SUMS OF FRACTIONS WITH BOUNDED NUMERATORS

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1. Introduction. The general problem considered in this paper is that of sums of a finite number of reduced fractions whose numerators are elements of a finite set S of integers, and whose denominators are distinct positive integers. Egyptian, or unit, fractions are merely the case $S = \{1\}$. Problems concerning these fractions have been treated extensively. Another specific case $S = \{1, -1\}$ has been treated by Sierpinski **(2)**.

2. General results. The following theorem completely characterizes those sets $S = \{a_1, \ldots, a_n\}$ for which every rational number can be expressed in the form

$$\frac{a'_1}{b_1} + \frac{a'_2}{b_2} + \ldots + \frac{a'_m}{b_m}, \qquad a'_i \in S, \tag{1}$$

where the b_i are distinct positive integers such that $(a'_i, b_i) = 1$. The b_i are taken to be positive, since allowing them to be negative is equivalent to including $-a_i$ in S.

THEOREM 1. *If $S = \{a_1, \ldots, a_n\}$, then every rational number a/b can be expressed in the form* (1) *if and only if $(a_1, \ldots, a_n) = 1$ and not all of the a_i are of the same sign. Moreover, a/b can be expressed in this way using each a_i equally often.*

Proof. For the sufficiency proof we construct integers A_i by specifying their prime factorizations.

Let $q_1, \ldots, q_n$ be distinct primes such that $(q_1 q_2 \ldots q_n, ba_1 a_2 \ldots a_n) = 1$, and let $q_i | A_i$. To be definite, let $q_i || A_i$. Note that the q_i may be chosen arbitrarily large.

If p is a prime that divides at least one of $b, a_1, \ldots, a_n$, then since $(a_1, \ldots, a_n) = 1$, there is at least one j such that $p \nmid a_j$. To be definite, let j be minimal.

(i) If $p^\alpha || b$, $\alpha \geqslant 1$, let $p^\alpha || A_j$ and $p \nmid A_i$, $i \neq j$.

(ii) If $p \nmid b$, let $p || A_j$ and $p \nmid A_i$, $i \neq j$.

Define $A = A_1 A_2 \ldots A_n$ and $K = a_1(A/A_1) + \ldots + a_n(A/A_n)$.

If $a > 0$, we want $K > 0$. (We assume $b > 0$.) There is at least one a_i, say a_k, that is positive. Since the q_i may be chosen arbitrarily large, we choose

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$q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_n$ so large that

$$|a_1|/A_1 + \ldots + |a_{k-1}|/A_{k-1} + |a_{k+1}|/A_{k+1} + \ldots + |a_n|/A_n < a_k/A_k.$$

This guarantees $K > 0$. Similarly, if $a < 0$, we choose the q_i such that $K < 0$.

For each p defined above we have $p \nmid a_j(A/A_j)$, but $p | a_i (A/A_i)$, $i \neq j$, so $p \nmid K$. Thus $(a_i, K) = 1, i = 1, \ldots, n$. By construction $(a_i, A_i) = 1$. Hence $(a_i, A_i K) = 1$, $i = 1, \ldots, n$. Also by construction if $p^\alpha \| b$, then $p^\alpha \| A_j$ (for some j); therefore $b|A$, and we write $A = bc$.

Combining the above we see that

$$\frac{1}{bc} = \frac{1}{A} = \frac{a_1}{A_1 K} + \ldots + \frac{a_n}{A_n K}.$$

If $a > 0$, express ac as a finite sum of distinct unit fractions whose denominators are elements of the arithmetic progression $|a_1 \ldots a_n| q_1 \ldots q_n x + 1$ with $x > 0$. This is possible by a theorem of P. J. van Albada and J. H. van Lint (**5**, p. 172, Theorem 3.4); see also R. L. Graham (**1**). If $a < 0$, express $-ac$ in this form. Thus

$$ac = \sum_{j=1}^{N} \frac{1}{v_j}$$

where the v_j are of the same sign as a. Then

$$\frac{a}{b} = \frac{ac}{bc} = \sum_{i=1}^{n} \left(\frac{a_i}{A_i K} \left(\sum_{j=1}^{N} \frac{1}{v_j} \right) \right). \tag{2}$$

Note that, for either $a > 0$ or $a < 0$, all denominators are positive when the sums are multiplied out, since all $A_i > 0$, and both K and v_j are of the same sign as a.

Since $(a_i, v_j) = 1$ for all i and j, and since $(a_i, A_i K) = 1$, each fraction in (2) is reduced. The denominators are distinct: for if $i \neq k$, then $A_i K v_r \neq A_k K v_s$ because $q_i | A_i$ but $q_i \nmid A_k K v_s$; and if $r \neq s$, then $A_i K v_r \neq A_i K v_s$, because $v_r \neq v_s$. We also note that each a_i is used the same number of times, namely N.

