Can. J. Math., Vol. XXIX, No. 5, 1977, pp. 1112–1119

JORDAN LOOPS AND DECOMPOSITIONS OF OPERATORS

ARLEN BROWN AND CARL PEARCY

1. Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. In what follows we shall denote the spectrum, essential spectrum, and left essential spectrum of an operator T in $\mathscr{L}(\mathscr{H})$ by $\sigma(T)$, $\sigma_e(T)$, and $\sigma_{le}(T)$, respectively. Furthermore, if $T_1 \in \mathscr{L}(\mathscr{H})$ and T_1 is unitarily equivalent to a compact perturbation of an operator T_2, then we write $T_1 \sim T_2$, and if the compact perturbation can be chosen to have norm less than ϵ, we write $T_1 \sim T_2(\epsilon)$.

One of the fundamental theorems that has proved to be an extremely valuable tool in the recent advances in the structure theory of operators is the following (cf. [**1**; **3**; **8**]).

THEOREM A. *Suppose $T \in \mathscr{L}(\mathscr{H})$ and B is an arbitrary nonempty closed subset of $\sigma_{le}(T)$. Then for every $\epsilon > 0$, there exists an operator T_ϵ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ such that $T \sim T_\epsilon(\epsilon)$ and such that*

$$(1) \quad T_\epsilon = \begin{bmatrix} N_\epsilon & A_\epsilon \\ 0 & S_\epsilon \end{bmatrix},$$

where N_ϵ is a diagonalizable normal operator of uniform infinite multiplicity in $\mathscr{L}(\mathscr{H})$ satisfying $\sigma(N_\epsilon) = \sigma_e(N_\epsilon) = B$.

One reason Theorem A has been so useful, especially in proving density theorems, is that the operator T in the hypothesis is completely arbitrary. Another is that it is sometimes possible, at the expense of another perturbation of small norm, to replace the normal operator N_ϵ appearing in (1) by a nearby normal operator whose essential spectrum has desirable properties. The main purpose of this note is to prove a theorem of this sort. We show (Theorem 3.1) that if B is taken to be the outer boundary of $\sigma_e(T)$ (see § 2 for definitions), then there exists a normal operator N arbitrarily close to N_ϵ such that $\sigma(N) = \sigma_e(N)$ and such that $\sigma_e(N)$ consists of the union of a finite number of piecewise smooth Jordan loops that surround $\sigma_e(T)$. Since the outer boundary of $\sigma_e(T)$ may have positive planar Lebesgue measure, the existence of such an N seems not to be completely obvious, and depends upon a result (Theorem 2.1) concerning the geometry of the plane. As consequences of our main theorem we obtain new proofs of the central theorems of [**5**] and [**7**].

Received November 23, 1976 and in revised form, March 29, 1977.

2. We begin with a preliminary discussion whose purpose is to fix terminology and notation. If K is a nonempty compact subset of the complex plane $\mathbf{C}$, then its complement $\mathbf{C}\backslash K$ is a countable union of pairwise disjoint domains (i.e., nonempty connected open subsets of $\mathbf{C}$) called the *complementary components* of K. Of these complementary components exactly one—call it $U_\infty(K)$—is unbounded. The other, bounded, components of $\mathbf{C}\backslash K$ are called *holes* in K. The set of points in K that belong to $U_\infty(K)^-$ will be called the *outer boundary* of K, and will be denoted by $\partial_0 K$. (The outer boundary is thus a part of the boundary ∂K; all other points of ∂K belong to the closure of one or more holes in K.)

In what follows we shall be concerned with matters having to do with sets of *Jordan loops* and the domains they bound. (We do not distinguish between a Jordan loop, which is a continuous mapping γ of a nondegenerate real interval $[a, b]$ into the complex plane that is one-to-one except for the fact that $\gamma(a) = \gamma(b)$, and the range $J = \gamma([a, b])$ of γ, which is a connected compact subset of $\mathbf{C}$). If J is a Jordan loop, then its complement $\mathbf{C}\backslash J$ is the union of exactly two components, an unbounded component $U_\infty(J)$ and one hole H, and $J = \partial U_\infty(J) = \partial H$. The domains $U_\infty(J)$ and H are called the *exterior* and *interior* domains of J, and will be denoted by Ext (J) and Int (J), respectively.

