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EXPONENTIALITY OF CERTAIN REAL SOLVABLE LIE GROUPS

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ABSTRACT. In this article, making use of the second author's criterion for exponentiality of a connected solvable Lie group, we give a rather simple necessary and sufficient condition for the semidirect product of a torus acting on certain connected solvable Lie groups to be exponential.

In [13], the second author established a criterion for a connected solvable real Lie group to be exponential. This was a generalization of the standard result of Dixmier and of Saito in the classical case of a simply connected solvable group (see [4] or [12]). One important virtue of this criterion is that it does not require the solvable group to be faithfully represented (which is automatically so in the simply connected case). Therefore one is not tied to the power series form of the exponential function. In contradistinction, using power series, it was observed by the first author [9] (in the course of his proof that the group, $\mathrm{SO}_0(n,1)$, of hyperbolic motions is exponential) that in an important case, namely the connected component of the Euclidean motion group, non-simply connected groups can be exponential even if the criterion of Dixmier-Saito fails.

Actually, in [9] the first author developed a method for showing that all centerless, non-compact, rank 1, real simple Lie groups are exponential and subsequently discovered that a part of the argument broke down except for the groups $\mathrm{SO}_0(n,1)$, the other cases remaining unresolved. The present results have been developed, among other reasons, because they are what is needed to complete the argument for the other rank 1 groups in [9]. This is done in [10] where it is proven that all rank 1, centerless, non-compact simple groups are exponential, save the exceptional group (which is not, this latter fact being due to Djokovic and Nguyen [5]). It is also expected that the present results will be useful in deciding whether the connected isometry groups of certain other Hadamard manifolds are exponential, or not.

Finally, the authors would like to take this opportunity to thank the referee for a number of useful comments, both in improving the exposition and in somewhat generalizing the results.

In our situation, *Cartan subgroups* play an important role. Whereas *Cartan subalgebras* are well understood and their definition will cause no difficulty—a Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which is equal to its normalizer—the definition of a Cartan subgroup is more complicated. Indeed, there are two different concepts for a Cartan subgroup. The first is due to C. Chevalley ([3, Definition VI.4.1]):

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DEFINITION 1. A subgroup H of G is called a *Cartan subgroup* if the following conditions are satisfied:

1. H is a maximal nilpotent subgroup of G.
2. Every normal subgroup U of finite index in H is of finite index in its normalizer $N_G(U)$.

We note that this definition makes sense for any group, not just a Lie group. The other is due to K. H. Hofmann and K.-H. Neeb: Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $N_G(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in G under the adjoint representation. If $\Lambda_{\mathfrak{h}} \subseteq \mathfrak{h}_{\mathbb{C}}^*$ denotes the set of roots belonging to $\mathfrak{h}$ and $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ is the root space with respect to λ, we define

$$C(\mathfrak{h}) = \left\{ g \in N_G(\mathfrak{h}) : \lambda \circ \mathrm{Ad}(g)|_{\mathfrak{h}_{\mathbb{C}}} = \lambda \text{ for all } \lambda \in \Lambda_{\mathfrak{h}} \right\}.$$

Here a subgroup H of G is called *Cartan subgroup* if and only if its Lie algebra, $\mathbf{L}(H)$, is a Cartan subalgebra of $\mathfrak{g}$ and $H = C\big(\mathbf{L}(H)\big)$. Moreover, since $\mathrm{Ad}(g^{-1}) \cdot \mathfrak{g}_{\mathbb{C}}^{\lambda} = \mathfrak{g}_{\mathbb{C}}^{\lambda \circ \mathrm{Ad}(g)}$ we have

$$C(\mathfrak{h}) = \left\{ g \in G : \mathrm{Ad}(g) \cdot \mathfrak{g}_{\mathbb{C}}^{\lambda} = \mathfrak{g}_{\mathbb{C}}^{\lambda} \text{ for all } \lambda \in \Lambda_{\mathfrak{h}} \cup \{0\} \right\}.$$

In [11, Theorem A.4], K.-H. Neeb proved that for connected Lie groups the two definitions are actually equivalent. As a result the criterion of Chevalley can be simplified in certain cases.

PROPOSITION 2. *Let G be a connected Lie group and suppose H is a maximal nilpotent subgroup satisfying:*

1. $[H : H_0]$ *is finite where H_0 is the identity component of H.*
2. *The index, $[N_G(H_0) : H_0]$, is finite.*

Then H is a Cartan subgroup of G.