It remains to discuss the case $a = 0$. By the above arguments if $a \neq 0$, then a/b can be represented in the form (2) with each numerator repeated N times. Similarly, it is possible to represent $-a/b$ in the form (2) with each numerator repeated N' times, using primes q'_i so large that every denominator used in representing $-a/b$ exceeds every denominator used in representing a/b. Hence a representation of $0 = a/b + (-a/b)$ in the form (1) is available with each numerator repeated $N + N'$ times.

For the necessity proof, first we assume that $(a_1, \ldots, a_n) = d > 1$. Suppose a/b to be represented in the form (1). Let B be the least common multiple of the b_i. Note that $(d, B) = 1$ because $d | a'_i$ and $(a'_i, b_i) = 1$. From $a/b = dC/B$ we have $dbC = aB$. Since $(d, B) = 1$, it follows that $d|a$. Hence if $d \nmid a$, then a/b cannot be represented in the form (1).

Secondly, even if $(a_1, \ldots, a_n) = 1$, but the a_i are all of one sign, it is obvious that fractions a/b of the other sign cannot be represented by (1).

COROLLARY. *If all the a_i are positive (negative), then any fraction $a/b > 0$ $(a/b < 0)$ can be expressed in the form* (1) *if and only if $(a_1, \ldots, a_n) = 1$.*

Proof. The proof is exactly the same as for Theorem 1, except that the sign of K becomes irrelevant.

3. The number of summands. Theorem 1 tells us when a fraction a/b can be expressed in the form (1). The next question that might be asked is: How many fractions of the desired type are necessary to express a given fraction? Obviously the number used in the proof of Theorem 1 is very large. The number of fractions necessary clearly depends on S, but we might ask if there is any set S such that for some fixed n_0 all fractions in some interval can be expressed in the form (1) using fewer than n_0 summands. In Theorem 2 we prove that no such set S exists.

Let $A_m(S)$ be the set of all a/b which can be expressed in the form (1) using m or fewer fractions.

LEMMA 1. *The set $A_m(S)$ is nowhere dense.*

Proof. Our original proof for any S followed the method of Sierpinski **(2)** for the case $S = \{1, -1\}$. For the case $S = \{1\}$ this method can be traced to the work of H. J. S. Smith **(3)**. The referee has suggested an alternative proof, which we present here.

If A is a set of real numbers, let $L(A) = L^1(A)$ be the set of limit points of A. Define $L^{s+1}(A) = L(L^s(A))$, $s \geqslant 1$. For the sets A and B define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

It can be shown that if at least one of the sets A or B is bounded, and if $L^s(A) = \emptyset$ and $L^t(B) = \emptyset$, then $L^{s+t-1}(A + B) = \emptyset$.

Let $H_1 = \{0, \pm 1, \pm 1/2, \pm 1/3, \ldots\}$ and let

$$H_k = \{d_1 + d_2 + \ldots + d_k \mid d_i \in H_1\}, \qquad k \geqslant 1.$$

Since $L^2(H_1) = \emptyset$ and $H_{k+1} = H_1 + H_k$, it follows from the above result, by induction on k, that $L^{k+1}(H_k) = \emptyset$.

For the set $S = \{a_1, \ldots, a_n\}$ let M be the maximum value of $|a_i|$. Then $A_m(S) \subseteq H_{mM}$. Since $L^{mM+1}(H_{mM}) = \emptyset$, it follows that $A_m(S)$ is nowhere dense.

THEOREM 2. *For any given set S, in every interval there exist rational numbers a/b whose representation in the form* (1) *requires arbitrarily many fractions.*

Proof. If all rational numbers in an interval were expressible in the form (1) using m or fewer fractions, then $A_m(S)$ would include the set of all rational numbers in the interval. But by Lemma 1 we know that $A_m(S)$ is nowhere dense, while the rational numbers are dense.

4. A specific case. In this section we consider the case $S = \{1, -1\}$.

LEMMA 2. *a/b is expressible in the form $(\pm 1/r_1) + (\pm 1/r_2)$ if and only if there exist d_1, d_2 such that $d_1|b, d_2|b$, and $d_1 \pm d_2 = ka$, $k \neq 0$.*

Proof. If such d_1 and d_2 exist, then $a/b = (1/k(b/d_1)) \pm (1/k(b/d_2))$.

Conversely, since a/b and $-a/b$ are both expressible, or both not expressible, we consider only the signs in the case $a/b = 1/r \pm 1/s$. Let $(r, s) = d$, $r = r'd$, $s = s'd$, so that $a/b = (s' \pm r')/r's'd$. Since $(r', s') = 1$, it follows that $(s' \pm r', r's') = 1$. Let $k = (s' \pm r', d)$. Since we may assume a/b to be reduced, it follows that $a = (s' \pm r')/k$ and $b = r's'(d/k)$. Thus we have $s' \pm r' = ka$, $k \neq 0$, and $r's'|b$, in agreement with the lemma.