If J_1 and J_2 are two disjoint Jordan loops, then each must be contained in a single complementary component of the other. If each lies in the exterior domain of the other, then J_1 and J_2 are *mutually exterior.* If J_1 lies in Int (J_2), then $J_2 \subset$ Ext (J_1), and we say that J_1 is *interior* to J_2. A *Jordan domain* is a domain D with the property that ∂D is the union of a finite number of pairwise disjoint Jordan loops. A *Jordan region* is the closure of a Jordan domain.

If J_0 is a Jordan loop and $J_1, \ldots, J_k$ are Jordan loops that are mutually exterior in pairs and all interior to J_0, then

$$D = \text{Int}\,(J_0) \cap \text{Ext}\,(J_1) \cap \ldots \cap \text{Ext}\,(J_k)$$

is a bounded Jordan domain, and every bounded Jordan domain is of this form. The compact Jordan region D^- has J_0 for outer boundary and holes Int (J_i), $i = 1, \ldots, k$. The Jordan loops $J_1, \ldots, J_k$ will be called the *inner* boundary loops of D^-.

Now let $\mathscr{S} = \{R_i\}_{i=1}^p$ be a finite set of pairwise disjoint, bounded Jordan regions, and let $E = R_1 \cup \ldots \cup R_p$. Clearly there must exist regions R_i belonging to $\mathscr{S}$ that can be joined to ∞ in $U_\infty(E)$, and we shall call such regions *primary* in the set $\mathscr{S}$. (An arc α is said to *join* a set R to ∞ in an open set U if α is defined on a ray $[a, +\infty)$, $\alpha(a) \in R$, $\alpha(t) \in U$ for $a < t < +\infty$, and $\lim_{t\to+\infty}|\alpha(t)| = +\infty$; the primary regions in $\mathscr{S}$ are simply those whose outer boundaries lie in the exterior domain of all the other outer boundary loops of the regions R_i, $i = 1, \ldots, p$.) If there are other, nonprimary, regions in $\mathscr{S}$, then each must lie in some one hole of a primary region in $\mathscr{S}$, and such nonprimary regions we call *secondary* in $\mathscr{S}$.

We are ready, at last, to prove the following result.

THEOREM 2.1. *Let K be a nonempty compact subset of* $\mathbf{C}$, *let $U_\infty(K)$ be the unbounded complementary component of K, and let ϵ be an arbitrary positive number. Then there exists a finite set $\mathscr{S} = \{R_i\}_{i=1}^p$ of pairwise disjoint, simply connected, bounded Jordan regions such that if we write $E = R_1 \cup \ldots \cup R_p$, then*

i) $K \subset E^0 = R_1{}^0 \cup \ldots \cup R_p{}^0$,

ii) *there exist mutually disjoint, piecewise smooth, Jordan loops $J_1, \ldots, J_p$, each lying in the exterior domain of all the others, such that for $i = 1, \ldots, p$, $J_i \subset U_\infty(K)$ and $R_i = J_i \cup \text{Int}(J_i)$,*

iii) *for each point λ in $\partial_0 K$ there is a point λ' in $\partial E = J_1 \cup \ldots \cup J_p$ such that $|\lambda - \lambda'| < \epsilon$, and*

iv) *for each point μ in ∂E there is a point μ' in $\partial_0 K$ such that $|\mu - \mu'| < \epsilon$.*

Proof. If μ_0 is any point of $\partial_0 K$ and η is an arbitrary positive number, then there exist a point μ_1 of $U_\infty(K)$ in the open disc of radius η centered at μ_0 and an arc α joining μ_1 to ∞ in $U_\infty(K)$. Let τ be the line segment joining μ_0 to μ_1 (linearly parametrized), and let μ_2 be the last point on τ that belongs to K. Then $\mu_2 \in \partial_0 K$, $|\mu_0 - \mu_2| < \eta$, and if τ_1 denotes the segment joining μ_2 to μ_1, then $\tilde{\alpha} = \tau_1 + \alpha$ is an arc joining μ_2 to ∞ and lying entirely in $U_\infty(K)$ except for the end point μ_2. If we agree to call a point λ of $\partial_0 K$ *accessible* if there exists such an arc joining λ to ∞, then the foregoing argument shows that the accessible points of $\partial_0 K$ are dense in $\partial_0 K$. Since $\partial_0 K$ is compact, we conclude that there exists a finite set of points $F = \{\lambda_1, \ldots, \lambda_n\}$ in $\partial_0 K$ such that each λ_j is accessible and such that if λ is any point of $\partial_0 K$, then there exists a point λ_i in F such that $|\lambda - \lambda_i| < \epsilon/2$.

