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ON THE MONOTONE SIMULTANEOUS APPROXIMATION ON [0,1]

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Let Ω denote the closed interval $[0, 1]$ and let bA denote the set of all bounded, approximately continuous functions on Ω. Let Q denote the Banach space (sup norm) of quasi-continuous functions on Ω. Let M denote the closed convex cone in Q comprised of non-decreasing functions. Let h_p, $1 < p < \infty$, denote the best L_p-simultaneaous approximation to the bounded measurable functions f and g by elements of M. It is shown that if f and g are elements of Q, then h_p converges uniformly to a best L_1-simultaneous approximation of f and g. We also show that if f and g are in bA, then h_p is continuous.

1. INTRODUCTION

Let f and g be bounded measurable functions on $[0, 1]$. It was shown in [**4**] that if $f \notin M$ or $g \notin M$, then there exists a unique $h_p \in M$ such that

$$(1) \qquad [\|f - h_p\|_p^p + \|g - h_p\|_p^p]^{1/p} = \inf_{h \in M} [\|f - h\|_p^p + \|g - h\|_p^p]^{1/p}.$$

We call h_p the best L_p-simultaneous approximation to f and g by elements of M and abbreviate this to b.s.a. In [**6**] it was shown that if f and g are in Q, then they have the so-called simultaneous Polya property, that is h_p converges uniformly as $p \to \infty$. In this paper we show that they have also the simultaneous Polya-one property, that is, h_p converges uniformly as p decreases to one to a best L_1-simultaneous approximation.

To establish this property, we start in Section 2 with the case when f and g are finite real vlaued functions. In Section 3 we generalise the results of Section 2 to the space of step functions, and then to the space of quasi-continuous functions.

In Section 4 we establish the continuity of h_p when both of f and g are in bA.

Throughout this paper we assume either f or g is not in M, unless otherwise stated.

2. CONVERGENCE OF B.S.A. ON A FINITE SET

Let $X = \{x_1, \ldots, x_n\}$ be a finite subset of $\mathbf{R}$ with $x_1 < x_2 < \ldots < x_n$. Let $B = B(X)$ be the linear space of bounded real functions on X and $M = M(X)$ the closed convex cone of nondecreasing functions in B, that is functions h satisfying

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$h(x) \leqslant h(y)$ whenever $x, y \in X$ and $x \leqslant y$. For each p, $1 < p < \infty$, define a weighted L_p-norm ${}_w\|\cdot\|_p$ by

$$(2) \qquad {}_w\|f\|_p = \left(\sum_{i=1}^{n} w_i\,|f_i|^p\right)^{1/p}$$

where $f = \{f_i\}_{i=1}^n = \{f(x_i)\}_{i=1}^n \in B$, and $w = \{w_i\}_{i=1}^n > 0$ is a given weight function satisfying $\sum_{i=1}^{n} x_i = 1$.

Let $f = \{f_i\}_{i=1}^n$ and $g = \{g_i\}_{i=1}^n$ in B be fixed. For each $p, 1 < p < \infty$, we call a function $h_p = \{h_{p,i}\}_{i=1}^n \in M$ the best weighted L_p-simultaneous approximation if

$$\left({}_w\|f - h_p\|_p^p +{}_w \|g - h_p\|_p^p\right)^{1/p} = \inf\{\left({}_w\|f - h\|_p^p +{}_w \|g - h\|_p^p\right)^{1/p} : h \in M\},$$

or,

$$(3) \qquad \left[\sum_{i=1}^{n} w_i(|f_i - h_{p,i}|^p + |g_i - h_{p,i}|^p)\right]^{1/p} \leqslant \left[\sum_{i=1}^{n} w_i(|f_i - h_i|^p + |g_i - h_i|^p)\right]^{1/p},$$

for all $h = \{h_i : i = 1, \ldots, n\} \in M$.