LEMMA 3. *a/b is expressible as $a/b = (\pm 1/r_1) + \ldots + (\pm 1/r_m)$ if there exist $d_1, \ldots, d_m$, divisors of b, such that*

$$(\pm d_1) + \ldots + (\pm d_m) = ka, \qquad k \neq 0.$$

Proof. From the hypotheses $a/b = (\pm 1/k(b/d_1)) + \ldots + (\pm 1/k(b/d_m))$.

THEOREM 3. *If $S = \{1, -1\}$ then $a/b \in A_2(S)$ for a fixed $a > 0$ and all b sufficiently large if and only if $a = 1, 2, 3, 4$, or 6.*

Proof. If $a \neq 1, 2, 3, 4$, or 6, there exists an r such that $(r, a) = 1$ and $r \not\equiv \pm 1 \pmod{a}$. By Dirichlet's theorem there exist infinitely many k such that $p = ak + r$ is a prime. Then for a/p the only divisors of the denominator are ± 1 and $\pm p$. Since $r \not\equiv \pm 1 \pmod{a}$, no combination of these divisors has a sum which is a non-zero multiple of a. By Lemma 2 it follows that $a/p \notin A_2(S)$.

Conversely, if $a = 1$, then $1/b \in A_1(S)$; hence $1/b \in A_2(S)$ trivially. If $a = 2, 3, 4$, or 6, we may assume a/b to be reduced; hence we may express $b = ak + r$, with $r = \pm 1$. Both $d_1 = b$ and $d_2 = r$ divide b and $d_1 - d_2 = b - r = ka$, with $k \neq 0$ if $b > 1$. Since $d_1 \neq d_2$, it follows from Lemma 2 that $a/b \in A_2(S)$ whenever $(a, b) = 1$ and $b > 1$.

THEOREM 4. *If $S = \{1, -1\}$, then $a/b \in A_3(S)$ for a fixed $a > 0$ and all b sufficiently large if $a < 36$.*

Proof. Using a different method, Sierpinski **(2)** was able to show Theorem 4 for $a \leqslant 20$. However, the result is easily obtained for $a < 30$ by the use of Lemma 2. We illustrate the method for $a = 22$.

We suppose that we have completed the proof of Theorem 4 for $0 < a < 22$. Hence we may assume that $(22, b) = 1$ and we write $b = 22q \pm r$, $r = 1, 3, 5, 7$, or 9. We note that $22/b = 1/q \mp r/qb$. We complete the proof by showing that r/qb is in $A_2(S)$ if $q > 1$, hence if $b > 31$.

The hypothesis $q > 1$ implies that $1, q, b, qb$ are distinct divisors of qb. If $r < 8$, we may consider the possibilities of $q \pmod{r}$ and show that the sum or difference of some two of these divisors is a non-zero multiple of r. Hence

Lemma 2 applies to show that r/qb is in $A_2(S)$. (When $r < 8$, this argument using combinations of 1, q, $b \equiv aq$, $bq \equiv aq^2 \pmod{r}$ is applicable for all a).

If $r = 9$, we consider the possibilities of $q \pmod{9}$. If $q \equiv 0, 3$, or $6 \pmod{9}$, then $qb \equiv 0 \pmod{9}$; hence $9/qb$ reduces to a unit fraction that is in $A_2(S)$ trivially. If $q \equiv 1 \pmod{9}$, then $9|q - 1$. If $q \equiv 2 \pmod{9}$, then $9|b + 1$. If $q \equiv 4$ or $5 \pmod{9}$, then $9|qb - 1$. If $q \equiv 7 \pmod{9}$, then $9|b - 1$. If $q \equiv 8 \pmod{9}$, then $9|q + 1$. Hence Lemma 2 applies to show that $9/qb$ is in $A_2(S)$. This completes the proof for $a = 22$.

For some of the cases in Theorem 4 there are new difficulties, but these may be circumvented by using $q + 1$ or $q - 1$, in place of q, in the first step of obtaining a representation.

Notice the difference between this case $S = \{1, -1\}$ and the case of Egyptian fractions $S = \{1\}$. For Egyptian fractions it is known that $a/b \in A_a(S)$ for b sufficiently large; but this is known to be a best possible result only for $a = 2$ and $a = 3$. It seems almost certain that $a/b \in A_t(S)$ for some $t < a$ if $a > 3$. For a discussion of the Erdös conjecture $4/b \in A_3(S)$ and the Sierpinski conjecture $5/b \in A_3(S)$ see **(4)**.

In contrast, for the case $S = \{1, -1\}$, Theorem 4 shows that the first unresolved situation appears at a considerably later stage, namely, $a = 36$.

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