

Multiscale modelling for tokamak pedestals

I. G. Abel^{1,2,†} and A. Hallenbert¹

¹Department of Physics, Chalmers University of Technology, Göteborg, SE-41296, Sweden

²Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544, USA

(Received 28 November 2017; revised 26 March 2018; accepted 26 March 2018)

Pedestal modelling is crucial to predict the performance of future fusion devices. Current modelling efforts suffer either from a lack of kinetic physics, or an excess of computational complexity. To ameliorate these problems, we take a first-principles multiscale approach to the pedestal. We will present three separate sets of equations, covering the dynamics of edge localised modes (ELMs), the inter-ELM pedestal and pedestal turbulence, respectively. Precisely how these equations should be coupled to each other is covered in detail. This framework is completely self-consistent; it is derived from first principles by means of an asymptotic expansion of the fundamental Vlasov–Landau–Maxwell system in appropriate small parameters. The derivation exploits the narrowness of the pedestal region, the smallness of the thermal gyroradius and the low plasma β (the ratio of thermal to magnetic pressures) typical of current pedestal operation to achieve its simplifications. The relationship between this framework and gyrokinetics is analysed, and possibilities to directly match our systems of equations onto multiscale gyrokinetics are explored. A detailed comparison between our model and other models in the literature is performed. Finally, the potential for matching this framework onto an open-field-line region is briefly discussed.

Key words: fusion plasma, plasma confinement

1. Introduction

The pedestal region of current fusion devices is both a boon and a serious obstacle to achieving sustained fusion power. Discovered by pioneering work on the ASDEX tokamak in Germany (Wagner *et al.* 1982; The ASDEX Team 1989), the ‘*H*-mode edge pedestal’ or ‘edge transport barrier’ is a narrow region at the edge of the tokamak where extremely steep gradients can be maintained (thus improving confinement) but which intermittently relaxes through the release of large amounts of hot plasma, known as ‘edge localised modes’ (Zohm 1996) (henceforth ELMs). The dichotomy between the confinement that can make a tokamak into a smaller and more efficient reactor and the ELMs that can severely damage that reactor has led to a vast amount of study of pedestals since their discovery.

Enormous strides have been made, including understanding of the fundamental instabilities that are responsible for ELMs (Connor *et al.* 1998; Snyder *et al.* 2002)

† Email address for correspondence: iga@physics.org

and incorporating this knowledge into successful analysis of experiments (Snyder *et al.* 2007, 2011). Experimentally, analyses have been made from data acquired on multiple machines (Beurskens *et al.* 2011), but the narrowness of the pedestal and the difficulty of making these measurements has hampered making predictions for future devices. Indeed, recent results and analyses (Hatch *et al.* 2017; Kotschenreuther *et al.* 2017) suggest that the scaling of the pedestal width with fundamental plasma parameter from current machines to ITER may be in doubt. More detailed modelling is thus essential.

Edge modelling comprises both first principles analysis and a large and established effort in trying to understand existing devices through interpretative modelling. Many difficulties arise in modelling this region of the plasma; any model must be able to handle large, rapidly evolving perturbations associated with ELMs as well as small-scale fluctuations that may regulate the residual transport through the inter-ELM pedestal. In addition, the collisionality of the plasma may vary by orders of magnitude across the pedestal. Taking data from a multi-machine comparison (Militello & Fundamenski 2011), the collisionality at the pedestal top for the joint European torus (JET) is $\nu_e^* \approx 4 \times 10^{-3}$ climbing to $\nu_e^* \approx 0.13$ at the separatrix. Using the predictions for ITER in the same paper, the collisionality at the pedestal top is now $\nu_e^* \approx 4 \times 10^{-4}$ and still $\nu_e^* \sim 0.2$ at the separatrix. This indicates a trend towards increasingly collisionless pedestals as machines get larger, driven in part by divertor heat load requirements preventing a high separatrix temperature. Suitable modelling must therefore be able to transition from collisionless to collisional physics across the pedestal region.

