.)

Two ranking functions are said to be "$n : r$ equivalent" if for any set of results for n competitors in r races they give the same final

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rankings. Clearly $n : r$ equivalence is an equivalence relation in the usual sense.

EXAMPLE 1.1. Suppose F is a ranking function for some fixed n and r and that m and c are positive constants. Then an $n : r$ equivalent ranking function E may be defined by setting

$$E(j) = mF(j) + c \text{ for } 1 \le j \le n .$$

EXAMPLE 1.2. Suppose $n = 3$, $r = 2$, and F, G, H are defined such that

$$F(1) + F(3) = 2F(2) ,$$

$$G(1) + G(3) > 2G(2) ,$$

$$H(1) + H(3) < 2H(2) .$$

Then it is easy to see that F, G, H are representatives of the three $3 : 2$ equivalence classes.

In Section 2 we shall obtain a formula for the number of $3 : r$ equivalence classes for general r. To simplify discussions we shall consider only the "normalised" ranking functions which satisfy the extra conditions that $F(n) = 0$ and $F(1) = 1$. By Example 1.1 it is sufficient to consider only normalised functions.

EXAMPLE 1.3. For $n = 3$ and $r = 2$ a normalised ranking function F is characterised by $F(2)$ which lies in the open interval $(0, 1)$ and the equivalence classes are $(0, \frac{1}{2})$, $\{\frac{1}{2}\}$, $(\frac{1}{2}, 1)$.

In general for $n \ge 3$ the normalised ranking functions are characterised by an open subset $S(n)$ of $\mathbf{R}^{n-2}$ corresponding to the possible values of $\left(F(2), F(3), \ldots, F(n-1)\right)$. We associate with $S(n)$ the usual topology of $\mathbf{R}^{n-2}$ restricted to $S(n)$.

DEFINITION 1.4. We say that a normalised ranking function F is $n : r$ stable if there exists an open set $U \subset S(n)$ such that $F \in U$ and U is contained in the $n : r$ equivalence class containing F.

In other words there exists a neighbourhood U of F such that all ranking functions in U always rank n competitors who compete in r races in exactly the same order as F.

EXAMPLE 1.5. Let F, G, H be defined as in Example 1.2. Then we note that G and H are $3:2$ stable and F is not $3:2$ stable.

It is easy to see that the $n:r$ stable normalised ranking functions are generic (that is they form an open dense set in $S(n)$). They are dense because the unstable functions lie on a finite set of hyperplanes which intersect $S(n)$.

EXAMPLE 1.6. Let $n = 4$, $r = 2$, $f(2) = x$, and $f(3) = y$. Then all the normalised ranking functions are in the open region bounded by the triangle with edges $x = y$, $x = 1$, $y = 0$. The complete set of results is $\{1, 1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 2\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 3\}$, $\{3, 4\}$, $\{4, 4\}$. We write down the pairs of results which are free to be ranked either way and the equation which gives them equal ranking:

$\{1, 3\}$	$\{2, 2\}$	$1 + y = 2x$,
$\{1, 4\}$	$\{2, 2\}$	$1 = 2x$,
$\{1, 4\}$	$\{2, 3\}$	$1 = x + y$,
$\{1, 4\}$	$\{3, 3\}$	$1 = 2y$,
$\{2, 4\}$	$\{3, 3\}$	$x = 2y$.

Any ranking function which satisfies any one of these equations is not stable. From Figure 1 we see that there are ten equivalence classes of stable ranking functions.

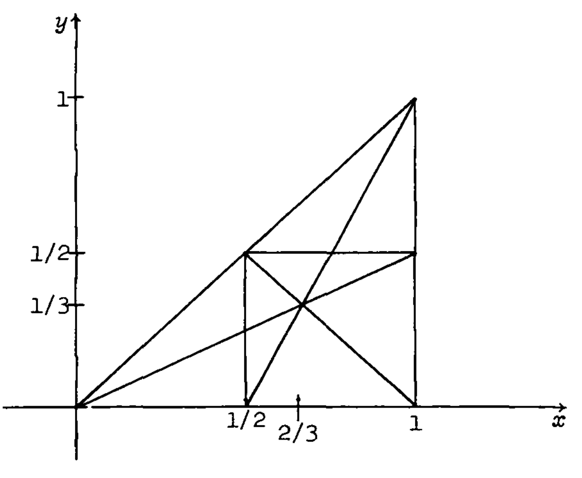

FIGURE 1

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For a fixed n, bifurcations of the equivalence classes occur as r increases and we have two related problems. To determine the bifurcations and the number of equivalence classes for each fixed r. This problem is solved for $n = 3$ in Section 2. It is interesting to note that the bifurcation set obtained (see Figure 2) is a ramified structure of the kind associated with a generalized catastrophe (see [1], p. 107). The corresponding problem for $n = 4$ is more complex. Algebraic and combinatorial properties of ranking functions are studied in [2].

2. The enumeration of $3 : r$ ranking functions

We let $\phi(h)$ denote Euler's ϕ-function. That is $\phi(h)$ is the number of natural numbers less than or equal to h relatively prime to h.

THEOREM 2.1. *The number of* $3 : r$ *equivalence classes of stable ranking functions is* $\sum_{j=1}^{r} \phi(j)$.

Proof. Suppose two results $\{\alpha_k\}_{1\le k\le r}$ and $\{\beta_k\}_{1\le k\le r}$ have the following properties:

(a) no α_k is the same as a β_k ;

(b) it is possible to rank $\{\alpha_k\}_{1\le k\le r}$ above or below $\{\beta_k\}_{1\le k\le r}$.

