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ON THE RANGE AND INVERTIBILITY OF A CLASS OF MELLIN MULTIPLIER TRANSFORMS III

To Professor Tim Rooney with best wishes

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ABSTRACT. We continue to develop the theory of previous papers concerning transforms corresponding to Mellin multipliers which involve products and/or quotients of Γ-functions. We show that, by working with certain subspaces of $L_{p,\mu}$ consisting of smooth functions, we can simplify considerably the restrictions on the parameters which were necessary in the $L_{p,\mu}$ setting. As a result, operators in our class become homeomorphisms on these subspaces under conditions of great generality.

1. In this paper we continue our investigations into Mellin multiplier transforms T satisfying a relation of the form

$$\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s) \tag{1.1}$$

under suitable conditions, where the multiplier h has the form

$$h(s) = \frac{\prod_{i=k+1}^{K} \Gamma(\eta_i + r_i s) \prod_{j=\ell+1}^{L} \Gamma(\xi_j - t_j s)}{\prod_{i=1}^{k} \Gamma(\eta_i + r_i s) \prod_{j=1}^{\ell} \Gamma(\xi_j - t_j s)}. \tag{1.2}$$

Here k, ℓ, K, L are non-negative integers satisfying $0 \le k \le K$, $0 \le \ell \le L$ (empty products being unity by convention), the numbers $r_1, \ldots, r_K$ and $t_1, \ldots, t_L$ are real and positive, and $\eta_1, \ldots, \eta_K$, $\xi_1, \ldots, \xi_L$ are complex numbers.

In [5] and [6] we characterised the range of the operators corresponding to the multipliers $\Gamma(\eta + s/m)$, $\Gamma(\xi - s/m)$ and $\Gamma(\eta + s/m)\Gamma(\xi - s/m)$, with $m > 0$, on the weighted spaces $L_{p,\mu}$. In the case of the third multiplier, the range was already becoming rather complicated and depended on η and ξ separately. In principle, it might be feasible to analyse the multiplier (1.2) relative to $L_{p,\mu}$ but, in practice, it would be a very tiresome business. A thorough investigation was carried out by Rooney in [9] but was restricted to the special case when all the numbers r_i and t_j are unity.

The purpose of the present paper is to develop a corresponding theory within the framework of certain subspaces $F_{p,\mu}$ of $L_{p,\mu}$ consisting of smooth functions. It turns out that the complexity associated with $L_{p,\mu}$ disappears and is replaced by a set of simple conditions of great generality. In particular the range on $F_{p,\mu}$ of the operator T corresponding to (1.2) is a certain subspace $F_{p,\mu,r}$ which depends essentially on only a particular combination of the numbers r_i and t_j, as given by (5.3).

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In § 2, we introduce the relevant spaces by examining the range on $F_{p,\mu}$ of the operator N_m^η corresponding to the simple multiplier $\Gamma(\eta + s/m)$ with $m > 0$. We obtain a characterisation of $N_m^\eta(F_{p,\mu})$ as a set and then equip it with an appropriate topology which turns the set into a Fréchet space. We then discover that the range is essentially independent of η and relabel it as $F_{p,\mu,r}$ with the number $r = 1/m$ often being more convenient to use than m itself. An equally simple multiplier which we might have tried at the start is $\Gamma(\xi - s/m)$. This gives rise to an operator M_m^ξ which was studied along with N_m^η in [5] but produced a different range for every ξ. In contrast we discover in § 3 that $M_m^\xi(F_{p,\mu})$ is independent of ξ and is just $F_{p,\mu,r}$ back again. This is the first hint of the intrinsic importance of $F_{p,\mu,r}$ for all operators with the same value of r given by (5.3). In § 4, we examine a few simple operators relative to $F_{p,\mu,r}$ and review some known results. Finally, in § 5, we reveal the full details of how the spaces $F_{p,\mu,r}$ emerge as the ranges of operators on $F_{p,\mu}$ with multipliers of the form (1.2). However, as we point out at the end, the theory can be extended in a number of ways and this we hope to do in a future paper.

Throughout the paper we shall make use of notation, terminology and results from [5] and [6] to which the reader should refer as necessary. In particular, we use the notation

$$\Omega = \{z \in \mathbf{C} : \operatorname{Re} z \neq 0, -1, -2, \ldots\}. \tag{1.3}$$

2. We begin by recalling the following result from [5].

THEOREM 2.1. *If $1 < p < \infty$, $m > 0$ and $\eta - \mu/m \in \Omega$, then $g \in N_m^\eta(L_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there exists a constant A_g such that*

$$\left\| \left[\Gamma\big(\operatorname{Re}(\eta + n - \mu/m)\big)\right]^{-1} K_m^{\eta+n,-n} g \right\|_{p,\mu} \leq A_g \text{ for } n = 0, 1, 2, \ldots \tag{2.1}$$

where $K_m^{\eta+n,-n}$ denotes the differential operator

$$K_m^{\eta+n,-n} = x^{m\eta+mn}(-D_m)^n x^{-m\eta}; \quad D_m = d/dx^m. \tag{2.2}$$

PROOF. See [5, Theorems 3.3 and 7.3].

Since the topology of $F_{p,\mu}$ is defined by a family of seminorms rather than a single norm as in the case of $L_{p,\mu}$, the next result represents the obvious modification to Theorem 2.1 which makes use of these seminorms.

THEOREM 2.2. *If $1 < p < \infty$, $m > 0$ and $\eta - \mu/m \in \Omega$, then $g \in N_m^\eta(F_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and, for each $i = 0, 1, 2, \ldots$, there exists a constant $A_g^{(i)}$, depending on g but independent of n, such that*

$$\gamma_i^{p,\mu}\left(\left[\Gamma\big(\operatorname{Re}(\eta + n - \mu/m)\big)\right]^{-1} K_m^{\eta+n,-n} g\right) \leq A_g^{(i)} \text{ for } n = 0, 1, 2, \ldots. \tag{2.3}$$

PROOF. Let $g = N_m^\eta f$ where $f \in F_{p,\mu}$. Then, for each $i = 0, 1, 2, \ldots$, $\delta^i g = N_m^\eta \delta^i f$ and since $\delta^i f \in L_{p,\mu}$, Theorem 2.1 shows that, for some constant $B_g^{(i)}$,

$$\left\| \left[\Gamma\big(\operatorname{Re}(\eta + n - \mu/m)\big)\right]^{-1} K_m^{\eta+n,-n} \delta^i g \right\|_{p,\mu} \leq B_g^{(i)} \text{ for } n = 0, 1, 2, \ldots. \tag{2.4}$$

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However the operators $K_m^{\eta+n,-n}$ and δ^i commute and the operator x^iD^i appearing in the seminorm $\gamma_i^{p,\mu}$ is a polynomial of degree i in δ. Hence (2.3) follows from (2.4) with the constant $A_g^{(i)}$ being a linear combination of $B_g^{(k)}$ $(k = 0, 1, \ldots, i)$.