Now choose a circle Z in $\mathbf{C}$ sufficiently large that $K \subset \text{Int}(Z)$ and dist $(K, Z) > 2\epsilon$. For each $i = 1, \ldots, n$, let α_i be an arc joining λ_i to ∞ in $U_\infty(K)$, let λ_i' denote the first point λ of α_i lying on the circle Z, and let $\tilde{\alpha}_i$ denote the subarc of α_i joining λ_i to λ_i' obtained by discarding the rest of α_i beyond λ_i'. Furthermore, for $i = 1, \ldots, n$, let λ_i'' denote the first point λ on the arc $\tilde{\alpha}_i$ such that $|\lambda_i - \lambda| = \epsilon/2$, and let $\hat{\alpha}_i$ denote the subarc of $\tilde{\alpha}_i$ joining λ_i'' to λ_i'. Finally let W_i denote the range of $\hat{\alpha}_i$, and set $M = W_1 \cup \ldots \cup W_n$. Then M is a compact subset of $U_\infty(K)$, so dist $(M, K) > 0$ and $\text{Int}(Z) \backslash M$ is an open neighborhood of K.

We now construct a set $\mathscr{T} = \{S_i\}_{i=1}^q$ of pairwise disjoint Jordan regions such that if we write $L = S_1 \cup \ldots \cup S_q$, then $K \subset L^0 \subset L \subset \text{Int}(Z) \backslash M$, dist $(\lambda, K) < \epsilon$ for every λ in L, and ∂L is the union of a finite set of pairwise disjoint, piecewise linear, Jordan loops (every point of each of which lies in $\mathbf{C} \backslash K$ and is at a distance less than ϵ from K). This is a standard construction and we omit the details; see [**9**, Theorem 13.5] or [**2**, Problems 5H-5K].

We next consider the primary regions $S_{i_1}, \ldots, S_{i_p}$ in the set $\mathscr{T}$, and for each such primary region S_{i_k}, we denote the piecewise linear Jordan loop that is the outer boundary of S_{i_k} by J_k. For $k = 1, \ldots, p$, we define $R_k = \text{Int}(J_k) \cup J_k$, and we shall show that the set $\mathscr{S} = \{R_k\}_{k=1}^p$ of Jordan regions has the desired properties.

In the first place, since the S_{i_k}, $k = 1, \ldots, p$, are the primary regions of $\mathscr{T}$, each of the Jordan loops J_k lies in the exterior region of all the Jordan loops $J_1, \ldots, J_{k-1}, J_{k+1}, \ldots, J_p$. Thus the Jordan regions $R_1, \ldots, R_p$ are mutually disjoint and simply connected. Furthermore if we set $E = R_1 \cup \ldots \cup R_p$, then $\partial E = J_1 \cup \ldots \cup J_p$, and it is obvious that $L \subset E$. Moreover, since $\partial E \subset \partial L$ and each point λ of L satisfies dist $(\lambda, K) < \epsilon$, it is clear that $\partial E \subset \text{Int}(Z)$ and thus $E \subset \text{Int}(Z)$. Also, since each point of M can be joined to Z by an arc lying entirely in M, it follows that $M \cap E = \emptyset$, for otherwise $M \cap \partial E$ would be nonvoid and this would contradict the facts that $\partial E \subset \partial L$ and $L \cap M = \emptyset$. Thus we have

$$K \subset L^0 \subset E^0 \subset E \text{ Int } (Z) \backslash M,$$

which proves (i). Since

$$\partial E = J_1 \cup \ldots \cup J_p \subset \partial L \subset \mathbf{C} \backslash K,$$

and each J_k can be joined to ∞ in $U_\infty(L) \subset U_\infty(K)$ (by definition of a primary region), each J_k can be joined to ∞ in $U_\infty(K)$ and thus must lie in $U_\infty(K)$, which proves (ii). To establish (iv), let $\mu \in \partial E = J_1 \cup \ldots \cup J_p \subset U_\infty(K)$, and recall that since $\mu \in L$, there exists a point μ_1 in K such that $|\mu - \mu_1| < \epsilon$. Let μ' be the last point on the line segment joining μ_1 to μ that belongs to K. Then $\mu' \in \partial K$, and since μ' is joined to μ in $\mathbf{C} \backslash K$ and $\partial E \subset U_\infty(K)$, it follows that $\mu' \in \partial_0 K$, and thus (iv) is proved.