PROOF. If $U \subseteq H$ be a normal subgroup with finite index, then $H_0 = U_0$. Because H_0 is characteristic in U, we get $N_G(U) \subseteq N_G(H_0)$. Since $N_G(H_0)/H_0$ is a finite group, $N_G(U)/H_0$ is also finite. As a subgroup of the finite group H/H_0 we see U/H_0 is also finite. Hence the quotient group $N_G(U)/U$ is finite and so H is a Cartan subgroup of G.

∎

An immediate consequence is:

PROPOSITION 3. *Let G be a connected Lie group and H be a maximal nilpotent subgroup which is connected and whose index, $[N_G(H) : H]$, is finite. Then H is a Cartan subgroup.*

In studying exponential Lie groups, a first step is to investigate *weakly exponential* Lie groups, *i.e.*, Lie groups where the range of the exponential map is dense. K. H. Hofmann and K.-H. Neeb observed the following ([11, Theorem I.2] and [6, Corollary 18]):

THEOREM. *A connected Lie group G is weakly exponential if and only if all its Cartan subgroups are connected.*

Thus a necessary condition for exponentiality is that the Cartan subgroups are connected. K.H. Hofmann and A. Mukherjea showed in [8, Proposition 3.4] that if a connected Lie group is solvable then it is weakly exponential. Hence in the solvable case all Cartan subgroups are connected and since the Cartan subgroups are nilpotent, they are themselves exponential. This means the Cartan subgroups of a connected solvable Lie group are exactly the exponential images of the Cartan subalgebras of the Lie algebra and it is this fact which we will use several times in the sequel.

Recall also that by Section 3, n° 4, Théorème 3 of [2] the Cartan subalgebras of a solvable Lie algebra are conjugate. Hence the Cartan subgroups of a connected solvable Lie group are also conjugate.

An element $x \in \mathfrak{g}$ for which $\operatorname{ad} x$ is nilpotent we will call, briefly, a *nilpotent* element. One for which $\operatorname{ad} x$ is semisimple we will call a *semisimple* element. We can now formulate the following criterion ([13]):

CRITERION. A connected solvable Lie group, G, is exponential if and only if for some Cartan subgroup H of G, $Z_H(x) := \{h \in H : \operatorname{Ad}(h)(x) = x\}$, the centralizer of x in H, is connected for each nilpotent element $x \in \mathfrak{g}$, the Lie algebra of G.

In order to apply this criterion we must first identify a Cartan subgroup of the group we want to establish is exponential. We shall consider the semidirect product of a torus acting on a connected nilpotent Lie group. Later we will pass to the situation where the group being acted upon is a simply connected solvable group of type E.

PROPOSITION 4. *Let N be a connected nilpotent Lie group, T be a torus acting on N via $\eta \colon T \longrightarrow \operatorname{Aut}(N)$, where $\operatorname{Aut}(N)$ is the automorphism group of N and G be the semidirect product. Then $H = T \cdot N_*$ is a Cartan subgroup of G, where N_* is the group of T-fixed points in N.*

PROOF. We look at the Lie algebra $\mathfrak{t}$ of T. This is abelian and consists only of semisimple elements. So, by Proposition VII.2.10 of [2] there is a Cartan subalgebra $\mathfrak{h}$ containing $\mathfrak{t}$. In particular, T is contained in a Cartan subgroup H. By the modular law, we get $H = H \cap (T \cdot N) = T \cdot (H \cap N)$. Moreover, since $T \cdot (H \cap N)$ is even a semidirect product and H and T are connected, $H \cap N$ is also connected. Since it is nilpotent, it is equal to $\exp(\mathfrak{h} \cap \mathfrak{n})$. On the other hand, each $t \in \mathfrak{t}$ acts semisimply and nilpotently on $\mathfrak{h}$, hence t is in the center of $\mathfrak{h}$. It follows that $(H \cap N) \subseteq N_*$ and $H \cong T \times (H \cap N)$. On the other hand, TN_* is nilpotent because T and N_* commute elementwise and N_* is nilpotent. Because of the maximality of H with respect to nilpotency, $H = TN_*$ and $N_* = (H \cap N)$. ∎

Having identified a Cartan subgroup we can now give a sharper criterion for the exponentiality for certain semidirect products. If $x \in \mathfrak{n}$, $T(x)$ will denote the subgroup of T fixing x.