Conversely, let $g \in F_{p,\mu}$ and let g satisfy (2.3) for certain constants $A_g^{(i)}$. Then g satisfies (2.4) for certain constants $B_g^{(i)}$. By Theorem 2.1, for each $i = 0, 1, 2, \ldots$ there exists $f_i \in L_{p,\mu}$ such that

$$\delta^i g = N_m^\eta f_i. \tag{2.5}$$

If $\operatorname{Re}\mu \neq 0$, δ is invertible on $F_{p,\mu}$ and, since δ^{-1} commutes with N_m^η on $L_{p,\mu}$,

$$g = (\delta^{-1})^i N_m^\eta f_i = N_m^\eta (\delta^{-1})^i f_i \quad (i = 0, 1, 2, \ldots). \tag{2.6}$$

Explicitly, δ^{-1} is given by

$$(\delta^{-1}h)(x) = \begin{cases} \int_0^x h(t)\,dt/t & (\operatorname{Re}\mu > 0) \\ -\int_x^\infty h(t)\,dt/t & (\operatorname{Re}\mu < 0) \end{cases} \tag{2.7}$$

and in either case defines a bounded integral operator on $L_{p,\mu}$. Hence $(\delta^{-1})^i f_i \in L_{p,\mu}$ for each $i = 0, 1, 2, \ldots$. Further, N_m^η is one-to-one on $L_{p,\mu}$ and (2.5), (2.6) therefore lead to

$$f_0 = (\delta^{-1})^i f_i \quad (i = 0, 1, 2, \ldots). \tag{2.8}$$

A standard argument based on (2.7) and (2.8) now shows that f_0 is infinitely differentiable and is a function in $F_{p,\mu}$. Hence $g = N_m^\eta f_0 \in N_m^\eta(F_{p,\mu})$ in this case. Finally, to deal with the case $\operatorname{Re}\mu = 0$, notice that (2.3) can be rewritten as

$$\gamma_i^{p,\mu-m}\Big(\big[\Gamma\big(\operatorname{Re}(\eta + n - \mu/m)\big)\big]^{-1} K_m^{\eta-1+n,-n} x^{-m} g\Big) \leq A_g^{(i)} \text{ for } n = 0, 1, 2, \ldots.$$

By the previous case with η, μ and g replaced by $\eta - 1$, $\mu - m$ and $x^{-m}g$ respectively, there exists $h \in F_{p,\mu-m}$ such that $x^{-m}g = N_m^{\eta-1}h$. A simple calculation shows that, as operators on $F_{p,\mu}$,

$$N_m^\eta = x^m N_m^{\eta-1} x^{-m}$$

under the stated conditions. Hence $g = N_m^\eta f$ where $f = x^m h \in F_{p,\mu}$. This completes the proof.

REMARK 2.3. The necessary and sufficient condition obtained in Theorem 2.2 is equivalent to another condition in which the non-negative integer n is replaced by a more general complex number λ. More precisely, under the hypotheses of Theorem 2.2, $g \in N_m^\eta(F_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there are constants $A_g^{(i)}$ independent of λ such that

$$\gamma_i^{p,\mu}\Big(\big[\Gamma\big(\operatorname{Re}(\eta + \lambda - \mu/m)\big)\big]^{-1} K_m^{\eta+\lambda,-\lambda} g\Big) \leq A_g^{(i)} \tag{2.9}$$

for all complex numbers λ such that $\operatorname{Re}(\eta + \lambda - \mu/m) \neq 0, -1, -2, \ldots$. The operator $K_m^{\eta+\lambda,-\lambda}$ is a general Erdélyi-Kober operator whose Mellin multiplier is

$\Gamma(\eta + \lambda + s/m)/\Gamma(\eta + s/m)$. The proof of (2.9) is omitted but we shall require this characterisation of the range in the sequel.

In [5], we noted that $N_m^\eta(L_{p,\mu})$ depends on η in the sense that, if $\operatorname{Re}\eta_1 < \operatorname{Re}\eta_2$ then $N_m^{\eta_1}(L_{p,\mu})$ is a subset of $N_m^{\eta_2}(L_{p,\mu})$ under appropriate conditions. See in particular [5, Corollary 3.6]. This situation arose because a certain Erdélyi-Kober operator was not invertible in the $L_{p,\mu}$ setting. However, when we work in $F_{p,\mu}$, this difficulty disappears.

THEOREM 2.4. *If $1 < p < \infty$, $m > 0$ and $\eta_j - \mu/m \in \Omega$ for $j = 1, 2$ then, as sets,*

$$N_m^{\eta_1}(F_{p,\mu}) = N_m^{\eta_2}(F_{p,\mu}). \tag{2.10}$$

PROOF. Under the stated conditions, the operator equation

$$N_m^{\eta_1} = N_m^{\eta_2} K_m^{\eta_1, \eta_2 - \eta_1}$$

holds on $F_{p,\mu}$, while $K_m^{\eta_1,\eta_2-\eta_1}$ is a homeomorphism from $F_{p,\mu}$ onto itself, with inverse $K_m^{\eta_2,\eta_1-\eta_2}$. The result follows immediately.

Our experience with $N_m^\eta(L_{p,\mu})$ suggests that, to turn N_m^η into a homeomorphism in the $F_{p,\mu}$ setting, we should make use of (2.3) and imitate the construction in [5]. This time, however, we shall obtain a whole family of new seminorms on the range rather than just a single new norm. In what follows, we shall often write

$$r = 1/m \quad (\text{where } m > 0). \tag{2.11}$$

DEFINITION 2.5. Let $m > 0$, $1 < p < \infty$ and $\eta - \mu/m \in \Omega$. For $i = 0, 1, 2, \ldots$ and $g \in N_m^\eta(F_{p,\mu})$ let

$$\gamma_i^{p,\mu,r,\eta}(g) = \inf\{A_g^{(i)} : (2.3) \text{ holds for this fixed } g \text{ and } i\}. \tag{2.12}$$

REMARK 2.6. It is easy to check that, under the stated conditions, $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^\infty$ is a countable multinorm in the sense of Zemanian [13]. Having shown in Theorem 2.4 that the set $N_m^\eta(F_{p,\mu})$ is independent of η, under the appropriate conditions, our aim is to show that $N_m^\eta(F_{p,\mu})$ equipped with the multinorm $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^\infty$ is independent of such η as a topological vector space.

