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EXAMPLES OF SUPERSOLUBLE LOCKETT SECTIONS

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Two Lockett sections of Fitting classes of supersoluble groups are determined. One of these sections has only one member, a minimal supersoluble nonnilpotent Fitting class.

1. INTRODUCTION

Recently a supersoluble Fitting class $\mathfrak{U}$ was presented [6]. This is a construction similar to that of Hawkes in [3], which yields metanilpotent Lockett classes, a fact proved by Brison [2]. The existence of supersoluble Lockett classes is assured by the following

THEOREM 1.1. *Let $\mathfrak{X}$ be a subgroup and quotient group closed class of groups and $\mathfrak{F}$ be a Fitting class with $\mathfrak{F} \subseteq \mathfrak{X}$. Then $\mathfrak{F}^* \subseteq \mathfrak{X}$. Therefore every supersoluble Fitting class $\mathfrak{F}$ has a supersoluble Lockett section $[\mathfrak{F}_*, \mathfrak{F}^*]$.*

PROOF: $\mathfrak{F}^* = \{G \in \mathfrak{S} \mid (G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}})\langle (g^{-1}, g) \mid g \in G\rangle\}$ due to [5].

$D := \langle (g^{-1}, g) \mid g \in G\rangle$ is a subgroup of $(G \times G)_{\mathfrak{F}} \in \mathfrak{X}$ and therefore $D \in \mathfrak{X}$. The projection π from D onto its second component is an epimorphism from D onto G. Hence $G \cong D/Ker(\pi) \in \mathfrak{X}$. ∎

In this paper we obtain two Lockett sections of Fitting subclasses of $\mathfrak{U}$, the one of $\mathfrak{U}$ itself and the Lockett section of a minimal nonnilpotent subclass $\mathfrak{W}$ of $\mathfrak{U}$. $\mathfrak{U}$ turns out to be a Lockett class but not to coincide with its lower star, whereas the Lockett section of $\mathfrak{W}$ has only one member.

All groups will be finite and soluble. $\mathfrak{F}(G)$ denotes the Fitting class generated by G. The notations and properties of Lockett's "star"-operators can be found in [5].

2. A LOCKETT SECTION WITH TWO MEMBERS

NOTATION. Let p be prime, $p \equiv 1 \bmod 3$, n a primitive 3rd unit root in $GF(p)$.

$$T := \langle a, b \mid a^p = b^p = [a,b,a,a] = [a,b,a,b] = [a,b,b,b] = 1\rangle$$
$$U := \langle T, s \mid s^3 = 1, a^s = a^n, b^s = b^n\rangle$$

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Then T is free with respect to the properties (nilpotent of class three, exponent p, two generators), $|T| = p^5$ and $Z(U) = Z(T) \cong C_p \times C_p$.

G is called a *central product* of normal subgroups $G_1, \dots, G_m$ if $G = G_1 \cdot \ldots \cdot G_m$ and $[G_i, G_j] = 1 \quad$ for $i \neq j$.

Let now $\mathfrak{U}_0$ be the class of all finite groups $G = XY$ with

(i) $X := O_p(G)$ is a central product of groups $T_i \cong T$ (the empty product, that is, $O_p(G) = 1$, being admitted).

(ii) $Y \in Syl_3(G)$, $\forall i : \left[Y/C_Y(T_i) \cong C_3 \quad \text{and} \quad T_i \rtimes \left(Y/C_Y(T_i)\right) \cong U\right]$.

Further let

$$G \in \mathfrak{U} :\Longleftrightarrow G = O_p(G) \cdot O^p(G) \quad \text{and} \quad O^p(G) \in \mathfrak{U}_0$$
$$\Longleftrightarrow G \in \mathfrak{S}_p\mathfrak{S}_3 \quad \text{and} \quad O^p(G) \in \mathfrak{U}_0,$$

where $O^p(G)$ is the largest p-perfect normal subgroup of G. Then $\mathfrak{U} = \mathfrak{F}(U)$, and $\mathfrak{U}$ is supersoluble according to [**6**, (4.1) and (4.2)].