We complete the proof of the theorem by establishing (iii). Since every point in $\partial_0 K$ is within $\epsilon/2$ of some point λ_i in F, it suffices to show that each such λ_i is within $\epsilon/2$ of a point of ∂E. Consider the arc α_i, constructed earlier, that joins λ_i to ∞ in $U_\infty(K)$. Since $\lambda_i \in E^0$ by (i), the range of α_i must intersect $\partial E = J_1 \cup \ldots \cup J_p$. But this intersection cannot take place on or outside the circle Z, since dist $(K, Z) > 2\epsilon$ and every point of ∂E is within ϵ of a point of K. Nor can the intersection take place in range $\hat{\alpha}_i = W_i \subset M$, since by construction $\partial E \subset \text{Int}(Z) \backslash M$. Thus the intersection must take place on the subarc of α_i joining λ_i to λ_i''. But by construction, all points in the range of this subarc are interior to the circle with center λ_i and radius $\epsilon/2$. Thus there is a point of ∂E interior to this circle, and the theorem is proved.

3. In this section we put together Theorem A and Theorem 2.1 to obtain a decomposition theorem for arbitrary operators (Theorem 3.1). This result is then employed to give different proofs of the main results of [**5**] and [**7**]. We begin by introducing certain operators that are pertinent to the discussion. Let $J_1, \ldots, J_p$ be pairwise disjoint, piecewise smooth, Jordan loops in $\mathbf{C}$, each of which lies in the exterior domain of all the others, and for $j = 1, \ldots, p$, let R_j be the simply connected Jordan region $R_j = J_j \cup \text{Int}(J_j)$. We set $\Omega = R_1 \cup \ldots \cup R_p$, and observe that $\partial\Omega = J_1 \cup \ldots \cup J_p$ and $\Omega^0 = \text{Int}(J_1) \cup \ldots \cup \text{Int}(J_p)$. Let μ denote arc length measure on $\partial\Omega$, and let $L_2(\partial\Omega)$ be the Hilbert space consisting of all functions defined on $\partial\Omega$ that are square integrable with respect to μ. The normal operator on $L_2(\partial\Omega)$ that is multiplication by the coordinate function (i.e., the operator $f(\lambda) \to \lambda f(\lambda)$) will be denoted by $M(\partial\Omega)$, and it is clear that $\sigma(M(\partial\Omega)) = \sigma_e(M(\partial\Omega)) = \partial\Omega$. The direct sum of $\aleph_0$ copies of $M(\partial\Omega)$ acting on the direct sum $\tilde{L}_2(\partial\Omega)$ of $\aleph_0$ copies of $L_2(\partial\Omega)$ will be denoted by $\tilde{M}(\partial\Omega)$. Once again it is easy to verify that $\sigma(\tilde{M}(\partial\Omega)) = \sigma_e(\tilde{M}(\partial\Omega)) = \partial\Omega$.

We shall denote by Rat (Ω) the algebra of all rational functions $r(z)$ whose poles lie outside Ω, and by $H_2(\partial\Omega)$ the subspace of $L_2(\partial\Omega)$ consisting of the closure in $L_2(\partial\Omega)$ of the linear manifold $\{r|\partial\Omega: r \in \text{Rat}(\Omega)\}$. It is clear that $H_2(\partial\Omega)$ is an invariant subspace of $M(\partial\Omega)$, and we set $M_+(\partial\Omega) = M(\partial\Omega)|H_2(\partial\Omega)$. One knows from [**3**, Proposition 9.1] that $H_2(\partial\Omega)$ is a proper subspace of $L_2(\partial\Omega)$, that the subnormal operator $M_+(\partial\Omega)$ satisfies $\sigma_e(M_+(\partial\Omega)) = \partial\Omega$ and $\sigma(M_+(\partial\Omega)) = \Omega$, and that the self-commutator of $M_+(\partial\Omega)$ is compact.

We are now prepared to prove the basic decomposition theorem.

THEOREM 3.1. *Suppose $T \in \mathscr{L}(\mathscr{H})$ and ϵ is an arbitrary positive number. Then there exists an operator T' in $\mathscr{L}(\mathscr{H})$ such that $\|T - T'\| < \epsilon$ and such that T' is unitarily equivalent to an operator of the form*

$$T'' = \begin{bmatrix} \tilde{M}(\partial\Omega) & A' \\ 0 & S \end{bmatrix},$$

where

a) *Ω is the union $\Omega = R_1 \cup \ldots \cup R_p$ of a finite number of pairwise disjoint, simply connected, bounded Jordan domains R_j with boundaries $\partial R_j = J_j$ that are piecewise smooth Jordan loops, each of which lies in the exterior domain of all the others,*

b) $\sigma_e(S) \subset \sigma_e(T) \subset \Omega^0$,

c) $\sigma(S) \cap \partial\Omega = \emptyset$.