To compute h_p explicitly, we first define $L \subseteq X$ to be a lower subset if $x_i \in L$ and $x_j \in X$, $x_j \leqslant x_i$, implies that $x_j \in L$. Similarly $U \subseteq X$ is an upper subset if $x_i \in L$ and $x_j \in X$, $x_j \geqslant x_i$, implies that $x_j \in U$. For simplicity we will write $i \in Y \subseteq X$ instead of $x_i \in Y$. Fix $p \in (1, \infty)$. If $L \cap U$ is non-empty, define $\mu_p(L \cap U)$ to be the unique real number minimising $\sum_j \{w_j[|f_j - u|^p + |g_j - u|^p] : j \in L \cap U\}$. Let $h_p = \{h_{p,i} : i = 1, 2, \ldots, n\}$ be the function defined on X by

$$(4) \qquad \begin{aligned} h_{p,i} &= \max_{\{U:\ i \in U\}} \min_{\{L:\ i \in L\}} \mu_p(L \cap U), \\ &= \min_{\{L:\ i \in L\}} \max_{\{U:\ i \in U\}} \mu_p(L \cap U). \end{aligned}$$

It is shown in [6] that h_p is the unique solution satisfying (3).

Definition: Let $a = \min\{-\|f\|_\infty, -\|g\|_\infty\}$ and $b = \max\{\|f\|_\infty, \|g\|_\infty\}$, and define functions

$$\tau_p(\overline{u}) = \sum_{i=1}^{n} w_i(|f_i - u_i|^p + |g_i - u_i|^p),$$

$$\kappa_p(u) = \sum_{i=1}^{n} w_i(|f_i - u|^p + |g_i - u|^p),$$

where $\overline{u} = (u_1, u_2, \ldots, u_n) \in [a, b]^n$ and $u \in [a, b]$.

Remark. By [5, Lemma 2], for each $p \in (1, \infty)$, κ_p is strictly convex and has a unique minimiser $u_p \in [a, b]$.

LEMMA 1. *Under the above hypothesis, we have*

$$\lim_{p \downarrow 1} (\tau_p(\overline{u}))^{1/p} = \tau_1(\overline{u}), \tag{5}$$

and,

$$\lim_{p \downarrow 1} (\kappa_p(u))^{1/p} = \kappa_1(u) \tag{6}$$

the convergence being uniform on the compact sets $[a, b]^n$ *and* $[a, b]$ *respectively.*

PROOF: For $\overline{u} \in [a, b]^n$, $1 \leqslant i \leqslant n$ and $p < 2$ we have

$$|f_i - u_i|^p \leqslant 2^p[|f_i|^p + |u_i|^p] \leqslant 2^{p+1}b^p \leqslant B,$$

where $B = 2^3 \max\{b^2, 1\}$. Similarly

$$|g_i - u_i|^p \leqslant 2^{p+1}b^p \leqslant B.$$

Let $\varepsilon > 0$ be given. We show that for $\overline{u} \in [a, b]^n$, there exists $\alpha_0 \in (0, 1)$ such that

$$\left|(\tau_{1+\alpha}(\overline{u}))^{1/(1+\alpha)} - \tau_1(\overline{u})\right| < \varepsilon \tag{7}$$

whenever $\alpha \in (0, \alpha_0)$.

Notice that

$$\left|(\tau_{1+\alpha}(\overline{u}))^{1/(1+\alpha)} - \tau_1(\overline{u})\right| \leqslant \left|(\tau_{1+\alpha}(\overline{u}))^{1/(1+\alpha)} - (\tau_1(\overline{u}))^{1/(1+\alpha)}\right| + \left|(\tau_1(\overline{u}))^{1/(1+\alpha)} - \tau_1(\overline{u})\right|. \tag{8}$$

Since the map $s \mapsto s^{1/(1+\alpha)}$ is continuous for $x \geqslant 0$, there exists $\delta > 0$ such that

$$\left|x^{1/(1+\alpha)} - y^{1/(1+\alpha)}\right| < \varepsilon/2,$$

whenever

$$|x - y| < \delta. \tag{9}$$

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Let $x = \tau_{1+\alpha}(\overline{u})$ and $y = \tau_1(\overline{u})$. Then the first summand of (8) is less that $\varepsilon/2$ provided we show there is α small enough to satisfy (9). Indeed

$$\begin{aligned}|x-y| &= \left|\sum_{i=1}^{n} w_i\,|f_i-u_i|^{1+\alpha} + \sum_{i=1}^{n} w_i\,|g_i-u_i|^{1+\alpha} - \sum_{i=1}^{n} w_i\,|f_i-u_i| - \sum_{i=1}^{n} w_i\,|g_i-u_i|\right| \\ &\leqslant \left|\sum_{i=1}^{n} w_i\,|f_i-u_i|^{1+\alpha} - \sum_{i=1}^{n} w_i\,|f_i-u_i|\right| + \left|\sum_{i=1}^{n} w_i\,|g_i-u_i|^{1+\alpha} - \sum_{i=1}^{n} w_i\,|g_i-u_i|\right|\end{aligned}$$