LEMMA 2.7. *If $1 < p < \infty$, $m > 0$ and $\eta_j - \mu/m \in \Omega$ for $j = 1, 2$ then*

$$N_m^{\eta_1}(F_{p,\mu}) \textit{ is continuously imbedded in } N_m^{\eta_2}(F_{p,\mu})$$

with respect to the topologies generated by the multinorms $\{\gamma_i^{p,\mu,r,\eta_1}\}_{i=0}^\infty$ and $\{\gamma_i^{p,\mu,r,\eta_2}\}_{i=0}^\infty$.

PROOF. Let $g \in N_m^{\eta_1}(F_{p,\mu}) = N_m^{\eta_2}(F_{p,\mu})$. We shall make use of basic properties of the Erdélyi-Kober operators. Firstly, for $n = 0, 1, 2, \ldots$

$$K_m^{\eta_2+n,-n} g = K_m^{\eta_1,\eta_2-\eta_1} K_m^{\eta_2+n,\eta_1-\eta_2-n} g.$$

Under the given conditions, $h_n = K_m^{\eta_2+n,\eta_1-\eta_2-n}g \in F_{p,\mu}$ and $K_m^{\eta_1,\eta_2-\eta_1}$ is a continuous linear mapping from $F_{p,\mu}$ into itself. By [13, Lemma 1.10-1], with η_1 and η_2 fixed, for each $i = 0, 1, 2, \ldots$, there exist a non-negative integer N_i and non-negative constants C_j $(j = 0, 1, \ldots , N_i)$ which are independent of n such that

$$\gamma_i^{p,\mu}(K_m^{\eta_2+n,-n}g) \leq \sum_{j=0}^{N_i} C_j\gamma_j^{p,\mu}(K_m^{\eta_2+n,\eta_1-\eta_2-n}g)$$

for all $n = 0, 1, 2, \ldots$. Now divide both sides by the quantity

$$\left|\Gamma\big(\mathrm{Re}\,(\eta_2 + n - \mu/m)\big)\right| = \left|\Gamma\Big(\mathrm{Re}\,\big(\eta_1 + (\eta_2 - \eta_1 + n) - \mu/m\big)\Big)\right|,$$

invoke Remark 2.3 with η and λ replaced by η_1 and $\eta_2 - \eta_1 + n$ to handle the righthand side and take infima to get

$$\gamma_i^{p,\mu,r,\eta_2}(g) \leq \sum_{j=0}^{N_i} C_j\gamma_j^{p,\mu,r,\eta_1}(g). \tag{2.13}$$

The result now follows.

THEOREM 2.8. *For fixed m, p and μ such that $m > 0$, $1 < p < \infty$ and $\mu \in \mathbf{C}$, the topological vector space consisting of the set $N_m^{\eta}(F_{p,\mu})$ and the multinorm $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^{\infty}$ is independent of $\eta \in \mathbf{C}$ satisfying $\eta - \mu/m \in \Omega$.*

PROOF. The two multinorms give equivalent topologies in view of (2.13) and a similar inequality with η_1 and η_2 interchanged. This, together with (2.10), completes the proof.

NOTATION 2.9. Under the conditions of Theorem 2.8, we shall write

$$N_m^{\eta}(F_{p,\mu}) \equiv F_{p,\mu,r} \text{ and } \gamma_i^{p,\mu,r,\eta} \equiv \gamma_i^{p,\mu,r} \quad (i = 0, 1, 2, \ldots) \tag{2.14}$$

to indicate independence of η, subject to the condition

$$\eta - r\mu \in \Omega \tag{2.15}$$

which will be assumed throughout.

REMARK 2.10.

(i) Since we have lost dependence on η, we have the first indication that the operator N_m^{η} is not the only candidate which can be used to generate $F_{p,\mu,r}$. The space depends intrinsically on r (equivalently on m) and we have one such space for every $r > 0$. We shall continue to use N_m^{η} a little longer to develop properties of $F_{p,\mu,r}$ and return later to dependence on r only (for fixed p and μ).

(ii) A fairly routine calculation shows that, with the relevant topologies

$$F_{p,\mu,r} \text{ is homeomorphic to } F_{p,\mu r,1} \tag{2.16}$$

under the mapping P_r where $(P_r\phi)(x) = \phi(x^r)$. Hence any two spaces $F_{p,\mu,r} (r > 0)$ are homeomorphic to each other.

(iii) It will be convenient to write

$$F_{p,\mu} \equiv F_{p,\mu,0} \tag{2.17}$$

i.e. to regard our original $F_{p,\mu}$ space as corresponding in some sense to $r = 0$. The reason for this will become clearer later.

THEOREM 2.11. *The space $F_{p,\mu,r}(r > 0)$ is a Fréchet space with respect to* $\{\gamma_i^{p,\mu,r}\}_{i=0}^{\infty}$.

PROOF. Only completeness has to be established and in view of Remark 2.10(ii) it is sufficient to prove the result for $r = 1$. Choose any $\eta \in \mathbf{C} : \eta - \mu \in \Omega$.

Let $\{g_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $F_{p,\mu,1}$. By (2.3) and (2.12),

$$\gamma_i^{p,\mu}\Big(K_1^{\eta+n,-n}(g_j - g_k)\Big) \leq \gamma_i^{p,\mu,1}(g_j - g_k)\Big|\Gamma\Big(\operatorname{Re}(\eta + n - \mu)\Big)\Big| \tag{2.18}$$

for each fixed i, $n = 0, 1, 2, \ldots$ and all $j, k = 1, 2, \ldots$. Hence for each fixed n, $\{K_1^{\eta+n,-n}g_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $F_{p,\mu}$. Since the latter space is complete, for each $n = 0, 1, 2, \ldots$ we may define

$$h_n = \lim_{k\to\infty} K_1^{\eta+n,-n}g_k \tag{2.19}$$

where the limit is with respect to the $F_{p,\mu}$ topology. In particular $g_k \longrightarrow h_0$ as $k \longrightarrow \infty$, and, since $K_1^{\eta+n,-n}$ is a homeomorphism on $F_{p,\mu}$ under the given conditions, (2.19) shows that

$$h_n = K_1^{\eta+n,-n}h_0 \text{ for all } n = 0, 1, 2, \ldots . \tag{2.20}$$

Next observe that, for fixed $i = 0, 1, 2, \ldots$, there is a constant C_i such that, for $k \geq 1$,

$$\gamma_i^{p,\mu}(K_1^{\eta+n,-n}g_k) \leq C_i\Big|\Gamma\Big(\operatorname{Re}(\eta + n - \mu)\Big)\Big| \text{ for all } n = 0, 1, 2, \ldots .$$

