Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 475–480

THE CONVERGENCE OF SERIES FOR VARIOUS CHOICES OF SIGN IN BANACH SPACES

JAMES SHIREY

1. Let (x_n, X_n) denote a basis for a Banach space $(X, ||\cdot||)$ of measurable functions in (0, 1).

It is shown in [**2**] and [**9**] that the equivalence of the norms

$$||(\sum_1^\infty X_n^2(\cdot)x_n^2)^{\frac{1}{2}}||$$

and $||\cdot||$ is equivalent to the unconditionality of the basis (x_n, X_n). In [**8**] a weaker relationship between these norms is exploited to establish the existence of an element of $L_1(E)$ for each $E \subset (0, 1)$, $|E| > 0$, whose Haar series expansion is conditionally convergent in the norm of $L_1(E)$.

In this note, a Lemma of Orlicz [**7**] is generalized to provide a relationship between $||(\sum_1^\infty y_n^2)^{\frac{1}{2}}||$, $y_n \in X$, and the changes in sign that are tolerated in $\sum_1^\infty y_n$ without disruption of norm convergence. Some applications to the Haar and Walsh systems are given.

Given a set $H \subset L_1(0,1)$ of non-negative functions, define for each measurable real-valued function x on (0, 1),

$$||x|| = \sup\left\{\int_0^1 |x(t)h(t)|dt : h \in H\right\}, \quad \text{and}$$

$$X = \{x : ||x|| < \infty\}.$$

The functional $||\cdot||$ is said to have the "Fatou property" whenever it follows from $0 \leqq u_1 \leqq u_2 \leqq \ldots \uparrow u$, with all u_n measurable, that $||u_n|| \uparrow ||u||$. In all that follows we assume that $||\cdot||$ has the Fatou property, which guarantees the norm-completeness of $(X, ||\cdot||)$ [**10**, Chapter 15]. This may be ensured by less stringent conditions on $||\cdot||$, but the Fatou property is easy to verify in cases and pertains to most of the important examples. In [**5**, p. 66] various conditions on H are listed whose fulfillment causes $||\cdot||$ to have this property.

Given a space $(X, ||\cdot||)$ of the type described above and a series $x = \sum x_i$, $x_i \in X$, define $G(x) = ||(\sum_1^\infty x_i^2)^{\frac{1}{2}}||$ and $C(x) = \{\theta : \sum_1^\infty r_i(\theta)x_i \text{ converges in } ||\cdot||\}$ where $\{r_i\}$ denotes the Rademacher system.

If $|C(x)| = 0$ $(= 1)$, then the series $\sum_1^\infty x_i$ is said to "diverge (respectively converge) for almost every choice of sign". The obvious justification for such terminology is the possibility of obtaining any desired choice of signs in the series $\sum_1^\infty \pm x_i$ by a proper selection of θ in $\sum_1^\infty r_i(\theta)x_i$.

Received November 6, 1973 and in revised form, March 26, 1974.

https://doi.org/10.4153/CJM-1975-056-2 Published online by Cambridge University Press

2. THEOREM 1. *Let* $x \equiv \sum_1^\infty x_i$ *be convergent in* $(X, ||\cdot||)$. *If* $|C(x)| > 0$, *then* $G(x) < \infty$.

Remarks. This result was given by Orlicz [**7**] for the "Orlicz spaces" under the assumption that $C(x) = [0, 1]$, and in [**9**] for the Banach function spaces of the type defined above under the same assumption. Gelbaum [**2**] has shown that $|C(x)| = 1$ through the use of the "0 − 1" law, provided that $|C(x)| > 0$.

Proof of Theorem 1. It is easily verified that $C(x)$ is a Borel set (see, for example, the proof of Theorem 6 in [**8**]) and that there exists a Borel set $S \subset C(x)$, $|S| > 0$, and an $M > 0$ such that

$$(1)\quad ||\sum_m^n r_n(\theta)x_n|| \leqq M \text{ for all } n, m > 0 \text{ and all } \theta \in S.$$

In reference to Lemma 4 of [**8**], there exist constants A and N depending only on the set S such that

$$A\left(\sum_N^n x_i^2(t)\right)^{1/2} h(t) \leqq \int_S \left|\sum_N^n x_n(t)r_n(\theta)h(t)\right| d\theta$$

for any $t \in (0, 1)$, $h \in H$, and $n > N$.