Now we use the same technique as was used in [**5**, Lemma 3] to obtain an $\alpha_1 > 0$ such that (9) holds for all $\alpha \in (0, \alpha_1)$.

For the second summand in (8) we give more details following the same line of proof in [**5**, Lemma 3]. So let $x = \tau_1(\overline{u})$. Then

$$\begin{aligned}0 < x &= \sum_{i=1}^{n} w_i(|f_i-u_i| + |g_i-u_i|) \\ &< \sum_{i=1}^{n} w_i(2b+2b) = 4b = B^*.\end{aligned}$$

Define G by

$$G(x,\,\alpha) = x^{1/(1+\alpha)} - x.$$

Then $\partial G/\partial x = (1+\alpha)^{-1}x^{-\alpha/(1+\alpha)} - 1 = 0$ only when $x = x_0 = (1+\alpha)^{-(1+1/\alpha)}$, and $G(x_0,\,\alpha) = (1+\alpha)^{-1/\alpha} - (1+\alpha)^{-1(+1/\alpha)} = (1+\alpha)^{-1/\alpha}\Big(1-(1-\alpha)^{-1}\Big) = -\alpha(1+\alpha)^{-(1+1/\alpha)}$ so $\lim\limits_{\alpha\downarrow 0} G(x_0,\,\alpha) = 0$. Let

$$T(\alpha) = 2\max\{|G(x_0,\,\alpha)|,\,\left|B^* - {B^*}^{1/(1+\alpha)}\right|\}.$$

Then $\sup\{|G(x,\,\alpha)| : 0 < x < B^*\} < T(\alpha)$. But $\lim\limits_{\alpha\downarrow 0} G(x_0,\,\alpha) = 0$, and $\lim\limits_{\alpha\downarrow 0}\left|B^* - {B^*}^{1/(1+\alpha)}\right| = 0$ implies the existence of $\alpha_2 > 0$ such that $|T(\alpha)| < \varepsilon/2$ for all $\alpha \in (0,\,\alpha_2)$. Consequently

$$|G(x,\,\alpha)| = \left|x^{1/(1+\alpha)} - x\right| < \varepsilon/2$$

which is what we need when we substitute for $x = \tau_1(\overline{u})$.

Finally take $\alpha_0 = \min(\alpha_1,\,\alpha_2)$ and the proof of (5) is complete. To obtain (6) take $\overline{u} = (u,\,u,\,\ldots,\,u)$ in (5). This establishes the lemma. ∎

Remark. Let M_n denote the space M as defined in the beginning of this section. For $1 \leqslant p < \infty$, let

$$\begin{aligned} d_n(p) &= \inf\{{}_w\|f - \overline{u}\|_p +_w \|g - \overline{u}\|_p : \overline{u} \in M_n\} \\ &= \inf\{{}_w\|f - \overline{u}\|_p +_w \|g - \overline{u}\|_p : \overline{u} \in M_n \cap [a, b]^n\}. \end{aligned}$$

Then it follows from (7) that

$$\lim_{p \downarrow 1} d_n(p) = d_n(1). \tag{10}$$

By putting $\overline{u} = (u, u, \ldots, u)$ it also follows that

$$\lim_{p \downarrow 1} d(p) = d(1) \tag{11}$$

where

$$d(p) = \inf\{{}_w\|f - u\|_p +_w \|g - u\|_p : u \in [a, b]\}.$$

THEOREM 2. *For $p \in (1, \infty)$, let y_p be the unique minimiser of κ_p. Then $\lim_{p \downarrow 1} u_p = u_1$ exists. Moreover u_1 is a minimiser of κ_1.*