LEMMA 2.1.

(a) *Let* $G \in \mathfrak{S}_p\mathfrak{S}_3$. *Then* $G \in \mathfrak{U} \Longleftrightarrow G/O_3(G) \in \mathfrak{U}$.

(b) $\mathfrak{U} = \mathfrak{H} := \left\{G \in \mathfrak{S}_p\mathfrak{S}_3 \mid O^p\left(G/O_3(G)\right) \in \mathfrak{U}_0\right\}$.

PROOF: Set $\overline{G} := G/O_3(G)$. Since $O_p\left(O^p(G)\right) \cong O_p\left(O^p(\overline{G})\right)$, we have to check only the operations induced on the central factors of $O_p\left(O^p(G)\right)$ and $O_p\left(O^p(\overline{G})\right)$ by the 3-elements of G and $\overline{G}$. By assumption $G \in \mathfrak{S}_p\mathfrak{S}_3$, so $O_3(G)$ centralises all p-elements of G and the assertion (a) is clear. Let $G \in \mathfrak{H}$. $G/O_3(G) = O_p\left(G/O_3(G)\right) \cdot O^p\left(G/O_3(G)\right)$ and $O^p\left(G/O_3(G)\right) \in \mathfrak{U}_0$, so $G/O_3(G) \in \mathfrak{U}$ and therefore $G \in \mathfrak{U}$. The converse direction is clear because of (a). ∎

Part (b) of the preceeding lemma shows that the class $\mathfrak{U}$ has a structure similar to that of the metanilpotent Fitting classes constructed by Hawkes [3], although item *d* of hypothesis 5.1 is not fulfilled. In fact $\mathfrak{U} = \mathfrak{F}(3', p', \mathfrak{U}_0) \cap \mathfrak{S}_p\mathfrak{S}_3$ in the notation of Hawkes. Now we can use Brison's proof almost verbatim to show that the supersoluble Fitting class $\mathfrak{U}$ is a Lockett class, too. In order to avoid the rewriting, the reader is referred to [3] and [2] for the necessary parts of the cited proofs.

LEMMA 2.2. *$\mathfrak{U}$ is a Lockett class.*

PROOF: Using the notation of [3] and [2] it remains only to show that $C_S(Q_j) \cong C_3$ and $S/C_S(Q_j)$ operates on Q_j/Q_j' by a power automorphism. Let $j \in \mathcal{S}_i$. It is already known that $S = AC_i$ and $[Q_j, C_i] \leqslant [F_i, C_i] = 1$, so $C_S(Q_j) \geqslant C_i$. Let $y := \left(g^{-1}, g\right)$ be the generating element of A. By hypothesis y raises the elements of $P_j/P_j' = P_j/Q_j'$ to the power of n. Let $\{a, b\}$ be a generating system of Q_j. Since $Q_j \leqslant P_jH_i$, there are elements $a', b' \in P_j$ and $h_1, h_2 \in H_i$ with $a = a'h_1, b = b'h_2$. $H_i \leqslant K'$, hence

a' and b' are not elements of P_j'. Therefore we have $z_1, z_2 \in K'$ with $(a')^y = (a')^n z_1$ and $(b')^y = (b')^n z_2$. Now $a^y = (a')^y h_1^y = (a')^n z_1 h_1^y$ and $z_1 h_1^y \in K' \cap Q_j = Q_j'$. That means: y raises a modulo Q_j' to the n-th power. The same holds for b. Moreover $|S| = 3^2$ and $|A| = 3$ so $C_i = C_S(Q_j)$ and $S/C_i \cong C_3$. □

THEOREM 2.3. Blessenohl and Gaschütz [1, Satz 3.3]. *Let p be a prime, S a subgroup of the multiplicative subgroup $GF(p)^*$ of $GF(p)$. Let G be a soluble group, $g \in G$. Fix a principal series of G and denote by $\mathcal{P}$ the set of the p-factors of this series. Then there is (uniquely determined up to similarity) a matrix $M(g; H/K)$, that describes the operation of g on the p-chief factor H/K. Define further a mapping $w_G : G \longrightarrow GF(p)^*$ by*

$$g \longmapsto \prod_{H/K \in \mathcal{P}} det(M(g; H/K)).$$

Then $\mathfrak{G}_p(S) := \{G \in \mathfrak{S} \mid \forall g \in G : w_G(g) \in S\}$ is a normal Fitting class.