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THEOREM 5. *Let N be a connected nilpotent Lie group and $G = T \times_\eta N$ be a semidirect product of N with a torus. Then G is exponential if and only if $T(x)$ is connected for each $x \in \mathfrak{n}$.*

PROOF. By the proposition above, $H = T \cdot N_*$ is a Cartan subgroup of G, where N_* is the T-fixed points in N. We apply the criterion of [13], which, in this context, states that G is exponential if and only if $Z_H(x)$ is connected for each $x \in \mathfrak{n}$. Let $x \in \mathfrak{n}$ be fixed. As above, $\eta\colon T \longrightarrow \mathrm{Aut}(N) \subseteq \mathrm{Aut}(\mathfrak{n})$ and $\eta^{\cdot}\colon \mathfrak{t} \longrightarrow \mathrm{Der}(\mathfrak{n})$.

Making use of the fact that the nilpotent group, N, is exponential, a direct calculation in the semidirect product and then taking differentials yields:

$$Z_H(x) = \left\{(t, \exp n_*) : \operatorname{Exp} D_t(x) = \mathrm{Ad}_{\exp n_*^{-1}}(x)\right\},$$

where $n_* \in \mathfrak{n}_*$, $t \in T$ and $\eta(t) = \operatorname{Exp} D_t$, where $D_t \in \mathrm{Der}(\mathfrak{n})$. Since $\mathfrak{n}_* = \bigcap_{t \in T} \operatorname{Ker} D_t$, we see that for each D_t, $[D_t, \mathrm{ad}_{n_*}] = \mathrm{ad}_{D_t(n_*)} = 0$, so that D_t and ad_{n_*} commute for every $n_* \in \mathfrak{n}_*$ and $t \in T$. Hence $\operatorname{Exp} D_t$ and $\operatorname{Exp} \mathrm{ad}_{n_*} = \mathrm{Ad}_{\exp n_*}$ also commute for every $n_* \in \mathfrak{n}_*$ and $t \in T$. Consider the pairs, $(t, \exp n_*)$, where $t \in T$ and $n_* \in \mathfrak{n}_*$, for which $\operatorname{Exp} D_t(x) = \mathrm{Ad}_{\exp n_*^{-1}}(x)$.

Now an easy argument shows if U and S are commuting operators on a vector space with U unipotent and S semisimple, then the set of elements on which they are equal forms a subspace on which each restricts to the identity. Since T is compact and acts continuously on $\mathfrak{n}$, it acts orthogonally with respect to some inner product so T acts semisimply. On the other hand, each $\mathrm{Ad}_{\exp n_*^{-1}}$ is unipotent and $\operatorname{Exp} D_t$ and $\mathrm{Ad}_{\exp n_*^{-1}}$ commute so the remark above tells us that $\operatorname{Exp} D_t$ and $\mathrm{Ad}_{\exp n_*^{-1}}$ are each the identity on the vectors where they agree. In particular, since this is so at x, $\operatorname{Exp} D_t(x) = x = \mathrm{Ad}_{\exp n_*^{-1}}(x)$. But, if $\exp n_*$ centralizes x, the whole 1-parameter subgroup of N_* through $\exp n_*$ also centralizes x. This means $Z_{N_*}(x)$ is connected and $Z_H(x) = T(x) \cdot Z_{N_*}(x)$. It follows that $Z_H(x)$ is connected if and only if $T(x)$ is. ∎

In the case when the normal subgroup S is solvable, but not nilpotent we can extend Theorem 5 in certain cases. We denote the nil-radical of S by N and the Lie algebra of N by $\mathfrak{n}$. We observe that if T is a maximal torus in a connected Lie group G, there is always a Cartan subgroup of G containing it. This is because T is exponential and its Lie algebra, $\mathfrak{t}$, is abelian and consists of semisimple elements. By Section 2, n° 3, Prop. 10 of [2], there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the Lie algebra of G, containing $\mathfrak{t}$. Let H be the Lie subgroup of G with Lie algebra $\mathfrak{h}$. Since H_0 contains T, H is a Cartan subgroup of G containing T.