PROOF: Minor changes are needed on the proof of [**5**, Theorem 4] to obtain the desired results. ∎

THEOREM 3. *The solution h_p (given by (4)) which satisfies (3) converges as $p \downarrow 1$ to a solution $h_1 = \{h_{1,i} : i = 1, 2, \ldots, n\}$ satisfying*

$$\sum_{i=1}^{n} w_i(|f_i - h_{1,i}| + |g_i - h_{1,i}|) \leqslant \sum_{i=1}^{n} w_i(|f_i - h_i| + |g_i - h_i|) \tag{12}$$

for all $h = \{h_i : i = 1, \ldots, n\} \in M_n$.

PROOF: Similar to the proof of Theorem 5 in [**5**] with the role of g_p played by h_p. ∎

3. GENERALISATIONS OF QUASI-CONTINUOUS FUNCTIONS

DEFINITION: Let π be a finite partition of $[0, 1]$ with points $\{t_i : 0, 1, \ldots, n\}$ such that $0 = t_0 < t_1 < \ldots < t_n = 1$. Let I_E denote the indicator function of a subset E of $[0, 1]$. Let S_π be the linear space comprised of all step functions of the form

$$f = f_i I_{[0,t_1]} + \sum_{i=2}^{n} f_i I_{(t_{i-1},t_i]},$$

where $f_i \in R$ for every i.

We recall the following four results from [**6**].

LEMMA 4. *Let f and g be in S_π. Let h_p, $1 < p < \infty$, be the b.s.a. to f and g by elements of M. Then $h_p \in S_\pi$.*

LEMMA 5. *Fix $p \in (1, \infty)$. Let f_1, f_2, g_1 and g_2 be elements of S_π. Let h_1 and h_2 be the b.s.a. to f_1, g_1 and f_2, g_2 respectively. If $f_1 \le f_2$ and $g_1 \le g_2$, then $h_1 \le h_2$.*

LEMMA 6. *Let f and g be elements of S_π. If h_p is the b.s.a. to f and g, then $h_p + c$ is the b.s.a. to $f + c$ and $g + c$.*

THEOREM 7. *Let f and g be elements of S_π given by*

$$f = f_1 I_{[0,t_1]} + \sum_{i=2}^{n} f_i I_{(t_{i-1},t_i]},$$

and

$$g = g_1 I_{[0,t_1]} + \sum_{i=2}^{n} g_i I_{(t_{i-1},t_i]}.$$

For every $p \in (1, \infty)$, let $w_p = \{w_{p,i} : i = 1, \ldots, n\}$ be defined by

$$w_{p,i} = t_i - t_{i-1}$$

for all i. Let $h_p = \{h_{p,i} : i = 1, 2, \ldots, n\}$ be given by (4). Then the b.s.a. to f and g is given by

$$h_p^* = h_{p,i} I_{[0,t_1]} + \sum_{i=2}^{n} h_{p,i} I_{(t_{i-1},t_i]}. \tag{13}$$

The next theorem establishes the convergence of h_p^* as $p \to 1$.

THEOREM 8. *Let f and g in S_π and h_p^* be as given in Theorem 7 above. Then h_p^* converges as $p \downarrow 1$ to the monotone non-decreasing function h_1^* in S_π given by*

$$h_1^* = h_{1,i} I_{[t_0,t_1]} + \sum_{i=2}^{n} h_{2,i} I_{(t_{i-1},t_i]} \tag{14}$$

where $h_{1,i} = \lim_{p\downarrow 1} h_{p,i}$ is as described earlier in Theorem 3. Moreover, h_1^* is a best L_1-simultaneous approximation of f and g by non-decreasing functions.

PROOF: For each $i = 1, \ldots, n$, let $x_i = (t_i + t_{i-1})/2$ and let $X = \{x_1, \ldots, x_n\}$. Consider $\{f_i = f(x_i) : i = 1, 2, \ldots, n\}$ and $\{g_i = g(x_i) : i = 1, 2, \ldots, n\}$ as finite real valued functions on X. Let $w = \{w_i : i = 1, \ldots, n\}$ be as defined above. Then

Theorem 3 implies that h_p^* converges to h_1^*. Therefore $\lim_{p\downarrow 1} h_p^*$ exist and is given by (14).