DEFINITION 2.4: $\mathfrak{V} := \mathfrak{U} \cap \mathfrak{G}_p(1)$; $G_m := (T_1 \times \ldots \times T_m) \rtimes \langle s \rangle$, where all the T_i are normal in G_m and isomorphic to T, $\langle s \rangle \cong C_3$, and s raises the elements of T_i/T_i' to the n-th power.

Since $\mathfrak{V}$ contains the nonnilpotent group G_3 but not the group U, $\mathfrak{V}$ lies strictly between $\mathfrak{S}_p\mathfrak{S}_3 \cap \mathfrak{N}$ and $\mathfrak{U}$. Moreover it is the lower bound of the Lockett section of $\mathfrak{U}$ (Lemma 2.5).

In the following proofs automorphisms α are considered which raise the elements of the commutator factor group T_i/T_i' of a central product $T_1 \cdot \ldots \cdot T_m$ to some power n^{λ_i}. Such automorphisms will be denoted briefly by $(n^{\lambda_1}, \ldots, n^{\lambda_m})$.

LEMMA 2.5. $\mathfrak{V} = \mathfrak{F}(G_3) = \mathfrak{U}_*$

PROOF:

(i) We have to show that all groups $G \in \mathfrak{U}$ with "trivial determinant" are contained in $\mathfrak{F}(G_3)$. Because $G = O_p(G) \cdot O^p(G)$, it is sufficient to prove that the p-perfect $\mathfrak{V}$-groups are in $\mathfrak{F}(G_3)$. Moreover the problem can be reduced to the following case: $G = X\langle s \rangle$, $X = T_1 \cdot \ldots \cdot T_m$ is a central product of $T_i \cong T, \langle s \rangle \cong C_3$, and s operates on X as $(n^{\lambda_1}, \ldots, n^{\lambda_m})$ with $\lambda_i \neq 0$.

(ii) Set $W := (T_1 \times T_2) \rtimes \langle s \rangle$, where s acts as (n, n^2). In order to prove $W \in \mathfrak{F}(G_3)$, we consider the extension of a direct product $P := T_1 \times T_2 \times T_3 \times T_4$ of four copies of T by automorphisms $\alpha = (n, n, n, 1)$ and $\beta = (n^2, n^2, 1, n^2)$. $\langle P, \alpha, \beta \rangle$ is a normal product of $\langle T_1, T_2, T_3, \alpha \rangle \cong G_3$ and $\langle T_1, T_2, T_4, \beta \rangle \cong G_3$ and possesses a subnormal subgroup $\langle T_3, T_4, \alpha\beta \rangle \cong W$.

(iii) Now (ii) will be generalised to groups $H := (T_1 \cdot T_2) \rtimes \langle s \rangle$, where $T_1 \cdot T_2$ is a central product of the T_i : Take a third copy T_3 of T and form the direct product $P := (T_1 \cdot T_2) \times T_3$. Then there exist automorphisms $\alpha = (n, 1, n^2)$ and $\beta = (1, n^2, n)$.

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$\langle P, \alpha, \beta\rangle$ is a normal product of $\langle T_1, T_3, \alpha\rangle \cong W$ and $\langle T_2, T_3, \beta\rangle \cong W$, so it is contained in $\mathfrak{F}(G_3)$ by (ii). Moreover $\langle T_1, T_2, \alpha\beta\rangle$ is a subnormal subgroup of $\langle P, \alpha, \beta\rangle$ and isomorphic to H. Hence $H \in \mathfrak{F}(G_3)$.