By letting $k \longrightarrow \infty$ and using (2.19) and (2.20), we obtain

$$\gamma_i^{p,\mu}(K_1^{\eta+n,-n}h_0) \leq C_i\Big|\Gamma\Big(\operatorname{Re}(\eta + n - \mu)\Big)\Big| \text{ for all } n = 0, 1, 2, \ldots$$

so that $h_0 \in F_{p,\mu,1}$, with $\gamma_i^{p,\mu,1}(h_0) \leq C_i$. Also, from (2.18), for any $\varepsilon > 0$, there exists a positive integer N, independent of n, such that

$$\gamma_i^{p,\mu}\Big(K_1^{\eta+n,-n}(g_j - g_k)\Big) \leq \varepsilon\Big|\Gamma\Big(\operatorname{Re}(\eta + n - \mu)\Big)\Big| \text{ for all } j, k \geq N.$$

If we let $j \longrightarrow \infty$ and use (2.19) and (2.20) again, we obtain for $n = 0, 1, 2, \ldots$

$$\gamma_i^{p,\mu}\Big(K_1^{\eta+n,-n}(h_0 - g_k)\Big) \leq \varepsilon\Big|\Gamma\Big(\operatorname{Re}(\eta + n - \mu)\Big)\Big| \text{ for all } k \geq N$$
$$\Rightarrow \gamma_i^{p,\mu,1}(h_0 - g_k) \leq \varepsilon \text{ for all } k \geq N.$$

Hence $\{g_k\}_{k=1}^{\infty}$ converges to h_0 with respect to $\{\gamma_i^{p,\mu,1}\}_{i=0}^{\infty}$. This completes the proof.

THEOREM 2.12. *Under condition (2.15), N_m^{η} is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$ with respect to the multinorms $\{\gamma_i^{p,\mu}\}_{i=0}^{\infty}$ and $\{\gamma_i^{p,\mu,r}\}_{i=0}^{\infty}$ respectively.*

PROOF. N_m^{η} is one-to-one (by the corresponding result in $L_{p,\mu}$) and onto (by construction). As regards continuity of N_m^{η}, note that, as operators on $F_{p,\mu}$,

$$K_m^{\eta+n,-n}N_m^{\eta} = N_m^{\eta+n}; \quad x^iD^iN_m^{\eta+n} = N_m^{\eta+n}x^iD^i \quad (i = 0, 1, 2, \ldots)$$

so that, for $f \in F_{p,\mu}$,

$$\begin{aligned}\gamma_i^{p,\mu}(K_m^{\eta+n,-n}N_m^\eta f) &= \gamma_i^{p,\mu}(N_m^{\eta+n}f) = \|x^iD^iN_m^{\eta+n}f\|_{p,\mu}\\ &= \|N_m^{\eta+n}x^iD^if\|_{p,\mu} \le \left|\Gamma\big(\mathrm{Re}\,(\eta+n-\mu/m)\big)\right| \|x^iD^if\|_{p,\mu}\end{aligned}$$

where we have used [5, Theorem 7.1]. It then follows that

$$\gamma_i^{p,\mu,r}(N_m^\eta f) \le \gamma_i^{p,\mu}(f) \text{ for all } i = 0, 1, 2, \ldots$$

from which continuity of N_m^η follows. Finally, continuity of $(N_m^\eta)^{-1}$ is now automatic by Theorem 2.11 and the Open Mapping Theorem for Fréchet spaces [12, Theorem 17.1].

3. In [5], we obtained characterisations of $N_m^\eta(L_{p,\mu})$ and $M_m^\xi(L_{p,\mu})$ and discovered that these spaces were not the same. In contrast, we shall show that the space $F_{p,\mu,r}$, which represents the range of N_m^η on $F_{p,\mu}$ for all η satisfying (2.15), is also the range of M_m^ξ on $F_{p,\mu}$ for all ξ satisfying an analogue of (2.15). We achieve this by making use of the theory of multipliers developed by Rooney [8].

THEOREM 3.1. *Let $1 < p < \infty$, $r > 0$ and $\mu \in \mathbf{C}$. If ξ is any complex number such that $\xi + r\mu \in \Omega$ then M_m^ξ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$ (where, as usual, $m = 1/r$).*

PROOF. Choose $\eta_1 \in \mathbf{C}$ such that $\mathrm{Re}\,(\eta_1 - r\mu) > \max\big(0, \mathrm{Re}\,(\xi + r\mu)\big)$ and let

$$h_1(s) = \Gamma(\xi - rs)/\Gamma(\eta_1 + rs). \tag{3.1}$$

Then we can find a strip $S = \{s \in \mathbf{C} : \alpha < \mathrm{Re}\, s < \beta\}$ such that

(i) S contains the line $\mathrm{Re}\, s = -\mathrm{Re}\,\mu$

(ii) $\mathrm{Re}\,(\eta_1 + rs) > \mathrm{Re}\,(\xi - rs)$ for all $s \in S$

(iii) h is analytic on S.

By using the strip S we can check that h is in the class $\mathcal{A}$ introduced in [8, Definition 3.1]. Indeed, by [1, 1.18(6)]

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\,|y|^{x-1/2}e^{-\pi|y|/2} \tag{3.2}$$

as $|y| \longrightarrow \infty$, uniformly with respect to x in a bounded interval, and since $\mathrm{Re}\,(\xi - rs) < \mathrm{Re}\,(\eta_1 + rs)$, we can deduce that h is bounded on any substrip of the form $\alpha' \le \mathrm{Re}\, s \le \beta'$ with $\alpha < \alpha' < \beta' < \beta$. The condition that

$$|h'(s)| = O(|\mathrm{Im}\, s|^{-1}) \quad (s \in S)$$

can be checked similarly by using the formula [1, 1.18(7)] for the asymptotic behaviour of the function $\psi = \Gamma'/\Gamma$. Thus, by [8, Theorem 1], h is an $L_{p,\mu}$ multiplier and hence, by [4, Theorem 3.3], an $F_{p,\mu}$ multiplier. Hence there is a continuous linear mapping T_1 from $F_{p,\mu}$ into $F_{p,\mu}$ such that

$$\big(\mathcal{M}(T_1 f)\big)(s) = h(s)(\mathcal{M}f)(s) \tag{3.3}$$

whenever $f \in F_{p,\mu} \cap F_{2,\mu}$ and $\operatorname{Re} s = -\operatorname{Re} \mu$.