Integrate this inequality with respect to t and reverse the order of integration on the right-hand side. This yields

$$(2)\quad A\int_0^1 \left(\sum_N^n x_i^2(t)\right)^{1/2} h(t)dt \leqq \int_S \left[\int_0^1 \left|\sum_N^n x_n(t)r_n(\theta)h(t)\right| dt\right] d\theta$$

where $n > N$ and $h \in H$.

Combine inequalities (1) and (2):

$$A\int_0^1 \left(\sum_N^n x_i^2(t)\right)^{1/2} h(t)dt \leqq M\int_S d\theta = M|S|.$$

It follows that $G(x) = ||(\sum_1^\infty x_i^2)^{\frac{1}{2}}||$ is finite.

COROLLARY 1. *Let* (x_n, X_n) *be a sequence of elements of* X *and of continuous linear functionals, respectively, such that* $x = \sum_1^\infty X_n(x)x_n$ *for all* x *in* X.

If $|C(x)| > 0$ *for each* x *in* X, *there exists a constant* $A > 0$ *such that* $G(x) \leqq A||x||$ *for all* x *in* X.

Proof. Theorem 1 implies that $G(x) < \infty$ for each x in X. It is now not hard to verify that $(X, ||\cdot|| + G(\cdot))$ is a Banach space (one may employ the notions in [**10**, Chapter 15] in lieu of a direct computation). The natural embedding $(X, ||\cdot|| + G(\cdot)) \to (X, ||\cdot||)$ is thereby a one-one and onto map of Banach spaces. By the open mapping theorem, there then exists $B > 0$ such that

$$||x|| + G(x) \leqq B||x|| \text{ for all } x \text{ in } X,$$

which proves the corollary.

We remark that in Corollary 4 below it is shown that this need not imply the equivalence of $G(\cdot)$ and $||\cdot||$. Were this the case, (x_n, X_n) would be an unconditional basis [**9**].

Example. We construct an element of $L_1(0, 1)$ whose Haar series expansion diverges in the L_1-norm for almost every choice of signs.

Let $\{h_{np}\}$ be the usual enumeration of the Haar system in which the support of h_{1p} is adjacent to "0", and let $f = \sum_1^\infty 2^k k^{-1} h_{1k}$. To see that this series actually converges in L_1, estimate

$$\int_0^1 \left| \sum_n^m 2^k k^{-1} h_{1k}(t) \right| dt$$

by partitioning the unit interval into the subintervals on which the integrand is constant and than add up the integrals on each subinterval. This gives

$$\begin{aligned}
&2^{m-n-1} \textstyle\sum_{s=0}^{m-n} 2^{n+s}(n+s)^{-1} \\
&\quad + \textstyle\sum_{k=1}^{m-1} 2^{-n-k}[2^{n+k-1}(n+k-1)^{-1} - \sum_{i=0}^{k-2} 2^{n+i}(n+i)^{-1}] \\
&\quad = o(1) + \tfrac{1}{2} \textstyle\sum_{r=0}^m (n+r)^{-1}(1 - \sum_{r=1}^{m-r} 2^{-k}) \\
&\quad = o(1) + \textstyle\sum_{r=0}^m 2^{r-m}(n+r)^{-1} \leqq o(1) + n^{-1} \sum_{r=0}^\infty 2^{-r} \\
&\quad = o(1).
\end{aligned}$$

Hence, the series converges in L_1.

To show that f has the desired property it is sufficient by Theorem 1, to show that $G(f) = \infty$.

$$\begin{aligned}
G(f) &= 2^{-1} \textstyle\sum_1^\infty 2^{-n} (\sum_1^n 4^k k^{-2})^{\frac{1}{2}} \\
&\geqq 2^{-1} \textstyle\sum_1^\infty 2^{-n} (\sum_1^n 4^k n^{-2})^{\frac{1}{2}} \\
&= 2^{-1}\, 3^{-\frac{1}{2}} \textstyle\sum_1^\infty 2^{-n} n^{-1} (4^{n+1} - 4)^{\frac{1}{2}} \\
&= 3^{-\frac{1}{2}} \textstyle\sum_1^\infty 2^{-n} n^{-1} (4^n - 1)^{\frac{1}{2}} \\
&= \infty,
\end{aligned}$$

which completes the example.