(iv) Using the same method as in (iii) one obtains $H := (T_1 \cdot T_2 \cdot T_3) \rtimes \langle s\rangle \in \mathfrak{F}(G_3)$, where $T_1 \cdot T_2 \cdot T_3$ is a central product and all factors T_i/T_i' are raised to the n-th power by s.

(v) In the general case the group G can be built by the prototypes constructed in (iii) and (iv): Imagine the factors T_i numbered in such a way that s raises $T_1, \dots, T_k$ modulo the respective commutator subgroups to the power of n and $T_{k+1}, \dots, T_m$ to the power of n^2. Without loss of generality $k \geqslant 1$. Setting $l := m - k$ one obtains $w_G(s) = n^{k+2l} \equiv 1 \bmod 3$; that means $k \equiv l \bmod 3$. There are integers κ, λ and μ with $k = 3\kappa + \mu$, $l = 3\lambda + \mu$. Hence G is isomorphic to a normal product of $\kappa + \lambda$ groups of type (iv) and μ groups of type (iii).

(vi) $\mathfrak{U} = \mathfrak{F}(U)$, $exp(U/U') = 3$ and G_3 is isomorphic to the group $U(3)$ defined in [4]. According to [4, Satz 1.3,b] we get therefore $\mathfrak{U}_* = \mathfrak{F}(G_3) = \mathfrak{V}$. ∎

Lemma 2.6. *Let $G \in \mathfrak{U}$, $O_p(O^p(G))$ be a direct product of copies T_i of T. $G \in \mathfrak{U} \setminus \mathfrak{V} \iff \mathfrak{F}(G) = \mathfrak{U}$ and $G \in \mathfrak{V} \setminus \mathfrak{N} \iff \mathfrak{F}(G) = \mathfrak{V}$. Therefore $\mathfrak{F}(G_m) = \mathfrak{U}$ if $m \not\equiv 0 \bmod 3$, and $\mathfrak{F}(G_m) = \mathfrak{V}$ if $m \equiv 0 \bmod 3$.*

Proof: $G \in \mathfrak{U} \setminus \mathfrak{V}(\mathfrak{V}$ respectively) if and only if $O^p(G) \in \mathfrak{U} \setminus \mathfrak{V}(\mathfrak{V}$ respectively). Hence G can be reduced to the following form: $G := X \rtimes \langle s\rangle, X = T_1 \times \dots \times T_m$ all $T_i \lhd G$ and $T_i \cong T$, $\langle s\rangle \cong C_3$, s acts on X like $(n, \dots, n, n^2, \dots, n^2)$, the number of entries "n" being k. Set $l := m - k$ and assume without loss of generality $1 \leqslant l \leqslant k$. $G \in \mathfrak{V} \iff w_G(s) = 1 \iff k - l \equiv 0 \bmod 3$. $\mathfrak{F}(G)$ also contains the extension of X by $\alpha := (\underbrace{n^2, \dots, n^2}_{l}, \underbrace{n, \dots, n}_{k})$ and $\beta := (n, \underbrace{n^2, \dots, n^2}_{l-1}, \underbrace{n, \dots, n}_{k-1}, n^2)$ and therefore the extension of $T_1 \times T_2$ by (n, n^2). Anyway G_3 can be found in $\mathfrak{F}(G)$ and subsequently we get $\mathfrak{V} \subseteq \mathfrak{F}(G)$. If $G \notin \mathfrak{V}$ then $k - l \not\equiv 0 \bmod 3$ and $\mathfrak{F}(G)$ contains a group isomorphic to G_{k-l} and then the group $G_1 \cong U$ or the group G_2. Since $G_3 \in \mathfrak{F}(G)$, $G_2 \in \mathfrak{F}(G)$ implies $G_1 \in \mathfrak{F}(G)$ and finally $\mathfrak{F}(G) = \mathfrak{U}$. ∎