Several main branches of research into near-edge (i.e. regions at the edge of the plasma, but still inside the closed-field-line region) modelling currently exist. The most experimentally successful approach has been that taken by the EPED collaboration (Snyder *et al.* 2011, 2007). This consists of linear magnetohydrodynamic (MHD) modelling (with the ELITE code) of the peeling-ballooning mode to determine the onset of an ELM, the use of the infinite- n ballooning stability criterion as a proxy for the stability of the inter-ELM pedestal to microturbulence and an assumption that the pedestal is pinned to this stability threshold between ELMs. Despite the success of this model, the *ad hoc* marginal stability criterion is required to constrain the width of pedestals, the linear stability calculation may neglect important kinetic effects and the model has no capacity for handling time-dependent inter-ELM dynamics. The dominant paradigm for first-principles edge modelling is that of ‘drift reduced’ or other anisotropic fluid equations (Zeiler, Drake & Rogers 1997). These are the basis of many current simulation codes, such as GBS (Ricci *et al.* 2012), and also older two-dimensional codes including SOLPS and others (Schneider *et al.* 2006), and have been used to study the pedestal directly (Nielsen 2016). Gyrofluid equations, originally developed for the core (Hammett *et al.* 1993) have also found much use in edge modelling (Scott 1997, 2007). Gyrofluid models are derived by taking moments of gyrokinetic equations and then applying one of a variety of closure conditions. These models have the advantage that a limited number of fluid moments is numerically tractable even in complex edge geometries. However, the strong requirements on collisionality for Braginskii equations to be applicable are often not satisfied in high-performance pedestals. Gyrofluid equations at low collisionality become increasingly plagued by questions of the validity of the chosen closure scheme, and lack a first-principles justification for the closures used. Thus, to handle a potentially trans-collisional pedestal region, we will take a kinetic approach.

Recently, kinetic simulations of the plasma edge have been performed. The first category of these are the global gyrokinetic simulations (Ku *et al.* 2006; Chang *et al.*

2017; Shi *et al.* 2017). These endeavour to capture all the dynamics of the edge region, from the macro scales down to the ion gyroradius scale. The computational expense is titanic, and thus parameter scans, and predictive modelling are currently out of reach for most applications. The second category of kinetic simulations only seeks to tackle turbulence in the edge, and have shed some light upon the inter-ELM pedestal (Dickinson *et al.* 2012, 2013; Hatch *et al.* 2017; Kotschenreuther *et al.* 2017). These simulations have been conducted despite the fact that the levels of $\mathbf{E} \times \mathbf{B}$ shear and flow velocities in the pedestal are not consistent with the assumptions under which gyrokinetics is usually derived (Abel *et al.* 2013). Thus, we seek to embed gyrokinetics within a more complete pedestal model.

Bridging the gap between fluid models that cannot fully capture the kinetic physics, and gyrokinetic modelling that is extremely expensive and hard to interpret, we instead use a first-principles multiscale approach. Our formulation neatly separates the physics of ELMs, coherent oscillations and inter-ELM turbulence. This permits models for each component that are only as complex as needed, rather than a model that endeavours to rigorously include all possible physics. This conceptual simplification should enable more detailed studies of each separate part of the pedestal physics than is currently possible. However, in this paper we only present a new model for the pedestal, which, in this work, we consider to be the steep gradient part of the closed-field-line region – we leave the issues of extending this to an open-field-line region at the top of the scrape-off layer to future work.

The structure of the remainder of the paper, which lays this model out in detail, is as follows. In §2 we discuss the physics needed to capture ELMs. Our physical requirements are translated into formal ordering assumptions in §2.1. In §2.2 we follow the well-trodden paths of asymptotic theory to turn our ordering assumptions into a closed set of equations for ELM dynamics.

Proceeding to the longer, inter-ELM, time scale in §3 we follow the same procedure, presenting our orderings in §3.1 and the equations for the inter-ELM evolution in §3.2. However, in this procedure we have to introduce an explicit multiple-scales approach to handle inter-ELM turbulence – governed in turn by orderings and equations that are detailed in §3.3.

Finally, in §4 we make detailed contact between our model and existing models of the pedestal. In §4.1 we describe how our model can be integrated with existing validated models for the core and scrape-off layer, acting as a boundary condition for both. Then, in §4.4, we compare our model to the models currently used to study the pedestal. Ultimately, we end the paper with a summary of our results in §5 and some discussion of exploration of new avenues this model may open up.

2. Fast and furious: the physics of edge localised modes

As alluded to in the introduction, ELMs are the large-amplitude convulsions of the edge pedestal that pose a great threat to current fusion devices. For this reason, ELM stability must be comprehensively addressed in any plausible pedestal model. To this end, in §2.1 we codify the physics of ELMs in terms of the temporal and spatial scales of interest, and relations between plasma parameters. These are then applied in §2.2 to obtain the dynamical equations governing ELMs.

