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A theorem of Thompson on isomorphisms induced by automorphisms

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Using homological methods (and in particular the 5-term sequence of Hochschild-Serre in the homology of groups) a generalization of the theorem of John G. Thompson (*J. Austral. Math. Soc.* 16 (1973), 16-17) referred to in the title is proved.

1. Introduction

In [3] Thompson proved that if G is a finite, perfect, centrally closed group and L, K are central subgroups of G such that $G/L \cong G/K$, then $\alpha(L) = K$ for some automorphism α of G. Thompson's proof is group theoretic and uses a free presentation F/R of G. Using homological methods we here prove

THEOREM 1.1. *Let G be a centrally closed group (not necessarily finite), L, K subgroups of G both contained in $G' \cap Z(G)$. Suppose that*

(i) $\operatorname{ext}(G/G', L) = 0$ *or* $\operatorname{ext}(G/G', K) = 0$ *and*

(ii) $G/L \cong G/K$.

Then $\alpha(L) = K$ *for some automorphism* α *of* G.

2. Notations and preliminaries

For a group G, G' denotes the derived group of G, $Z(G)$ the center of G and $H_2(G, Z)$ the second homology group of G with integer coefficients. Also for a group G and a subgroup K of G, $[K, G]$

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denotes the subgroup of G generated by the commutators $[k, x] = k^{-1}x^{-1}kx$, $k \in K$, $x \in G$.

Recall that an extension $0 \to A \to M \to G \to 1$ of an abelian group A by a group G is called a stem extension if and only if $A \subseteq M' \cap Z(M)$, [2].

Corresponding to every extension

$$(2.1) \qquad 0 \to K \to G \to Q \to 1$$

we have a 5-term sequence in low dimensional homology ([2], p. 17):

$$(2.2) \quad H_2(G, Z) \xrightarrow{\text{coinf}} H_2(Q, Z) \xrightarrow{\delta} K/[K, G] \xrightarrow{\text{cores}} G/G' \xrightarrow{\text{coinf}} Q/Q' \to 0 .$$

Observe that if (2.1) is a stem extension, then

$$\text{cores}: K/[K, G] \to G/G' \text{ is the zero map}$$

and the sequence (2.2) reduces to

$$(2.3) \qquad H_2(G, Z) \xrightarrow{\text{coinf}} H_2(Q, Z) \xrightarrow{\delta} K \to 0 .$$

3. Proof of the theorem

Let Q be a group such that $H_2(Q, Z) = 0$. Suppose that

$$(3.1) \qquad 0 \to A \to M \to Q \to 1$$

is a stem extension. The sequence (2.3) associated with this extension and the hypothesis on Q imply that $A = 0$. Thus Q is centrally closed.

Suppose now that Q is a centrally closed group. Let U be a subgroup of $H_2(Q, Z)$. By Proposition V.5.1 of [2], there exists a stem extension $0 \to N \to M \to Q \to 1$ with $N = H_2(Q, Z)/U$. Therefore $N = 0$. Thus $H_2(Q, Z)$ equals every subgroup U of itself. Hence $H_2(Q, Z) = 0$. This proves

LEMMA 3.2 [1]. *A group Q is centrally closed if and only if $H_2(Q, Z) = 0$.*

As an immediate consequence of this result and the sequence (2.3) we

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obtain

LEMMA 3.3. *For a subgroup* $L \subseteq G' \cap Z(G)$ *of a centrally closed group* G *the cotransgression* $\delta : H_2(G/L, Z) \to L$ *is an isomorphism.*

Proof of Theorem 1.1. Let $\theta : H_2(G/L, Z) \to H_2(G/K, Z)$ be the isomorphism induced by a given isomorphism $\psi : G/L \to G/K$. It follows from Lemma 3.3 that θ induces an isomorphism $\phi : L \to K$ such that the diagram

$$\begin{array}{ccc} H_2(G/L, Z) & \xrightarrow{\delta} & L \\ \downarrow \theta & & \downarrow \phi \\ H_2(G/K, Z) & \xrightarrow{\delta} & K \end{array}$$

commutes. If $\mathrm{ext}(G/G', K) = 0$ (we can interchange the roles of L and K otherwise), it follows from Proposition V.6.1 of [2] that there exists a homomorphism $\alpha : G \to G$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & G & \to & G/L & \to & 1 \\ & & \downarrow \phi & & \downarrow \alpha & & \downarrow \psi & & \\ 0 & \to & K & \to & G & \to & G/K & \to & 1 \end{array}$$

commutes. Since ϕ and ψ are isomorphisms and the rows in the diagram are exact, α is an automorphism of G.

References

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[3] John G. Thompson, "Isomorphisms induced by automorphisms", *J. Austral. Math. Soc.* 16 (1973), 16-17.

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