Lemma 2.7. *Let $T_1 \cdot T_2$ be a central product of two copies of T and $G := (T_1 T_2) \rtimes \langle s\rangle$ with $\langle s\rangle \cong C_3$, where s raises the elements of T_i/T_i' to the power of n^{λ_i}.*

(a) *If $\lambda_1 = \lambda_2 \neq 0$, then $\mathfrak{F}(G) = \mathfrak{U}$.*

(b) *If $\lambda_1 = 1$, $\lambda_2 = 2$ and $|T_1 \cap T_2| = p$, then $\mathfrak{F}(G) = \mathfrak{V}$.*

Proof:

(a) We can assume $\lambda_1 = 1$. The case $T_1 \cap T_2 = 1$ is done in (2.6), so there remain two cases:

Case 1. $|T_1 \cap T_2| = p$. Let X be a central product of four copies of T with the following intersections: $|T_1 \cap T_2| = |T_2 \cap T_3| = |T_3 \cap T_4| = p$; $T_1 \cap T_4 = 1$; $(T_1 \cap T_2) \times (T_2 \cap T_3) = Z(T_2)$; $(T_2 \cap T_3) \times (T_3 \cap T_4) = Z(T_3)$. Then $\mathfrak{F}(G)$ contains the extensions $H_1 := \langle X, \alpha \rangle$, $H_2 := \langle X, \beta \rangle$ and $H_3 := \langle X, \gamma \rangle$ with $\alpha := (n, n, 1, 1)$, $\beta := (1, 1, n, n)$ and $\gamma := (1, n^2, n^2, 1)$. $\alpha\beta\gamma = (n, 1, 1, n)$, and $H := \langle X, \alpha\beta\gamma \rangle$ possesses a normal subgroup $N \cong G_2$. According to (2.6) we get $\mathfrak{F}(G) = \mathfrak{U}$.

CASE 2. $|T_1 \cap T_2| = p^2$. Let again X be a central product of four copies of T, but now $Z(T_1) = T_i \cap T_j$ for all $i, j \in \{1, \dots, 4\}$. $\mathfrak{F}(G)$ contains the extensions of X by $\alpha := (n, n, 1, 1)$, $\beta := (1, 1, n^2, n^2)$, $\gamma := (n, 1, 1, n)$ and $\varepsilon := (n, 1, n, 1)$, and therefore with $\alpha\beta\gamma\varepsilon = (1, n, 1, 1)$ and at last a group isomorphic to U.

(b) There exists a central product $P = T_1 \cdot T_2 \cdot T_3$ of three copies of T with $T_2 \cdot T_3 \cong T_1 \cdot T_2$ and $T_1 \cap T_3 = 1$. Set $\sigma := (n, n^2, 1)$ and $\tau := (1, n, n^2)$. Hence $\langle P, \sigma, \tau \rangle$ is a normal product of two copies of G and has a subnormal subgroup $N = \langle T_1 \times T_3, \sigma\tau \rangle$ with $\mathfrak{F}(N) = \mathfrak{V}$ according to (2.6). ∎

Finally the Lockett section of $\mathfrak{U}$ is found, and it turns out that $\mathfrak{U}$ fulfills the Lockett conjecture:

THEOREM 2.8. *The Lockett section of* $\mathfrak{U}$ *is* $\{\mathfrak{V}, \mathfrak{U}\}$.

PROOF: Let $\mathfrak{X}$ be a Fitting class with $\mathfrak{U}_* = \mathfrak{V} \subsetneq \mathfrak{X} \subseteq \mathfrak{U}$. $\mathfrak{X}$ contains a group $G \in \mathfrak{U}_0 \setminus \mathfrak{V}$ constructed analogously to the group G in the proof of (2.6) with the only alteration that $X = T_1 \cdot \ldots \cdot T_m$ is now a central (eventually not a direct) product. Define k, m, l as in (2.6). Now $k - l \not\equiv 0 \bmod 3$. Let further α_j be the automorphism of X which centralises $T_1, \dots, T_j$ and operates on the remaining factors T_i in the same way as s. One obtains $X \rtimes \langle \alpha_1 \rangle \in \mathfrak{V}$ if $k - l \equiv 1 \bmod 3$, and $X \rtimes \langle \alpha_2 \rangle \in \mathfrak{V}$ if $k - l \equiv 2 \bmod 3$. In the first case $X \rtimes \langle s\alpha_1^{-1} \rangle \in \mathfrak{X}$ has a normal subgroup isomorphic to U, and $\mathfrak{X} = \mathfrak{U}$. In the second case $X \rtimes \langle s\alpha_2^{-1} \rangle \in \mathfrak{X}$ has a normal subgroup that fulfills the hypothesis of (2.7), and again $\mathfrak{X} = \mathfrak{U}$. ∎