Recall that the operators $N_m^{\eta_1}$ and M_m^{ξ} have the respective multipliers

$$\Gamma(\eta_1 + s/m) \equiv \Gamma(\eta_1 + rs) \text{ and } \Gamma(\xi - s/m) \equiv \Gamma(\xi - rs).$$

It follows from (3.1) and (3.3) that

$$\big(\mathcal{M}(N_m^{\eta_1} T_1 f)\big)(s) = \big(\mathcal{M}(M_m^{\xi} f)\big)(s)$$

whenever $f \in F_{p,\mu} \cap F_{2,\mu}$ and $\operatorname{Re} s = -\operatorname{Re} \mu$. By a standard continuity and density argument we deduce that, under the given hypotheses,

$$N_m^{\eta_1} T_1 f = M_m^{\xi} f \text{ for all } f \in F_{p,\mu} \tag{3.4}$$

so that M_m^{ξ} is a continuous linear mapping from $F_{p,\mu}$ into $N_m^{\eta_1}(F_{p,\mu}) \equiv F_{p,\mu,r}$ this being valid since $\eta_1 - r\mu \in \Omega$ by choice of η_1.

To prove that M_m^{ξ} is a homemorphism we choose $\eta_2 \in \mathbf{C}$ such that $\operatorname{Re}(\eta_2 - r\mu) < \operatorname{Re}(\xi + r\mu)$ and $\eta_2 - r\mu \in \Omega$. Consider the multiplier

$$h_2(s) = \Gamma(\eta_2 + rs)/\Gamma(\xi - rs)$$

in a suitable strip containing the line $\operatorname{Re} s = -\operatorname{Re} \mu$, throughout which h_2 is analytic and $\operatorname{Re}(\eta_2 + rs) < \operatorname{Re}(\xi - rs)$. By proceeding as above, we obtain a continuous linear operator T_2 from $F_{p,\mu}$ into $F_{p,\mu}$ such that

$$M_m^{\xi} T_2 f = N_m^{\eta_2} f \text{ for all } f \in F_{p,\mu}$$

by analogy with (3.4). This shows that $F_{p,\mu,r} \equiv N_m^{\eta_2}(F_{p,\mu}) \subseteq M_m^{\xi}(F_{p,\mu})$ and hence from above we obtain $M_m^{\xi}(F_{p,\mu}) = F_{p,\mu,r}$. Furthermore, M_m^{ξ} is a homeomorphism by the Open Mapping Theorem for Fréchet spaces [12, Theorem 17.1] and the proof is complete.

REMARK 3.2. Our proof above required us to choose two separate values of η in order to use the asymptotics of Γ and ψ and obtain a multiplier in the class $\mathcal{A}$ for both h_1 and h_2. An alternative approach can be found in [10, Theorem 6.22]. We shall make further use of multipliers in subsequent work which will reveal that Theorem 3.1 is only the tip of the iceberg. There are many continuous linear mappings and homeomorphisms, all mapping $F_{p,\mu}$ onto $F_{p,\mu,r}$ and all having their behaviour dictated by a common value of a parameter r. Any one of these operators could be used to study the space $F_{p,\mu,r}$ but N_m^{η} and M_m^{ξ} (with $m = 1/r$) are the two which have the simplest multipliers.

4. Having obtained the topological structure of the spaces $F_{p,\mu,r}$, we shall now look at a few simple operators relative to these spaces. Some of our results point the way ahead to more substantial results in § 5.

THEOREM 4.1. *For $1 < p < \infty$, $r > 0$ and any complex numbers λ and μ, the mapping x^λ is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,\mu+\lambda,r}$ with inverse $x^{-\lambda}$. [Here, as usual, we are talking of the mapping which sends $g(x)$ to $x^\lambda g(x)$.]*

PROOF. Choose any $\eta \in \mathbf{C}$ such that $\eta - r\mu \in \Omega$. Then, with $m = 1/r$ and $g \in F_{p,\mu,r} = N_m^\eta(F_{p,\mu})$, we obtain

$$K_m^{\eta+n,-n} g = x^{-\lambda} K_m^{\eta+r\lambda+n,-n}(x^\lambda g) \text{ for } n = 0, 1, 2, \ldots .$$

Hence, for $i = 0, 1, 2, \ldots$,

$$\begin{aligned}
\gamma_i^{p,\mu+\lambda}&\Big(\Big[\Gamma\Big(\operatorname{Re}\big((\eta + r\lambda) + n - r(\mu + \lambda)\big)\Big)\Big]^{-1} K_m^{\eta+r\lambda+n,-n}(x^\lambda g)\Big)\\
&= \big\|x^i D^i\Big(\Big[\Gamma\Big(\operatorname{Re}(\eta + n - r\mu)\Big)\Big]^{-1} x^{-\lambda} K_m^{\eta+r\lambda+n,-n}(x^\lambda g)\Big)\big\|_{p,\mu}\\
&= \big\|x^i D^i\Big(\Big[\Gamma\Big(\operatorname{Re}(\eta + n - r\mu)\Big)\Big]^{-1} K_m^{\eta+n,-n} g\Big)\big\|_{p,\mu}\\
&= \gamma_i^{p,\mu}\Big(\Big[\Gamma\Big(\operatorname{Re}(\eta + n - r\mu)\Big)\Big]^{-1} K_m^{\eta+n,-n} g\Big).
\end{aligned}$$

It now follows that

$$\gamma_i^{p,\mu+\lambda,r,\eta+r\lambda}(x^\lambda g) = \gamma_i^{p,\mu,r,\lambda}(g) \tag{4.1}$$

the lefthand side being well defined, as $\operatorname{Re}\big((\eta + r\lambda) - r(\mu + \lambda)\big) = \operatorname{Re}(\eta - r\mu) \neq 0, -1, -2, \ldots$. (4.1) proves that x^λ is a continuous mapping from $F_{p,\mu,r}$ into $F_{p,\mu+\lambda,r}$. A similar argument shows that $x^{-\lambda}$ is a continuous mapping from $F_{p,\mu+\lambda,r}$ into $F_{p,(\mu+\lambda)-\lambda,r}$ $= F_{p,\mu,r}$ and the required result follows at once.

THEOREM 4.2. *For $1 < p < \infty$, $r > 0$ and $\mu \in \mathbf{C}$, the operator U defined by*

$$(Ug)(x) = g(1/x) \quad (x > 0) \tag{4.2}$$

is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,-\mu,r}$ and $U^{-1} = U$.