It is not such an easy matter to find an example of an element of $L_1(E)$, $|E| > 0$, having the same property. Nevertheless, the results above entail the existence of such functions.

COROLLARY 2. *For any $E \subset (0, 1)$, $|E| > 0$, there exists an f in $L_1(E)$ whose Haar series expansion diverges in the norm of $L_1(E)$ for almost every choice of sign. Such functions constitute all of $L_1(E)$ with the exception of a set of first category.*

Proof. In the proof of Theorem 9 of [**8**] it is demonstrated that the norm in $L_1(E)$ defined by

$$G(\cdot) = \int_E \left| \sum (h_{np}, \cdot)^2 h_{np}^2(t) \right|^{1/2} dt$$

https://doi.org/10.4153/CJM-1975-056-2 Published online by Cambridge University Press

is not dominated by the norm of $L_1(E)$. By Corollary 1 this establishes the existence of the desired functions.

Let $X = \{f: f \text{ is in } L_1(E) \text{ and } G(f) < \infty\}$, and let $||\cdot||$ denote the $L_1(E)$ norm. As noted in the proof of Corollary 1, $(X, ||\cdot|| + G(\cdot))$ is a Banach space continuously embedded in $L_1(E)$. This embedding has just been shown to be non-surjective, so the image of X must be of the first category in $L_1(E)$. Thus, for each f in $L_1(E)\backslash X$, $G(f) = \infty$, and the conclusion follows from an application of Corollary 1.

As a partial converse of Theorem 1:

THEOREM 2. *Given: a formal series* $x = \sum_1^\infty x_i$ *of elements of* $L_p(E)$, $E \subset (0,1)$, $|E| > 0$, $1 \leqq p < \infty$.

If $G(x) < \infty$, *then there is an increasing sequence* $\{n_i\}$ *of positive integers such that the sequence* $\sum_1^{n_i} r_k(\theta)x_k$ *converges in the norm of* L_p *for almost every* θ.

Proof. The Khintchine Inequality [**3**] implies the existence of a $B > 0$ for which

$$\int_0^1 \left|\sum_n^m r_k(\theta)x_k(t)\right|^p d\theta \leqq B\left(\sum_n^m x_k^2(t)\right)^{p/2}.$$

Integrate this inequality with respect to t and change the order of integration. This gives

$$\int_0^1 \left(\int_E \left|\sum_n^m r_k(\theta)x_k(t)\right|^p dt\right) d\theta = o(1).$$

There is then an element g of $L_p([0, 1] \times [0, 1])$ for which

$$\int_0^1 \left(\int_E \left|\sum_1^n r_k(\theta)x_k(t) - g(\theta, t)\right|^p dt\right) d\theta = o(1).$$

It follows that there is an increasing sequence $\{n_i\}$ of positive integers such that for almost every θ,

$$\int_E \left|\sum_1^{n_i} r_k(\theta)x_k(t) - g(\theta, t)\right|^p dt = o(1),$$

which proves the theorem.

COROLLARY 3. *Let* $\{x_i, x_i^*\}$ *be a basis for* $L_p(E)$, *and let* $x \in L_p(E)$. *If* $G(x) < \infty$, *then* $|C(x)| = 1$.

Proof. Let $x(n, m, \theta, t) = \sum_n^m r_k(\theta)x_k^*(x)x_k(t)$. By Theorem 2 there is a set S, $|S| = 1$, such that if $\theta \in S$, there is a $g(\theta, \cdot)$ in $L_p(E)$ for which

$$(*) \quad \int_E |x(1, n_k, \theta, t) - g(\theta, t)|^p dt = o(1).$$

For each pair of positive integers n and m, $n > m$, define

$$n_i = \min \{n_k : n_k \geqq n\}, \quad m_i = \max \{n_k : n_k \leqq m\},$$

and let K denote the norm of the given basis. Then

$$\begin{aligned} ||x(m, n, \theta, \cdot)||_p &\leqq K||x(m, n_i, \theta, \cdot)||_p \\ &\leqq K||x(m_i, n_i, \theta, \cdot)||_p + K||x(m_i, m-1, \theta, \cdot)||_p \\ &\leqq (K + K^2)||x(m_i, n_i, \theta, \cdot)||_p \end{aligned}$$

for almost every θ. (*) implies that the last term tends to 0 as m and n tend to ∞, which proves the corollary.