3. A LOCKETT SECTION WITH ONE MEMBER

The Fitting class $\mathfrak{V}$ arose from intersecting $\mathfrak{U}$ with the normal Fitting class $\mathfrak{S}_p(1)$. Since all intersections of $\mathfrak{U}$ with normal Fitting classes will appear in the Lockett section of $\mathfrak{U}$, there is no hope for finding more subclasses of $\mathfrak{U}$ by constructing such intersections. However there is at least one other nonnilpotent subclass of $\mathfrak{U}$, and that class coincides with its upper and lower star. The Fitting classes generated by the groups G_m can be determined easily because the central factors T_i of $O_p(G_m)$ intersect trivially. A similar situation arises when these intersections are as great as possible.

DEFINITION 3.1:

(a) Let $H_m := (T_1 \cdot \ldots \cdot T_m) \rtimes \langle s \rangle$ fulfill the following hypothesis:

(i) $T_1 \cdot \ldots \cdot T_m$ is a central product of normal subgroups $T_i \cong T$ of H_m.

(ii) For all $i, j \in \{1, \cdots, m\}$: $T_i \cap T_j = Z(T_i) = Z(T_j)$.

(iii) $\langle s \rangle \cong C_3$ and s raises the elements of T_i/T_i' to the power of n.

(b) Let $K := (T_1 \cdot T_2) \rtimes \langle s \rangle$ fulfill hypotheses (i) and (ii) of (a), $\langle s \rangle \cong C_3$ and s operating on $T_1 T_2$ as (n, n^2).

Analogously to (2.6) one easily obtains:

Lemma 3.2. *$\mathfrak{F}(H_m) = \mathfrak{U}$ if $m \not\equiv 0 \bmod 3$ and $\mathfrak{F}(H_m) = \mathfrak{F}(H_3) = \mathfrak{F}(K)$ if $m \equiv 0 \bmod 3$.*

Definition 3.3: Let $\mathcal{M}$ be the class of all central products of copies T_i of T with $T_i \cap T_j = Z(T_i) = Z(T_j)$ for all i and j and $\mathfrak{W}$ be the class of all $G \in \mathfrak{U}$ such that $O_p(O^p(G))$ is a central product of maximal $\mathcal{M}$-groups $M_1, \ldots, M_m$ with the additional property that $\langle M_i, s \rangle \in \mathfrak{V}$ for all $i \in \{1, \ldots, m\}$ and all 3-elements $s \in G$.

The case $m = 0$, that means $O^p(G) \in \mathfrak{S}_3$, is admitted. In accordance with [**6**, 3.5], the factors M_i are uniquely determined up to permutation.

Proposition 3.4. *$\mathfrak{W}$ is a Fitting class, and $\mathfrak{W} = \mathfrak{F}(H_3) \subsetneq \mathfrak{V}$.*

Proof:

(i) Let $G_1 \cdot G_2$ be a normal product of groups $G_i \in \mathfrak{W}$. Without loss of generality we can assume $G_1, G_2 \in \mathfrak{U}_0$. If at least one of the two factors G_i is a 3-group, then $G_1 G_2 \in \mathfrak{W}$ is clear. Let be $O_p(G_1) = M_1 \cdot \ldots \cdot M_m$, $O_p(G_2) = L_1 \cdot \ldots \cdot L_l$, $m, l \geqslant 1$, as in (3.3).