For the second part of the theorem, notice that (12) holds for any positive weight function $w = \{w_i : i = 1, \ldots, n\}$ satisfying $\sum_{i=1}^{n} w_i = 1$. Thus for each i, $1 \leqslant i \leqslant n$, let $w_i = 1/n$; then (12) implies that

$$\sum_{i=1}^{n} n^{-1}(|f_i - h_{1,i}| + |g_i - h_{1,i}|) \leqslant \sum_{i=1}^{n} n^{-1}(|f_i - h_i| + |g_i - h_i|)$$

for all $h = \{h_i : i = 1, \ldots, n\} \in M_n$. Hence

$$\sum_{i=1}^{n} (|f_i - h_{1,i}| + |g_i - h_{1,i}|) \leqslant \|f - h\|_1 + \|g - h\|_1$$

for all $h \in M_n$, so h_1^* is a best L_1-simultaneous approximation to f and g by elements of S_π. Now let h be a non-decreasing function on $[0, 1]$. We show that there is a non-decreasing function $h^* \in S_\pi$ such that

$$\|f - h_1^*\|_1 + \|g - h^*\|_1 \leqslant \|f - h\|_1 + \|g - h\|_1 .$$

Indeed if h is not constant on $(t_{i-1}, t_i]$, assume without loss of generality that $f_i > g_i$. We have two different cases to consider:

Case 1. If $g_i \leqslant h(t_{i-1}^+) < h(t_i) \leqslant f_i$, then clearly taking $h^* = c$ for any $c \in [g_i, f_i]$ will imply

$$\int_{t_{i-1}}^{t_i} (|f_i - c| + |g_i - c|)dt = \int_{t_{i-1}}^{t_i} (|f_i - h(t)| + |g_i - h(t)|)dt.$$

Case 2. If $h(t_i^-) > f_i$, then $h(t) > f_i$ on a sub-interval $(t_i - \delta, t_i]$, whence taking $h^* = h$ on $(t_{i-1}, t_i - \delta]$ and $h^* = f_i$ on $(t_i - \delta, t_i]$ implies

$$\int_{t_{i-1}}^{t_i} (|f_i - h^*| + |g_i - h^*|)dt < \int_{t_{i-1}}^{t_i} (|f_i - h(t)| + |g_i - h(t)|)dt.$$

The case $h(t_{i-1}^+) < g_i$ is similar. All other cases are treated similarly. This establishes the theorem. ∎

We finish this section with an outline of the case when f and g are quasi-continuous, that is, functions having discontinuities of the first kind only. More precisely we shall assume that $f \in Q$ if $f(0) = f(0^+)$ and $f(x) = f(x^-)$, $0 < x \leqslant 1$.

DEFINITION: Let f be a bounded measurable function on $[0, 1]$, and let π be a partition of $[0, 1]$. Then $\overline{f}_\pi$ in S_π is defined by

$$\overline{f}_\pi(x) = \begin{cases} \sup\{f(y):\quad 0 \leqslant y \leqslant t_1\}, & x \in [0, t_1] \\ \sup\{f(y): t_{i-1} < y \leqslant t_i\}, & x \in (t_{i-1}, t_i], i > 1. \end{cases}$$

$\underline{f}_\pi$ is defined similarly by replacing sup with inf. A bounded function f is in Q if and only if, for any $\varepsilon > 0$, there exists a partition π of $[0, 1]$ such that $0 \leqslant \overline{f}_\pi - \underline{f}_\pi < \varepsilon$. Thus $\lim_\pi \overline{f}_\pi = \lim_\pi \underline{f}_\pi = f$. This characterisation enables us to use the previous results for step functions.

To this end we adapt the proofs of Lemma 6 and Theorem 4 of [**6**] which were based on the results of [**3**] to yield the principal result of this section.