The following corollary is an immediate consequence of Theorem 1 and Corollary 3.

COROLLARY 4. *Let* $\sum_1^\infty y_i$ *be a Schauder basis expansion of an element of* $L_p(E)$, $|E| > 0$, $1 \leqq p < \infty$.
Then the series $\sum \pm y_i$ *converges (diverges) in* $L_p(E)$ *for almost every choice of sign if and only if* $||(\sum y_i^2)||_p < \infty$ $(= \infty)$.

The orthonormal system of Walsh is known to be a basis for each reflexive $L_p(0, 1)$ space [**6**]. When $G(\cdot)$ as defined in § 1 is formed with respect to this system, $G(f)$ turns out to be the l_2-norm of the coefficient sequence in the Walsh expansion of f in $L_p(0, 1)$. This fact is used to establish the following corollary.

COROLLARY 4. (I) *For* $2 < p < \infty$, *the Walsh Series expansion of any element of* $L_p(0, 1)$ *is norm convergent for almost every choice of sign. However, the Walsh system is a conditional basis for* $L_p(0, 1)$.
(II) *For* $1 \leqq p \leqq 2$, *the Walsh Series expansion of any element of* $L_p(0, 1)$ *is unconditionally convergent if f is also in* $L_2(0, 1)$. *Otherwise, the series diverges for almost every choice of sign.*

Proof. For any given f in $L_p(0, 1)$, $f = \sum (W_{ij}, f) W_{ij}$ where $\{W_{ij}\}$ denotes the Walsh system. Since $W_{ij}(t) = \pm 1$, $G(f) = (\sum (W_{ij}, f)^2)^{\frac{1}{2}}$. Let $f \in L_p(0, 1)$ for $2 < p < \infty$. Then $f \in L_2(0, 1)$ as well, and so $G(f) < \infty$. By Theorem 2, $\sum (W_{ij}, f) W_{ij}$ converges for almost every choice of sign.

If $p \neq 2$, then $L_p(0, 1) \neq L_2(0, 1)$ and the norms $G(\cdot)$ and $||\cdot||_p$ are not equivalent. This implies by [**2**] or [**9**] the conditionality of the Walsh system as a basis for $L_p(0, 1)$. (A much more general statement can be made. See, for example, Corollary 9 of [**4**]).

Let $f \in L_p(0, 1)$ for $1 \leqq p < 2$. Since $||\cdot||_p \leqq ||\cdot||_2$, the Walsh expansion of f is unconditionally convergent of $f \in L_2(0, 1)$. Otherwise, $G(f) = \infty$ and an application of Theorem 1 establishes (II).

Finally, we remark that for $2 < p < \infty$ similar considerations would show any L_p-convergent trigonometric series to be L_p-convergent for almost every choice of sign.

https://doi.org/10.4153/CJM-1975-056-2 Published online by Cambridge University Press

References

1. V. F. Gaposhkin, *On unconditional bases in the space L_p $p > 1$*, Uspehi. Mat. Nauk. *13* (1958), 179–184.
2. B. R. Gelbaum, *Conditional and unconditional convergence in Banach spaces*, An. Acad. Brasil. Ci. *30* (1958), 21–27.
3. S. Kacmarz and H. Steinhaus, *Theorie der Orthogonalreihen* (Chelsea, New York, 1951).
4. M. J. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. *21* (1962), 161–176.
5. G. G. Lorentz, *Bernstein polynomials* (Univ. of Toronto Press, Toronto, 1953).
6. J. Marcinkiewicz, *Quelques théorèmes sur les series orthogonals*, Ann. Polon. Math. *16* (1938), 84–95.
7. W. Orlicz, *Uber unbedingte Konvergenz in Funktionraumen*, I, Studia Math. *4* (1933), 33–37.
8. J. Shirey, *Restricting a Schauder basis to a set of positive measure*, Trans. Amer. Math. Soc. *184* (1973), 61–71.
9. J. Shirey and R. Zink, *On unconditional bases in certain Banach function spaces*, Studia Math. *36* (1970), 169–175.
10. A. C. Zaanen, *Integration* (John Wiley and Sons Inc., New York, 1967).

Ohio University,
Athens, Ohio