(1) For all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, l\}$, either $M_i L_j$ is a maximal $\mathcal{M}$-normal subgroup of $G_1 G_2$, or M_i and L_j are situated in different maximal $\mathcal{M}$-normal subgroups of $G_1 G_2$. The first case arises if and only if in the decomposition of L_j into its factors $T_k \cong T$ there is a factor T_{k_0} with $T_{k_0} \cap M_i = Z(M_i)$. Then this equation is valid also for all the other factors T_{k_i} of L_j, because $L_j \in \mathcal{M}$.

(2) By (1) $O_p(G_1 G_2)$ is a central product of maximal $\mathcal{M}$-groups $K_1, \ldots, K_k$, where K_i satisfies one of the following conditions:

(I) $K_i = M_i$ and $K_i \cap G_2 \subseteq Z(G_2)$,

(II) there is a $j \in \{1, \ldots, l\}$ with $K_i = L_j$ and $K_i \cap G_1 \subseteq Z(G_1)$,

(III) there is a $j \in \{1, \ldots, l\}$ with $K_i = M_i L_j$.

(3) It remains to check the determinant of the operation induced by a 3-element $s \in G_1 G_2$ on K_i. Since $Y \in Syl_3(G_1 G_2)$ is a product of Y_1 and Y_2 with $Y_i \in Syl_3(G_i)$, it suffices to consider $s \in Y_1$ and $t \in Y_2$.

Case I. For $s \in G_1$ the assertion is clear. According to [6, 3.7.c], $t \in G_2$ centralises K_i.

Case II. This case is analogous to case I.

Case III. Let $L_j = T_1 \cdot \ldots \cdot T_r$ be the decomposition of L_j, uniquely up to permutation. Again in consequence of [6, 3.7.c], a factor T_ν is either identical to some central factor of M_i or it is centralised by $s \in G_1$. Hence $\langle M_i L_j, s\rangle$ and $\langle M_i L_j, t\rangle$ are contained in $\mathfrak{V}$.

(ii) Let be $G \in \mathfrak{W}$ and let N be a normal subgroup of G. $O_p(O^p(N))$ is a central product of normal subgroups $T_1, \ldots, T_m$, isomorphic to T, and $\{T_1, \ldots, T_m\}$ is a subset of the set of the central factors T_i of $O_p(O^p(G))$. Fix a maximal $\mathcal{M}$-factor M_i of $O_p(O^p(G))$. Passing from G to N, due to [6, 3.8.a] only factors $T_j \leqslant M_i$ that are centralised by all 3-elements $s \in N$ can disappear. So $M_i \cap N$ is a $\mathcal{M}$-group with $\langle M_i \cap N, s\rangle \in \mathfrak{V}$ for all 3-elements $s \in N$. Moreover $M_i \cap N$ is a maximal $\mathcal{M}$-factor of N. Since $O_p(O^p(N)) = (M_1 \cap N) \cdot \ldots \cdot (M_m \cap N)$, we get $N \in \mathfrak{W}$.

(iii) $H_3 \in \mathfrak{W}$, and $G_3 \notin \mathfrak{W}$. Therefore $\mathfrak{N} \subsetneq \mathfrak{F}(H_3) \subseteq \mathfrak{W} \subsetneq \mathfrak{V}$. Suppose now $M \in \mathcal{M}$ and $\langle s\rangle \cong C_3$ with $\langle M, s\rangle \in \mathfrak{V}$. Since $\mathfrak{F}(H_3) = \mathfrak{F}(K)$, one gets $\langle M, s\rangle \in \mathfrak{F}(H_3)$ using the same method as in part (v) of (2.5). Analogously to [6, 3.3] this leads to $\mathfrak{W} \subseteq \mathfrak{F}(H_3)$. □

Proposition 3.5. *$\mathfrak{W}$ is the only minimal nonnilpotent Fitting subclass of $\mathfrak{U}$.*