2.1. Orderings for ELM dynamics

ELMs are now well understood to be nonlinear peeling-ballooning modes (Connor *et al.* 1998). These modes are driven by the combination of edge current density,

curvature and pressure gradients. They naturally occur on an Alfvénic time scale, that is one which is comparable to L_{\parallel}/v_A , where L_{\parallel} is the typical length scale of the instability along a field line and v_A is the Alfvén velocity. In this work we will use Gaussian units throughout and so, explicitly, $v_A = B/\sqrt{4\pi \sum_s m_s n_s}$ where B is the magnetic field strength, and the plasma is composed of several species with masses, m_s , and densities, n_s . The parallel length scale L_{\parallel} will be taken to be the longest length scale in our system, and is comparable to the connection length of the magnetic field. In general this means that the defining relationship is $L_{\parallel} \sim (\mathbf{b} \cdot \nabla \ln B)^{-1}$, whereas L_{\parallel} is only ever used in our asymptotic orderings prefactors of order unity do not matter and will be neglected.

In order to cause such violent instabilities, the pressure gradients must be large, with the pressure varying over a distance $L_{\perp} \ll L_{\parallel}$ perpendicular to the field lines. Indeed, pedestals are observed to be extremely narrow structures in current experiments (Beurskens *et al.* 2011). Hence we adopt

$$\epsilon = \frac{L_{\perp}}{L_{\parallel}} \ll 1, \quad (2.1)$$

as our fundamental small parameter.

The most basic stability limit for the ballooning part of the peeling-ballooning mode is a limit on $\beta = 8\pi \sum_s n_s T_s / B^2$, with T_s the temperature of species s .¹ This limit derives from the constraint that the energy required to bend field lines must be comparable to the energy released by relaxing the pressure gradient (Freidberg 2014). Estimating this balance, we obtain

$$\beta \sim \frac{L_{\perp}}{L_{\parallel}}, \quad (2.2)$$

which immediately implies that β must be small in these systems. Note, in our slightly unconventional usage $a \sim b$ means that a and b are of the same asymptotic order as $\epsilon \rightarrow 0$, but may differ by numerical prefactors of order unity. It is also important at this stage to note that we are expanding in the ratio of the pedestal width to the magnetic scale length, not in the aspect ratio of the magnetic surfaces.²

At this point, we shall not distinguish between profiles and fluctuations in terms of frequencies and length scales. Thus, all gradients will be ordered as L_{\perp}^{-1} or L_{\parallel}^{-1} . This is consistent with our next assumption: that the kinetic plasma variables (e.g. density, pressure) will have $O(1)$ variation. This is physically reasonable as ELMs are observed to result in filaments of hot dense plasma erupting into much colder, lower density regions. To support finite density variations on times comparable to the electron transit time $L_{\parallel}/v_{\text{the}}$, we require that the electrostatic potential also support large variations

$$\frac{e\phi}{T_e} \sim 1. \quad (2.3)$$

Strong radial electric fields are known to play a role in pedestals, and many empirical measurements find these electric fields to be present in the absence of large

¹We will later discover that our orderings do not constrain all species to be in local thermal equilibrium. Thus β should really be defined instead in terms of the pressure of species s perpendicular to the field lines. As we are only interested in asymptotic orderings in this section, we will retain T_s as a useful placeholder, that can be thought of as a mean kinetic energy per particle if species s is not in thermal equilibrium.

²Hence, our ordering for β is compatible with the usual tokamak estimate $\beta \lesssim (r/q^2 R) dp/dr$ as r/R and q are taken to be $O(1)$.

plasma flows. The simplest and most plausible mechanism for this to occur is that diamagnetic flows, for the bulk ion species, are comparable to, and may sometimes cancel, the $\mathbf{E} \times \mathbf{B}$ flow in the pedestal. To permit such cancellation we require the diamagnetic flow to be of the same order as the $\mathbf{E} \times \mathbf{B}$ flow:

$$\frac{T_s}{m_s \Omega_s L_\perp} \sim \mathbf{u}_E, \tag{2.4}$$

with $\Omega_s = Z_s e B / m_s c$ the cyclotron frequency of species s , c the speed of light and $Z_s e$ the charge of species s in terms of the unit charge e . Note that this is also the ordering that follows from (2.3) and assuming that φ varies on the L_\perp scale. In this, we assume $Z_s \sim 1$ with respect to the $\epsilon \ll 1$ expansion. Thus, this ordering does not describe the behaviour of an impurity species that is so highly charged that $Z_s \sim \epsilon^{-1}$ or even greater. We also will assume that all temperatures are comparable, in particular $T_e \sim T_i$, and will not keep track of factors of T_i/T_e in our orderings.