PROOF. Choose any $\eta \in \mathbf{C}$ such that $\operatorname{Re}(\eta - r\mu) \neq 0, -1, -2, \ldots$. A simple calculation involving multipliers [5, (8.2)] gives

$$M_m^\eta U f = U N_m^\eta f \quad (f \in F_{p,\mu}). \tag{4.3}$$

(We shall use U rather than R, as used in [5], to avoid any confusion with r.) The operator U is a homeomorphism from $F_{p,\mu}$ onto $F_{p,-\mu}$ so that both sides of (4.3) are well-defined under the given condition on η. We can rewrite (4.3) in the form

$$Ug = M_m^\eta U (N_m^\eta)^{-1} g \quad (g \in F_{p,\mu,r})$$

and by Theorem 2.12, Theorem 3.1 and our previous remark concerning U, the required result now follows.

REMARK 4.3. From earlier work, we know that Theorems 4.1 and 4.2 remain true when $r = 0$ if we make the convention in (2.17). Indeed such results justify the use of (2.17) to some extent and further justification will follow.

Suppose now that we look at things the other way. Under what circumstances will an operator which is well-behaved in the original $F_{p,\mu}$ setting ($r = 0$) continue to be well-behaved relative to the $F_{p,\mu,r}$ spaces for $r > 0$? The simplest situation is when an operator T is a continuous linear mapping (or homeomorphism) from $F_{p,\mu}$ into $F_{p,\mu}$ (i.e. no change in μ). For $r > 0$, $F_{p,\mu,r}$ is a subset of $F_{p,\mu}$ and we may ask when T restricted to $F_{p,\mu,r}$ gives a continuous linear mapping (or homeomorphism) from $F_{p,\mu,r}$ into $F_{p,\mu,r}$. There is a large class of operators T for which this is true, as we shall now see.

THEOREM 4.4. *For $1 < p < \infty$ and appropriate complex numbers μ let T be a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ corresponding to an $F_{p,\mu}$ Mellin multiplier h. Then (the restriction of) T is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ for all $r > 0$ (under the same conditions on p and μ).*

PROOF. The differential operator $K_m^{\eta+n,-n}$ appearing in (2.3) is a Mellin multiplier transform whose multiplier is $\prod_{j=1}^{n}(\eta + j - 1 + s/m)$ where, as usual, $m = 1/r$ and η is such that $\mathrm{Re}(\eta - r\mu) \neq 0, -1, -2, \ldots$. Since any two multiplier transforms commute, it follows that, in the notation of (2.3),

$$\begin{aligned}
&\gamma_i^{p,\mu}\Big(\big[\Gamma\big(\mathrm{Re}(\eta + n - \mu r)\big)\big]^{-1} K_m^{\eta+n,-n} Tg\Big) \\
(4.4)\qquad &= \gamma_i^{p,\mu}\Big(T\Big(\big[\Gamma\big(\mathrm{Re}(\eta + n - \mu/m)\big)\big]^{-1} K_m^{\eta+n,-n} g\Big)\Big) \\
&\leq \sum_{j=0}^{N(i)} \gamma_j^{p,\mu}\Big(\big[\Gamma\big(\mathrm{Re}(\eta + n - \mu/m)\big)\big]^{-1} K_m^{\eta+n,-n} g\Big)
\end{aligned}$$

for some non-negative integer $N(i)$ and constants C_j $\big(j = 0, 1, \ldots, N(i)\big)$ by [13, Lemma 1.10-1]. The inequality (4.4) now leads to

$$\gamma_i^{p,\mu,r,\eta}(Tg) \leq \sum_{j=0}^{N(i)} C_j \gamma_j^{p,\mu,r,\eta}(g) \quad \forall g \in N_m^{\eta}(F_{p,\mu}) \equiv F_{p,\mu,r}$$

and the result follows.

COROLLARY 4.5. *Let $1 < p < \infty$ and $\mu \in \mathbf{C}$. If T is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ and T, T^{-1} are Mellin multiplier transforms (corresponding to multipliers h and $1/h$, say) then (the restriction of) T is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,\mu,r}$ for all $r > 0$.*

PROOF. This is immediate on applying Theorem 4.4 to T and T^{-1}.

REMARK 4.6. Theorem 4.4 and Corollary 4.5 are further instances of results which hold for $r > 0$ as well as for $r = 0$, with the results for $r > 0$ being inherited from those for $r = 0$.

EXAMPLE 4.7. The spaces $F_{p,\mu} \equiv F_{p,\mu,0}$ studied in [2] were developed for the study of the Erdélyi-Kober operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ (the η here not being the same η as in N_m^{η}

necessarily!) In view of results obtained for $r = 0$, we can say that for $1 < p < \infty$, $r > 0$ and appropriate $\mu \in \mathbf{C}$ (and with $m = 1/r$),

(i) $I_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ if $\eta + 1 + r\mu \in \Omega$ and is a homeomorphism if, in addition, $\eta + \alpha + 1 + r\mu \in \Omega$.

(ii) $K_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ if $\eta - r\mu \in \Omega$ and is a homeomorphism if, in addition, $\eta + \alpha - r\mu \in \Omega$.

We therefore have a whole family of subspaces of the original $F_{p,\mu}$ spaces which are invariant under the Erdélyi-Kober operators, at least for $1 < p < \infty$.

Use of Theorem 4.1 enables us to handle operators where μ changes but p and r remain the same. An example of such operators is given by certain operators involving the ${}_2F_1$ hypergeometric function which were studied relative to the $F_{p,\mu}$ spaces in [2].

EXAMPLE 4.8. Consider the operator $H_1(a,b;c;m) = x^{mc}T_1(a,b;c;m)$ where $T_1(a,b;c;m)$ is a Mellin multiplier transform with multiplier

$$\Gamma(a+1-s/m)\Gamma(b+1-s/m)/\{\Gamma(a+b+1-s/m)\Gamma(c+1-s/m)\}. \tag{4.5}$$

Here a, b and c are suitably restricted complex numbers and $m > 0$. The multiplier in (4.5) is of a form which we shall be handling later in more generality. For the moment, we can proceed by factorising $H_1(a,b;c;m)$ in the form

$$H_1(a,b;c;m) = x^{mc-mb} I_m^{0,c-b} I_m^{a-b,b} x^{mb}$$

which can easily be checked via multipliers. Manipulations will be valid on $F_{p,\mu}$ provided that $a + 1 + \mu/m \in \Omega$ and $b + 1 + \mu/m \in \Omega$. Theorem 4.1 and Example 4.7 then show that $H_1(a,b;c;m)$ is a continuous linear mapping from $F_{p,\mu,1/m}$ into $F_{p,\mu+mc,1/m}$ for $1 < p < \infty$, $m > 0$ and μ as above. Furthermore, a homeomorphism will be obtained if, in addition,

$$c + 1 + \mu/m \in \Omega \text{ and } a + b + 1 + \mu/m \in \Omega.$$

When $\operatorname{Re}(a + 1 + \mu/m) > 0$ and $\operatorname{Re}(b + 1 + \mu/m) > 0$, the operator $H_1(a,b;c;m)$ is the integral operator given by

$$\big(H_1(a,b;c;m)f\big)(x) = \int_0^x (x^m - t^m)F^*(a,b;c;1 - x^m/t^m)f(t)\,d(t^m) \tag{4.6}$$

where $F^*(a,b;c;z)$ is an analytic continuation of ${}_2F_1(a,b;c;z)/\Gamma(c)$ [2, p. 88, 93]. These results for $r = 1/m > 0$ accord with those in [2] for $r = 0$. The other operators in [2, Chapter 4] can be treated similarly.