Proof: We fix $G \in \mathfrak{U} \setminus \mathfrak{N}$ and show $\mathfrak{W} \subseteq \mathfrak{F}(G)$. Let G fulfill the usual hypothesis (see (2.6)), the operation of s being described by $(n^{\lambda_1}, \ldots, n^{\lambda_m})$, $\lambda_i \neq 0$. Let F be free with respect to the properties (nilpotent of class 3, exponent p, $2(m+1)$ generators). It is possible to find a normal subgroup $N \leqslant Z(F)$ such that $P := F/N$ is a central product of copies $T_1, \ldots, T_m, T_{m+1}$ of T obeying the condition $T_1 \cdot \ldots \cdot T_m \cong T_2 \cdot \ldots \cdot T_{m+1} \cong O_p(G)$. Now take $\sigma, \tau \in Aut(P)$ defined by $\sigma = (n^{\lambda_1}, \ldots, n^{\lambda_m}, 1)$ and $\tau = (1, n^{-\lambda_2}, \ldots, n^{-\lambda_m}, n^{-\lambda_1})$. Hence $\langle P, \sigma, \tau\rangle$ is a normal product of two copies of G and has a subnormal subgroup $L = \langle T_1 \cdot T_{m+1}, \sigma\tau\rangle$, where $\sigma\tau$ operates as $(n^{\lambda_1}, n^{-\lambda_1})$. If $|T_1 \cap T_{m+1}| = p^2$, then $L \cong K$ and $\mathfrak{F}(L) = \mathfrak{W}$. Otherwise $\mathfrak{F}(L) = \mathfrak{V}$ due to (2.6) and (2.7.b). □

Theorem 3.6. $\mathfrak{W}_* = \mathfrak{W} = \mathfrak{W}^*$

Proof: $\mathfrak{W}_* = \mathfrak{W}$ is clear by (3.5). $\mathfrak{W} = \mathfrak{W}^*$: Since $\mathfrak{W}^* \subseteq \mathfrak{U}^* = \mathfrak{U}$, it suffices to show that $(G \times H)_{\mathfrak{W}} = G_{\mathfrak{W}} \times H_{\mathfrak{W}}$ for all $G, H \in \mathfrak{U}$. Further $O^p(G \times H) = O^p(G) \times O^p(H)$, so we can assume G and H to be p-perfect. $G, H \in \mathfrak{U}$, so $O_p(G) = T_1 \cdot \ldots \cdot T_l$ and $O_p(H) = U_1 \cdot \ldots \cdot U_k$ are central products of factors isomorphic to T with the usual operation of the 3-elements. Since $G \cap H = 1$, $O_p(G \times H) = T_1 \cdot \ldots \cdot T_l \cdot U_1 \cdot \ldots \cdot U_k$

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is the corresponding decomposition of $O_p(G \times H)$. Let $M_1, \dots, M_m$ be the maximal $\mathcal{M}-$normal subgroups of $O_p(G \times H)$. Each factor M_i is contained in G or in H. $(G \times H)_{\mathfrak{W}} \geqslant O_p(G \times H)$. Moreover M_i is either contained in $O^p\big((G \times H)_{\mathfrak{W}}\big)$ or is centralised by all 3-elements $s \in (G \times H)_{\mathfrak{W}}$. Each 3-element $s \in G \times H$ is a product of 3-elements $s_1 \in G$ and $s_2 \in H$. $s = s_1 s_2$ is an element of $(G \times H)_{\mathfrak{W}}$ if and only if it induces on every factor M_i an automorphism with "determinant" 1. In this case s_1 and s_2 have this property, too: If $M_i \leqslant G$, s_2 centralises M_i. If $M_i \leqslant H$, s_1 centralises M_i. Hence $s_1 \in G_{\mathfrak{W}}$ and $s_2 \in H_{\mathfrak{W}}$. Together we get $(G \times H)_{\mathfrak{W}} = G_{\mathfrak{W}} \times H_{\mathfrak{W}}$. ☐

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