THEOREM 9. *Let f and g be in Q. Let $\overline{f}_\pi, \overline{g}_\pi, \underline{f}_\pi, \underline{g}_\pi$ be as defined above, and let $\overline{h}_{\pi,p}, \underline{h}_{\pi,p}$ be the best L_p-simultaneous approximations of $\overline{f}_\pi, \overline{g}_\pi$ and $\underline{f}_\pi, \underline{g}_\pi$ respectively. If h_p is the b.s.a. to f and g, then $\lim_\pi \overline{h}_{\pi,p} = \lim_\pi \underline{h}_{\pi,p} = h_p$, and $\lim_{p \downarrow 1} h_p = h_1$ exists. Moreover h_1 is a best L_1-simultaneous approximation to f and g.*

Remark. Theorem 5 in [**6**] shows that the continuity of f and g implies that of h_p for all $p \in (1, \infty)$. Since $h_p \to h_1$ uniformly, then h_1 is continuous in this case.

4. APPROXIMATE CONTINUITY OF f AND g

DEFINITION: If A is a measurable subset of Ω and I is a subinterval of Ω, the relative measure of A in I is defined by

$$m(A, I) = m(A \cap I)/m(I).$$

The upper metric density of A at $x \in \Omega$ is defined by

$$\overline{m}(A, x) = \lim_{n\to\infty} \sup_I\{m(A, I) : I \text{ is an interval, } x \in I \text{ and } mI < \frac{1}{n}\}.$$

The lower metric density $\underline{m}(A, x)$ is defined similarly, with sup replaced by inf. A has a metric density at x only when $\overline{m}(A, x) = \underline{m}(A, x) = m(A, x)$.

DEFINITION: A function $f: \Omega \to R$ is said to be approximately continuous at $x \in \Omega$ if, for any $\varepsilon > 0$, the set

$$A_\varepsilon = \{y: |f(y) - f(x)| < \varepsilon\}$$

has metric density equal to 1 at x; f is said to be approximately continuous on Ω if it is approximately continuous at each point in Ω. Notationally we write $f \in bA$.

THEOREM 10. *Let f and g be elements of bA. Then for all $p \in (1, \infty)$, $h = h_p$ is continuous.*

PROOF: Suppose first that both f and g are in $M \cap bA$. Then both have at most discontinuities of the first kind, that is, for any $y \in (0, 1)$ the left and right hand limits exist. Thus $f(y^-) = \lim_{x \uparrow y} f(y)$ and $f(y^+) = \lim_{x \downarrow y} f(y)$ both exist. So if f is not continuous at y, then $f(y^-) < f(y^+)$, and so $f \notin bA$. This is a contradiction. Therefore f must be continuous and so is g. By [**6**, Section 4] we may consider f and g as limits of non-decreasing step functions having their mean as their best L_p-simltaneous approximations. Taking limits as in [**6**], we conclude that the b.s.a. to f and g is nothing but $(f+g)/2$ which is clearly in $M \cap bA$. Hence $(f+g)/2 = h$ is continuous. A similar argument works out if $h = f$ or $h = g$.

For the general case we have $f \notin M$, $g \notin M$. So $f \neq h \neq g$. We start with points $y \in (0, 1)$ where $g(y) < h(y) < f(y)$. The case $f(y) < h(y) < g(y)$ is similar. Suppose $f(y) - h(y) = \varepsilon_1 > \varepsilon_2 = h(y) - g(y)$. We may assume that

$$\begin{aligned} h(y) &= \lim_{x \to y^+} h(x) \\ &= h(y^+). \end{aligned}$$

Let $\varepsilon = (\varepsilon_1 - \varepsilon_2)/5 > 0$. Let $Q \in (0, 1)$ be a fixed real number which we specify later. Since f is approximately continuous at y, there exists $\delta_1 = \delta_1(Q) > 0$ such that

$$\mu(\{x \colon f(x) > f(y) - \varepsilon\}, I) > Q,$$

for any interval I containing y and $I \subseteq B(y, \delta_1) = (y - \delta_1, y + \delta_1)$. Similarly there exists $\delta_2 = \delta_2(Q)$ such that

$$\mu(\{x \colon |g(x) - g(y)| < \varepsilon\}, I) > Q,$$

for any interval I containing y and $I \subseteq B(y, \delta_2) = (y - \delta_2, y + \delta_2)$. Let $1 > \delta = \min(\delta_1, \delta_2)$ and $I = (y - \delta, y]$. Let

$$F = I \cap \{x \colon f(x) > f(y) - \varepsilon\},$$

and

$$G = I \cap \{x \colon |g(x) - g(y)| < \varepsilon\}.$$

Then both F and G have measures greater than δQ.