Hypergeometric functions also arise in our next example. In [11], differential operators of the form $x^{a_1}Dx^{a_2}Dx^{a_3}$ (of so-called Bessel type) were studied and in [3] these considerations were extended to n^{th} order expressions. We shall review just one of the results.

EXAMPLE 4.9. Consider the formal differential expression

$$T = x^{a_1}Dx^{a_2}D\cdots x^{a_n}Dx^{a_{n+1}}$$

of order n where $a_1, \ldots, a_{n+1}$ are complex numbers such that

$$m = n - a > 0, \text{ where } a = \sum_{i=1}^{n+1} a_i.$$

We showed that relative to $F_{p,\mu}$ spaces, it was possible to define an α^{th} power T^α of T to be such that

$$\big(\mathcal{M}(T^\alpha f)\big)(s + m\alpha) = m^{n\alpha} \prod_{k=1}^{n} \frac{\Gamma(b_k + 1 - s/m)}{\Gamma(b_k + 1 - \alpha - s/m)} (\mathcal{M}f)(s)$$

under appropriate conditions, where

$$b_k = \left(\sum_{i=k+1}^{n+1} a_i + k - n \right) / m \quad (k = 1, \ldots, n).$$

Again, the product involving gamma functions is a special case of the type of general multiplier we shall discuss later but here we can use the factorisation

$$T^\alpha = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k, -\alpha} \tag{4.7}$$

in terms of Erdélyi-Kober operators. Theorem 4.1 and Example 4.7 show that T^α defines a continuous linear mapping from $F_{p,\mu,1/m}$ into $F_{p,\mu-m\alpha,1/m}$ for every $m > 0$ provided that $1 < p < \infty$ and $b_k + 1 + \mu/m \in \Omega$ for $k = 1, \ldots, n$.

REMARK 4.10. Examples 4.8 and 4.9 both lead to multipliers which are special cases of the class we are interested in and, indeed, for $\operatorname{Re}\alpha < 0$ the operator in (4.7) is an integral operator involving Meijer's G-function $G_{n,n}^{n,0}$. However, in neither case do we effect a change in the value of r. This is because the product of quotients of gamma functions is "balanced" in a sense to be made precise. A change in the value of r occurs when the multiplier is not "balanced," as we shall discover when we develop the theory of our general class of Mellin multiplier transforms relative to the $F_{p,\mu,r}$ spaces ($r \geq 0$). We are now ready to embark upon this development.

5. Consider again the multiplier h in (1.2) and let

$$c = \sum_{i=k+1}^{K} r_i + \sum_{j=1}^{\ell} t_j - \sum_{i=1}^{k} r_i - \sum_{j=\ell+1}^{L} t_j \tag{5.1}$$

$$d = \operatorname{Re}\left\{ \sum_{i=k+1}^{K} \eta_i + \sum_{j=\ell+1}^{L} \xi_j - \sum_{i=1}^{k} \eta_i - \sum_{j=1}^{\ell} \xi_j \right\} + k + \ell - \frac{1}{2}(K + L) \tag{5.2}$$

$$r = \sum_{i=k+1}^{K} r_i + \sum_{j=\ell+1}^{L} t_j - \sum_{i=1}^{k} r_i - \sum_{j=1}^{\ell} t_j. \tag{5.3}$$

The use of r in (5.3) is deliberate and we shall reconcile this version of r with the previous version of $r = 1/m$ shortly. Of course, c, d and r all depend on the multiplier h. The relevance of these quantities is shown in the following lemma.

LEMMA 5.1. *For the multiplier h in (1.2), and with* $s = \sigma + i\tau$ $(\sigma, \tau$ *real)*

$$|h(s)| = O(|\tau|^{c\sigma+d}\exp\{-\pi r|\tau|/2\}) \text{ as } |\tau| \longrightarrow \infty, \tag{5.4}$$

the estimate holding uniformly for σ *in any compact subset of* $\mathbf{R}$.

PROOF. The result follows by applying (3.2) to each Γ-function appearing in h.

Recall next the class $\mathcal{C}$ of multipliers introduced in [7]. If we use the equivalent version obtained in [7, Theorem 4.4], we can deduce

COROLLARY 5.2. *If* $r \equiv r(h) \geq 0$, *then the multiplier h in (1.2) belongs to the class* $\mathcal{C}$.

PROOF. Chose any real numbers α and β such that the strip $\alpha < \operatorname{Re} s < \beta$ contains none of the poles of h (which are finite in number). Then h is analytic on this strip and if we choose a positive integer N such that

$$N > \sup\{c\sigma + d : \alpha < \sigma = \operatorname{Re} s < \beta\},$$

then, by (5.4), $|s^{-N}h(s)|$ will be bounded as $|s| \longrightarrow \infty$, uniformly with respect to σ in any closed substrip $\alpha' \leq \sigma \leq \beta'$ where $\alpha < \alpha' < \beta' < \beta$, i.e. $h(s)$ is uniformly of order $|s|^N$ as $|s| \longrightarrow \infty$ within such a strip. The result follows.

In view of [7, Theorem 4.3], we can conclude that h is the multiplier of a mapping T which maps $F_{p,\mu}$ into $F_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$ where α and β are any real numbers such that h is analytic on $\alpha < \operatorname{Re} s < \beta$. Also we may allow $\alpha = -\infty$ or $\beta = \infty$, as appropriate. It is convenient to introduce the following notation.

DEFINITION 5.3. For h as in (1.2), define the set $\Delta \equiv \Delta(h)$ by

$$\Delta(h) = \{x \in \mathbf{R} : \text{ no pole of } h(s) \text{ lies on } \operatorname{Re} s = x\}. \tag{5.5}$$

Our previous statement then becomes the statement that h is an $F_{p,\mu}$ multiplier for $1 < p < \infty$ and $-\operatorname{Re}\mu \in \Delta(h)$, provided that $r(h) \geq 0$.

We can think of multipliers (1.2) having $r(h) = 0$ as being "balanced" while those having $r(h) > 0$ are "top heavy." As might be expected, the properties of the multiplier transform T corresponding to h are simplest in the balanced case.