Next, requiring that the system be fully nonlinear, the typical decorrelation rate due to the $\mathbf{E} \times \mathbf{B}$ nonlinearity should be comparable to the frequency of interest, ω :

$$\frac{\mathbf{u}_E}{L_\perp} \sim \omega. \tag{2.5}$$

Hence our frequencies are comparable to the diamagnetic drift frequency ω_*

$$\omega \sim \omega_* = \frac{c T_e}{e B L_\perp^2}. \tag{2.6}$$

Using our assumption of Alfvénic time scales

$$\omega \sim \frac{v_A}{L_\parallel}, \tag{2.7}$$

and our ordering for β , we have

$$\omega_* \sim \frac{v_A}{L_\parallel} \sim \sqrt{\epsilon} \frac{v_{thi}}{L_\perp}, \tag{2.8}$$

which immediately gives

$$\frac{\rho_i}{L_\perp} \sim \sqrt{\epsilon}. \tag{2.9}$$

Similarly, we see that our frequencies are small compared to the cyclotron frequency

$$\frac{\omega}{\Omega_s} \sim \frac{\rho_i^2}{L_\perp^2} \sim \epsilon. \tag{2.10}$$

Substituting the definition of ϵ from (2.1) into (2.9) gives an estimate for the typical pedestal scale L_\perp in terms of $L_\parallel \sim R$ (a typical tokamak estimate, where R is the major radius of the device) and ρ_i :

$$L_\perp \sim \sqrt[3]{\rho_i^2 R}. \tag{2.11}$$

With this ordering in hand, we can also relate our ϵ to the ratio of the gyroradius to the system size:

$$\epsilon \sim \left(\frac{\rho_i}{R}\right)^{2/3}. \quad (2.12)$$

Let us discuss the impact of the scaling (2.11). Firstly, we note that our orderings have given us a pedestal width where diamagnetic stabilisation of the ballooning mode may be important for ELM stability (Rogers & Drake 1999). This scaling is more optimistic for future devices than a naïve neoclassical scaling which would predict a pedestal width L_\perp comparable to the poloidal Larmor radius³ ρ_θ – in our orderings the poloidal Larmor radius is equivalent to ρ_i – but much less optimistic than the experimental results which typically show remarkably little dependence upon ρ_i (Beurskens *et al.* 2011).

One might take these experimental data as discouragement from believing our orderings. At the same time some pedestals appear to be dominated by neoclassical transport. These two can be reconciled by noting that in current devices ρ_θ is not much smaller than the pedestal width. In addition $\rho_i^{2/3} qR$ is not much larger than ρ_θ . Both of these relationships need to be well satisfied for our theory to be valid. Hence, it is very likely that current machines do not operate at a small enough value of ϵ to see the asymptotic scaling in (2.11). Thus, (2.11) is a prediction for machines where ρ_i/R is smaller, with a prefactor that could be determined by simulation of the equations presented herein.

We still have a couple of parameters to order with respect to ϵ . Firstly, we choose to retain electron parallel kinetic effects (Landau resonances etc.) upon the ELMs and so order

$$\frac{m_e}{m_i} \sim \beta \sim \epsilon, \quad (2.13)$$

resulting in $v_{th_e} \sim v_A$. Secondly, in order to straddle the divide between collisional and collisionless physics, we make the maximal ordering of

$$v_{ee} \sim \frac{v_{th_e}}{L_\parallel} \sim \omega, \quad (2.14)$$

which naturally results in $v_{ii} \sim v_{th_i}/L_\parallel$. This concludes the ordering for the ELMs. Before moving on to turning these orderings into dynamical equations, we must consider the possibility of the existence of small-scale turbulence being present whilst an ELM is occurring.

2.1.1. Orderings for pedestal turbulence

Foreshadowing the results of § 3.3, it will turn out that our pedestal supports turbulent fluctuations, and the inter-ELM transport is dominated by them.⁴ The effects

³This being the width required to increase ion neoclassical transport to the level where it can provide transport on the v_{th_i}/L_\parallel time scale. This is necessary if one wishes to match to an open-field-line region, where parallel losses to material surfaces occur on this time scale. Such a scaling is also predicted for the near-plasma scrape-off layer from drift-orbit estimates in Goldston (2012). The full neoclassical theory for such a pedestal is detailed in Kagan & Catto (2008, 2010) and related publications.