To prove our final result we require Lemma 5 of [7] which we state below. As usual we shall call a point $x \in \mathfrak{g}$, exp-regular if exp is a local diffeomorphism in some neighborhood of x. This is equivalent to the differential being invertable and in the case of exp that the roots are not non-zero integer multiples of $2\pi i$.

LEMMA. *Suppose that for two elements x, y in the Lie algebra $\mathfrak{g}$ of a Lie group G we have* $\exp x = \exp y$ *and that* exp *is regular at x. Then* $[x, y] = 0$ *and $x - y$ is in the kernel of the restriction of* exp *to the space spanned by x and y.*

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THEOREM 6. *Let G be the semidirect product of a torus T acting on a connected solvable group S. Let be H a Cartan subgroup containing T. Assume that for each $s \in \exp(\mathfrak{h} \cap \mathfrak{s})$ there is a* exp-*regular $x \in \mathfrak{s} \cap \mathfrak{h}$ with $s = \exp(\mathfrak{s} \cap \mathfrak{h})$. Then G is exponential if and only if $T(x)$ is connected for each $x \in \mathfrak{n}$.*

PROOF. Let H be a Cartan subgroup of G containing T. Then $H = T \cdot (H \cap S)$, a direct product of spaces. Since G is solvable, H is connected, and hence so is $H \cap S$. But the latter is also nilpotent so is exponential. Clearly the same is true of T. Now, by the criterion of [13], as a connected solvable Lie group, G is exponential if and only if $Z_H(x)$ is connected for all nilpotent elements $x \in \mathfrak{g}$. Since $\mathfrak{g}$ is solvable, it follows from Section 2, n^o 3, Cor. 5 of [1] that every nilpotent element of $\mathfrak{g}$ is in $\mathfrak{n}$. Express each $h \in Z_H(x)$ as $h = \exp t \exp s$, with $t \in \mathfrak{t}$ and an exp-regular $s \in \mathfrak{s} \cap \mathfrak{h}$. Hence the following fixes x:

$$\mathrm{Ad}_{\exp t \exp s} = \mathrm{Ad}_{\exp t} \mathrm{Ad}_{\exp s} = \mathrm{Exp}(\mathrm{ad}\, t)\, \mathrm{Exp}(\mathrm{ad}\, s),$$

so that $\mathrm{Exp}(\mathrm{ad}\, s)x = \mathrm{Exp}(-\,\mathrm{ad}\, t)x$.

We now write x as sum of $x_\lambda \in \mathfrak{g}^\lambda$, where $\mathfrak{g}^\lambda$ is the weight space for the weight λ (including $\lambda = 0$). Let $\Lambda(x)$ be the set of all weights with $x_\lambda \neq 0$. For each $\lambda \in \Lambda(x)$ we have $e^{\lambda(s)} = e^{-\lambda(t)}$, so that $\lambda(s) + \lambda(t) \in 2\pi i\mathbb{Z}$. But $\lambda(t) \in i\mathbb{R}$. Hence, the same is true of $\lambda(s)$ and since s is exp-regular $\lambda(s)$ must be zero for each $\lambda \in \Lambda(x)$. Now because $\lambda(t) \in 2\pi i\mathbb{Z}$ and $\mathrm{ad}\, t$ is semisimple, it follows that $\mathrm{Exp}(\mathrm{ad}\, t)x = \mathrm{Ad}_{\exp t}\, x = x$ and hence also $\mathrm{Ad}_{\exp s}\, x = x$. Thus $\exp t \in Z_T(x) = T(x)$ and $\exp r \in Z_{H\cap S}(x)$ and so $Z_H(x) = T(x) \cdot Z_{H\cap S}(x)$. As these subgroups intersect trivially we have a direct product. Since s is exp-regular, by the above lemma we get $[s, x] = 0$. Thus $Z_{H\cap S}(x)$ is exponential and in particular it is connected. Hence, the direct product, $Z_H(x)$, is connected if and only if the $T(x)$ factor is. ∎

If we assume S as a solvable Lie group whose universal covering group is of type E, we see that the condition of Theorem 6 is satisfied.

COROLLARY 7. *Let G be the semidirect product of a torus and a solvable Lie group whose universal covering group is of type E. Then G is exponential if and only if $T(x)$ is connected for each $x \in \mathfrak{n}$.*

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