Proof. One knows that $\partial_0\sigma_e(T)$ is contained in $\sigma_{le}(T)$, and thus we can apply Theorem A to T to conclude that there exists an operator T_1 such that $T \sim T_1(\epsilon/3)$, and such that T_1 has the form

$$T_1 = \begin{bmatrix} N & A \\ O & S \end{bmatrix},$$

where N is a diagonalizable normal operator staisfying $\sigma(N) = \sigma_e(N) = \partial_0\sigma_e(T)$. Since $T \sim T_1$, we have $\sigma_e(T_1) = \sigma_e(T)$, and since $\sigma_e(N) = \partial_0\sigma_e(T_1)$, an easy calculation shows that $\sigma_e(S) \subset \sigma_e(T_1)$.

We now apply Theorem 2.1 with $K = \sigma_e(T_1) = \sigma_e(T)$ to conclude the existence of a finite collection $\{J_1, \ldots, J_p\}$ of pairwise disjoint, piecewise smooth Jordan loops, each lying in the exterior domain of all the others, such that if we set $R_j = J_j \cup \text{Int}\,(J_j)$, $j = 1, \ldots, p$, and $\Omega = R_1 \cup \ldots \cup R_p$, then (i) $\sigma_e(T) \subset \Omega^0$, (ii) every point λ in $\partial_0\sigma_e(T)$ is within $\epsilon/3$ of some point of $\partial\Omega$, and (iii) every point μ in $\partial\Omega$ is within $\epsilon/3$ of some point of $\partial_0\sigma_e(T)$. Moreover, since $\sigma_e(S) \subset \sigma_e(T) \subset \Omega^0$, it follows that dist $(\sigma_e(S), \partial\Omega) > 0$, and therefore that $\sigma(S) \cap \partial\Omega$ is a finite set. It is thus a simple matter to perturb the Jordan loops $J_1, \ldots, J_p$ slightly to arrange that $\sigma(S) \cap \partial\Omega = \emptyset$, and we shall assume that this has been done. Furthermore, in view of (ii) and (iii) and the fact that the eigenvalues of N are dense in $\partial_0\sigma_e(T)$, it is easy to see (cf. [**3**, Proposition 6.2]) that there exists a diagonalizable normal operator N' such that $\sigma_e(N') = \sigma(N') = \partial\Omega$ and such that $\|N' - N\| < \epsilon/3$. Moreover it follows from the strong converse of the Berg-Weyl-von Neumann theorem (cf. [**6**] or [**8**]) that $N' \sim \tilde{M}(\partial\Omega)$ (γ) for every positive number γ, and thus, in particular, $N' \sim \tilde{M}(\partial\Omega)$ $(\epsilon/3)$. If U is a unitary operator such that $\|UN'U^* - \tilde{M}(\partial\Omega)\| < \epsilon/3$ and we set $A' = UA$, then it follows by putting together the above facts that there is an operator T' such that $\|T - T'\| < \epsilon$ and such that T' is unitarily equivalent to the operator

$$T'' = \begin{bmatrix} \tilde{M}(\partial\Omega) & A' \\ O & S \end{bmatrix}.$$

Thus the proof of the theorem is complete.

As an immediate consequence of this result, we obtain the principal theorem of [**7**].

COROLLARY 3.2. *The set of operators in $\mathscr{L}(\mathscr{H})$ with disconnected spectrum [respectively, disconnected essential spectrum] is norm dense in $\mathscr{L}(\mathscr{H})$.*

Proof. If T'' is as in Theorem 3.1, easy calculations show that

$$\sigma(T'') = \sigma(\tilde{M}(\partial\Omega)) \cup \sigma(S) = \partial\Omega \cup \sigma(S)$$

and that

$$\sigma_e(T'') = \sigma_e(\tilde{M}(\partial\Omega)) \cup \sigma_e(S) = \partial\Omega \cup \sigma_e(S).$$

Since dist $(\partial\Omega, \sigma(S)) > 0$, the result follows.

COROLLARY 3.3. *Suppose $T \in \mathscr{L}(\mathscr{H})$ and ϵ is an arbitrary positive number. Then there exists an operator T' such that $\|T - T'\| < \epsilon$ and such that T' is similar to an operator of the form $\tilde{M}(\partial\Omega) \oplus S$ where $\tilde{M}(\partial\Omega)$ and S are as in Theorem* 3.1.