THEOREM 5.4. *Let T be the multiplier transform corresponding to the multiplier (1.2) with* $r(h) = 0$.

(i) *If* $1 < p < \infty$ *and* $-\operatorname{Re}\mu \in \Delta(h)$, *then T is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu}$.

(ii) *If, in addition,* $-\operatorname{Re}\mu \in \Delta(1/h)$, *then T is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu}$ *whose inverse is the multiplier transform corresponding to* $1/h$.

PROOF. This is almost immediate from the preamble and the observation that, in general,

$$r(1/h) = -r(h) \tag{5.6}$$

as is obvious from (5.3). In our case $r(1/h) = r(h) = 0$.

EXAMPLE 5.5. Individual Erdélyi-Kober operators are balanced as are products of such operators, a particular example being the hypergeometric operator $T_1(a, b; c; m)$ in Example 4.8 corresponding to the balanced multiplier (4.5).

However, the full significance of the $F_{p,\mu,r}$ spaces begins to become apparent when $r(h) > 0$. The crux of the proof of the following theorem is to make the original top-heavy multiplier balanced by introducing another Γ-function in the denominator, the extra Γ-function being the multiplier associated with N_m^η $(m = 1/r)$.

THEOREM 5.6. *Let T be the multiplier transform corresponding to the multiplier (1.2) with $r \equiv r(h) > 0$.*

(i) If $1 < p < \infty$ and $-\mathrm{Re}\,\mu \in \Delta(h)$, then T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu,r}$.

(ii) If, in addition, $-\mathrm{Re}\,\mu \in \Delta(1/h)$, then T is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$.

PROOF. For fixed μ such that $-\mathrm{Re}\,\mu \in \Delta(h)$, there exist numbers α and β such that $h(s)$ is analytic on a strip $\alpha < \mathrm{Re}\,s < \beta$ which contains the line $\mathrm{Re}\,s = -\mathrm{Re}\,\mu$. Choose any η such that $\mathrm{Re}(\eta + r\alpha) > 0$. Then $\Gamma(\eta + rs)$ is also analytic on the strip $\alpha < \mathrm{Re}\,s < \beta$. Let

$$g(s) = h(s)/\Gamma(\eta + rs). \tag{5.7}$$

Then $\Delta(h) \subseteq \Delta(g)$, since no new poles have been created and the multiplier g is balanced, since $r(g) = r(h) - r = 0$. By Theorem 5.4(i), g gives rise to a Mellin multiplier transform T_0 which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ for the given μ and for $1 < p < \infty$. However, (5.7) leads to

$$h(s) = \Gamma(\eta + rs)g(s)$$

and, under the given conditions, $\Gamma(\eta + rs)$ is the multiplier of N_m^η with $m = 1/r$, so that, as operators on $F_{p,\mu}$,

$$T = N_m^\eta T_0 \tag{5.8}$$

and since, by construction, N_m^η maps $F_{p,\mu}$ continuously onto $F_{p,\mu,r}$, (i) of the theorem is proved.

To prove (ii) assume also that $-\mu \in \Delta(1/h)$. Then

$$1/g(s) = \Gamma(\eta + rs)/h(s)$$

so that by choice of η, with α and β as above, $-\mu \in \Delta(1/g)$. By Theorem 5.4(ii), T_0 is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$. Also N_m^η is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$. Part (ii) of the theorem therefore follows from (5.8). This completes the proof of the theorem.

REMARK 5.7. We can now see clearly how the theory of the multiplier (1.2) becomes so much simpler if we work relative to $F_{p,\mu}$ rather than to $L_{p,\mu}$. In the case when the corresponding operator T is as well-behaved as possible, i.e. is a homeomorphism, the range depends only on the combination (5.3) of the parameters r_i, t_j $(1 \leq i \leq K$, $1 \leq j \leq L)$ and not on the parameters individually. Furthermore, for fixed $\mu \in \mathbf{C}$ and $1 < p < \infty$, the range is independent of the parameters $\eta_i, \xi_j (1 \leq i \leq K, 1 \leq j \leq L)$ provided only that these are chosen so that we avoid poles and ensure that $-\mu \in \Delta(h) \cap \Delta(1/h)$.

Theorem 5.6 can be extended further. Bearing in mind the convention adopted in (2.17), we have obtained a continuous linear mapping from $F_{p,\mu,0}$ into $F_{p,\mu,r}$ and, under additional conditions, a homeomorphism. Since any $F_{p,\mu,r}$ space is a subset of $F_{p,\mu,0}$ (with a different topology), we might enquire as to how the restriction of T to $F_{p,\mu,r'}$ behaves, for any $r' > 0$. This question can be answered completely. We can state the answer rather imprecisely in the form of the final theorem.

THEOREM 5.8. *Let T be the multiplier transform corresponding to the multiplier (1.2) with $r \equiv r(h) \geq 0$. Then under conditions of great generality, T (restricted where appropriate) is a continuous linear mapping from $F_{p,\mu,r'}$ into $F_{p,\mu,r'+r}$ for any $r' \geq 0$ and will be a homeomorphism under additional mild restrictions.*

REMARK 5.9.

(i) We shall not offer a proof of Theorem 5.8 here as a certain amount of extra machinery is needed. One approach is via duality and it seems appropriate to defer further details until a future paper where we hope to present a distributional analogue of the classical $L_{p,\mu}$ theory. Since the $F_{p,\mu}$ spaces are the underlying spaces of test-functions, the simple conditions on parameters exemplified in Theorem 5.6 will be retained in the distributional theory, again in contrast to the classical theory.

(ii) Theorem 4.4 and Example 4.7 provide an illustration of Theorem 5.8, with r' and r being replaced by r and 0 respectively. In general, each $F_{p,\mu,r}$ space will be invariant under any multiplier transform corresponding to a balanced multiplier.

(iii) The question arises as to what can be done when r is negative. The multiplier (1.2) is then bottom heavy, a typical example being $1/\Gamma(\eta + rs)$ which should correspond to $(N_m^\eta)^{-1}$ with $m = 1/r$. Since this operator maps the subspace $F_{p,\mu,r}$ onto $F_{p,\mu,0}$, it seems reasonable that $(N_m^\eta)^{-1}(F_{p,\mu})$ will be a larger set than $F_{p,\mu,0}$. This leads to an attempt to define a "negative space" $F_{p,\mu,r}$ for $r < 0$ as opposed to the "positive spaces" $F_{p,\mu,r}$ for $r > 0$. Such ideas are again related to duality and it is possible to mimic a construction often used for Hilbert spaces. We hope to pursue this topic also in a future paper.

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