Suppose h is not continuous at y. We show this assumption yields a contradiction. Let $\eta = \min\{h(y) - h(y^-), \varepsilon\} > 0$. Define $h^* \colon \Omega \to R$ by

$$h^*(x) = \begin{cases} h(x) + \eta, & \text{if } x \in (y - \delta, y), \\ h(x), & \text{otherwise.} \end{cases}$$

Apply the Mean Value Theorem to the functions $s \mapsto s^p$ where $p > 1$, so there exists $u \in (s, s+\sigma)$ such that

$$(s+\sigma)^p - s^p = pu^{p-1}\sigma \geqslant ps^{p-1}\sigma, \tag{15}$$

or,

$$(s+\sigma)^p - s^p \leqslant p(s+\sigma)^{p-1}\sigma. \tag{16}$$

Hence for $t \in F$, we obtain by applying (15)

$$|f(t) - h(t)|^p - |f(t) - h^*(t)|^p \geqslant p\,|f(t) - h^*(t)|^{p-1}\,\eta,$$

whence

$$\int_F |f-h|^p - \int_F |f-h^*|^p \geqslant p\eta \int_F |f-h^*|^{p-1}. \tag{17}$$

Similarly, for $t \in G$, we obtain by applying (16)

$$\int_G |g-h^*|^p - \int_G |g-h|^p \leqslant p\eta \int_G |g-h^*|^{p-1}. \tag{18}$$

Subtracting (17) from (18) we obtain

$$\int_F |f-h^*|^p + \int_G |g-h^*|^p \leqslant \int_F |f-h|^p + \int_G |g-h|^p - p\eta B, \tag{19}$$

where $B = \int_F |f-h^*|^{p-1} - \int_G |g-h^*|^{p-1} > 0$.

Notice also in general that

$$\big||f(t) - h^*(t)|^p - |f(t) - h(t)|^p\big| \leqslant p(2\,\|f\|_\infty)^{p-1}\,|h^*(t) - h(t)|,$$

thus,

$$\int_{I-F} |f-h^*|^p \leqslant \int_{I-F} |f-h|^p + p(2\,\|f\|_\infty)^{p-1}\eta\mu(I-F). \tag{20}$$

Similarly

$$\int_{I-G} |g-h^*|^p \leqslant \int_{I-G} |g-h|^p + p(2k)^{p-1}\eta\mu(I-F), \tag{21}$$

where $k = \max(\|g\|_\infty, \|h^*\|_\infty)$.
After observing that $\mu(I - F) < \delta(1 - Q) < 1 - Q$ we add (19), (20) and (21) to get

$$\begin{aligned}\int_I |f - h^*|^p + \int_I |g - h^*|^p &< \int_I |f - h|^p + \int_I |g - h|^p - p\eta B \\ &\quad + p\eta(1 - Q)[(2\|f\|_\infty)^{p-1} + (2k)^{p-1}], \\ &< \int_I |f - h|^p + \int_I |g - h|^p,\end{aligned}$$

provided Q was chosen so that

$$(1 - Q)[(2\|f\|_\infty)^{p-1} + (2k)^{p-1}] < B,$$

or,

$$1 > Q > 1 - B/[(2\|f_\infty\|)^{p-1} + (2k)^{p-1}] > 0.$$

Thus, h^* is a better simultaneous approximation to f and g. This is a contradiction. This verifies the continuity of h at y in this case.

For the case when $\varepsilon_1 \leqslant \varepsilon_2$ we argue similarly on $[y, y + \delta)$ to obtain a contradiction by lowering the value of h.

If $f(y) = h(y)$ or $g(y) = h(y)$ and h is not continuous at y, then $h(y) - h(y^-) = 3\varepsilon > 0$. We apply again a similar argument on $(y - \delta, y]$ when $h(y) = g(y) \leqslant f(y)$ and on $[y, y + \delta)$ when $f(y) = h(y) > g(y)$.

This establishes the theorem. ∎

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