Proof. It is a well known fact that since $\sigma(\tilde{M}(\partial\Omega)) \cap \sigma(S) = \emptyset$, the operator T'' of Theorem 3.1 is similar to $M(\partial\Omega) \oplus S$.

Recall that the set of biquasitriangular operators in $\mathscr{L}(\mathscr{H})$, denoted by (BQT), consists of all operators T in $\mathscr{L}(\mathscr{H})$ such that both T and T^* are quasitriangular. One knows from [**1**, Theorem 5.4] that $T \in (BQT)$ if and only if for every λ in $\mathbf{C}$ such that $T - \lambda$ is a semi-Fredholm operator, it is true that index $(T - \lambda) = 0$. We shall write $(BQT)_{qs}$ for the set of all operators S in $\mathcal{L}(\mathcal{H})$ such that S is quasisimilar to some biquasitriangular operator. Whether $(BQT)_{qs} = \mathcal{L}(\mathcal{H})$ is an important problem, since an affirmative answer to this question would allow one to reduce the hyperinvariant subspace problem for operators on separable Hilbert space to the case of operators in (BQT) without using the deep results of [**1**]. In [**5**] it was shown that $(BQT)_{qs}$ is at least norm dense in $\mathcal{L}(\mathcal{H})$, and Theorem 3.1 allows us to give a different proof of this result.

COROLLARY 3.4. *The set* $(BQT)_{qs}$ *is norm dense in* $\mathcal{L}(\mathcal{H})$.

Proof. By virtue of Corollary 3.3, it suffices to prove that every operator of the form $\tilde{M}(\partial\Omega) \oplus S \in (BQT)_{qs}$, where $\tilde{M}(\partial\Omega)$ and S are as in Theorem 3.1. If we write $L_2(\partial\Omega) = H_2(\partial\Omega) \oplus H_2(\partial\Omega)^\perp$, then, corresponding to this decomposition, the operator $M(\partial\Omega)$ can be written as a matrix

$$(2) \quad \begin{bmatrix} M_+(\partial\Omega) & G \\ 0 & M_-(\partial\Omega) \end{bmatrix}.$$

(See the discussion preceding Theorem 3.1.) Since $M(\partial\Omega)$ is normal and $M_+(\partial\Omega)$ has a compact self-commutator, it follows that G is compact. Moreover, it is obvious that for every positive integer n, the matrix (2) is similar to the matrix

$$\begin{bmatrix} M_+(\partial\Omega) & G/n \\ 0 & M_-(\partial\Omega) \end{bmatrix}.$$

Thus, applying Proposition 4.2 of [**4**] and the fact that $\tilde{M}(\partial\Omega)$ is the direct sum of $\aleph_0$ copies of $M(\partial\Omega)$, we conclude that $S \oplus \tilde{M}(\partial\Omega)$ is quasisimilar to

$$S \oplus \sum_{n=1}^{\infty} \oplus \begin{bmatrix} M_+(\partial\Omega) & G/n \\ 0 & M_-(\partial\Omega) \end{bmatrix}.$$

Since the operator

$$\sum_{n=1}^{\infty} \oplus \begin{bmatrix} 0 & G/n \\ 0 & 0 \end{bmatrix}$$

is obviously compact, and the class (BQT) is stable under compact perturbations, the proof can be completed by establishing that the operator $Q = S \oplus \sum_{n=1}^{\infty} \oplus (M_+(\partial\Omega) \oplus M_-(\partial\Omega))$ is biquasitriangular. Calculation shows that both the spectrum and the essential spectrum of the operator $\sum_{n=1}^{\infty} \oplus (M_+(\partial\Omega) \oplus M_-(\partial\Omega))$ are equal to Ω, and since $\sigma_e(S) \subset \Omega^0$ by b) of Theorem 3.1, it follows that $\sigma_e(Q) = \Omega$. Finally, for any λ in Ω^0, one knows from [**3**, Proposition 9.1] that $M_+(\partial\Omega)$ is a Fredholm operator of index -1 and $M_-(\partial\Omega)$ is a Fred-

https://doi.org/10.4153/CJM-1977-109-8 Published online by Cambridge University Press

holm operator of index $+1$. Thus it is impossible that $Q - \lambda$ be semi-Fredholm for any such λ, and that $Q \in (BQT)$ now follows from [**1**, Theorem 5.4].

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Indiana University,
Bloomington, Indiana;
University of Michigan
Ann Arbor, Michigan