It can be seen that it is precisely such a pair of results which leads to the bifurcation of an equivalence class of ranking functions which are $3 : (r-1)$ stable. Further it is clear that properties (a) and (b) can only be satisfied by the results $\{2, 2, \ldots, 2\}$ and $\{1, 1, \ldots, 1, 3, \ldots, 3\}$ where the second result contains q first places and $r - q$ third places. A ranking function F which ranks these two results equal must satisfy the equation

$$rF(2) = qF(1) + (r-q)F(3) .$$

Since we need only consider normalised ranking functions with $F(1) = 1$ and $F(3) = 0$ it follows that $F(2) = q/r$.

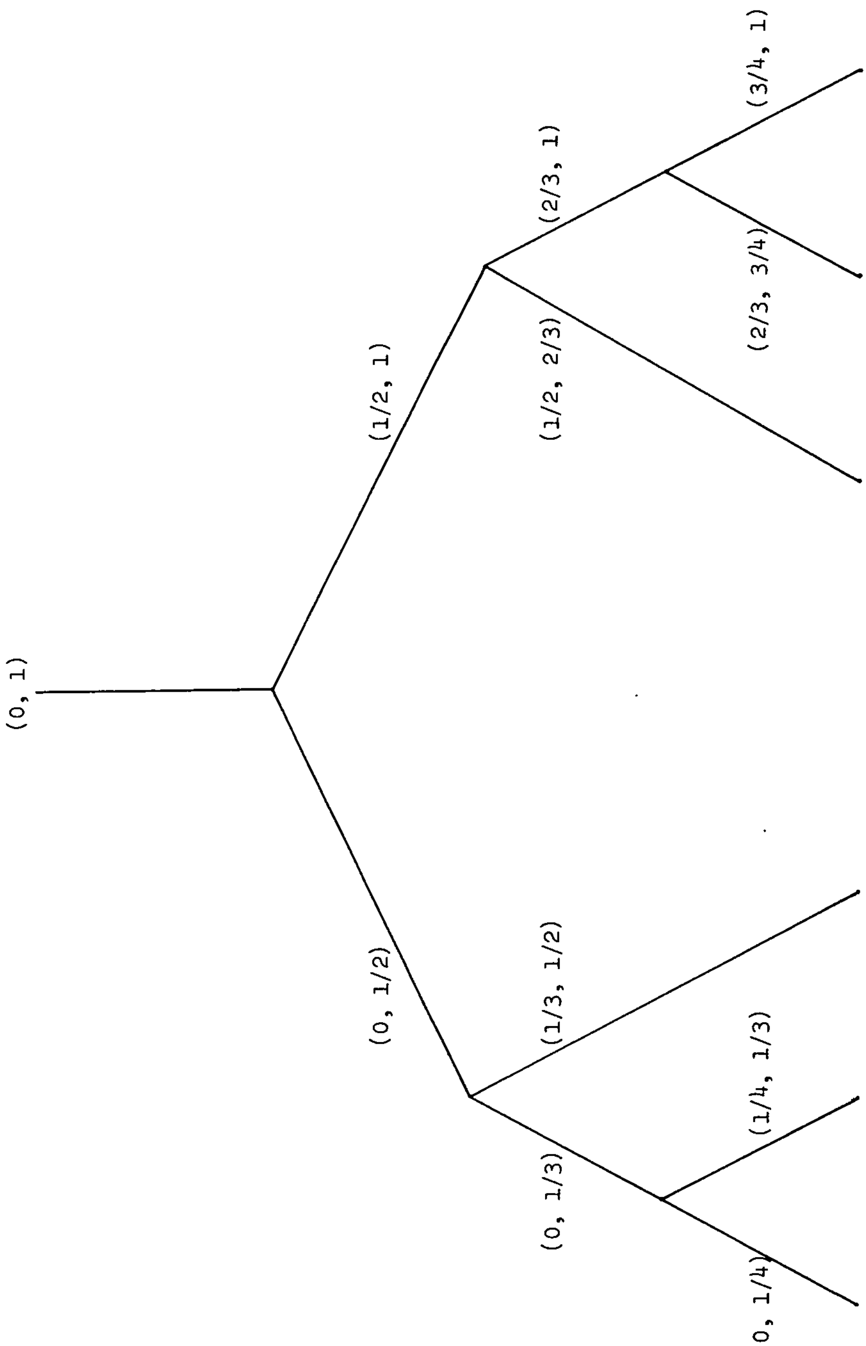

FIGURE 2

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Hence the $3 : r$ unstable normalised ranking functions are precisely those which satisfy the condition $F(2) = q/r$ where $1 \leq q \leq r-1$. A bifurcation occurs provided q and r are relatively prime. If q and r have a common factor the normalised ranking function F which satisfies $F(2) = q/r$ has already become unstable at a smaller value of r. In Figure 2 we show the bifurcations of the equivalence classes of $3 : r$ stable normalised ranking functions which occur at $r = 2, 3, 4$.

It is clear that the number of bifurcations for each r is in fact $\phi(r)$. Also the number of $3 : r$ unstable normalised ranking functions is $\sum_{j=2}^{r} \phi(j)$. The number of $3 : r$ equivalence classes of stable ranking functions exceeds the number of $3 : r$ unstable normalised ranking functions by one and hence can be written $\sum_{j=1}^{r} \phi(j)$. This completes the proof.

Corresponding results for $n > 3$ are more complex. We note however that it is easy to prove that an equivalence class of stable ranking functions in $S(n)$ is convex.

References

[1] René Thom, *Structural stability and morphogenesis: an outline of a general theory of models* (translated by D.H. Fowler. Benjamin, Reading, Massachusetts; London; Amsterdam; Don Mills, Ontario; Sydney; Tokyo; 1975).

[2] W.J. Walker, "Algebraic and combinatorial results for ranking competitors in a sequence of races", *Discrete Math.* **14** (1976), 297-304.

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