⁴Whilst it is often presumed that all turbulence is suppressed after a low to high confinement (L–H) transition, this is only possible for a pedestal that is on the scale of the (poloidal) gyroradius. For a pedestal that is asymptotically wider than this, turbulence is required to carry a finite heat flux through the pedestal. See the discussion at the end of § 2

of the turbulence are, *a priori*, strong enough that they could influence ELM evolution. In order to consistently include these effects, we will state here the assumptions we make about such turbulence. These assumptions will be justified in § 3.3.

The fluctuations are small, with

$$\frac{\delta f_s}{f_s} \sim \frac{e\delta\varphi}{T_e} \sim \sqrt{\epsilon}, \tag{2.15}$$

and

$$\frac{\delta B}{B} \sim \epsilon. \tag{2.16}$$

The fluctuations will have typical length scales, perpendicular to the field, given by

$$\nabla_{\perp} \ln \delta f_s \sim \rho_i^{-1}, \tag{2.17}$$

and along the field given by

$$\mathbf{b} \cdot \nabla \ln \delta f_s \sim (L_{\perp} L_{\parallel})^{-1/2}. \tag{2.18}$$

One might worry that the large drive in the pedestal will cause turbulence from $k_{\perp} \rho_i \sim 1$ to reach larger scales. However, in our ordering, this has already occurred. These fluctuations are naturally electron-scale fluctuations, forced out to ion scales by the strong drive. Indeed, the time scale of these fluctuations is assumed to be

$$\frac{\partial}{\partial t} \ln \delta f_s \sim \frac{v_{\text{the}}}{\sqrt{L_{\perp} L_{\parallel}}}. \tag{2.19}$$

As these fluctuations are faster than the ELMs and on a smaller scale, by the usual method of introducing intermediate temporal and spatial length scales we can assume the existence of an averaging operation $\langle \cdot \rangle_{\text{turb}}$ such that

$$\delta f_s = \tilde{f}_s - \langle \tilde{f}_s \rangle_{\text{turb}}, \tag{2.20}$$

where \tilde{f}_s is the exact (inclusive of fluctuations) distribution function. The averaged quantities are assumed to obey the orderings given above, and so we will use an undecorated notation for the averaged quantities, e.g.

$$f_s = \langle \tilde{f}_s \rangle_{\text{turb}}. \tag{2.21}$$

We note that, in these orderings, $\sqrt{\epsilon} \sim \rho_i/L_{\perp}$, and that L_{\perp} is the appropriate local scale length for the physics that drives the turbulence. Thus, our orderings say that the fluctuations are small in $\rho^* \equiv \rho_i/L_{\perp}$. It is this relationship that will give rise to the similarity between the equations governing the turbulence and gyrokinetics.

2.2. *Dynamical equations*

Having argued for our consistent orderings for the ELM dynamics, we now apply them to the governing equations of the plasma to obtain our reduced model. The fundamental equations we will work from are the Vlasov–Landau equation for the exact distribution function \tilde{f}_s , in terms of the exact fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$:

$$\frac{\partial \tilde{f}_s}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f}_s + \frac{Z_s e}{m_s} \left(\tilde{\mathbf{E}} + \frac{1}{c} \mathbf{v} \times \tilde{\mathbf{B}} \right) \cdot \frac{\partial \tilde{f}_s}{\partial \mathbf{v}} = C[\tilde{f}_s] + S_s, \tag{2.22}$$

where \mathbf{v} is the particle velocity and the collision operator on the right-hand side is the Landau operator. We have also added an arbitrary source S_s that we will keep in our inter-ELM equations (using the maximal ordering $S_s \sim f_s v_{\text{th}_i} / L_{\parallel}$ so that the source can compete with inter-ELM transport). We will assume a non-relativistic plasma and also that all frequencies are low compared to the plasma frequency and all lengths long compared to the Debye length. Thus, to close the system for the fields, we have the quasineutrality constraint

$$\sum_s Z_s e n_s = 0, \tag{2.23}$$

and the pre-Maxwell Ampère’s law:

$$\nabla \times \tilde{\mathbf{B}} = \frac{4\pi}{c} \tilde{\mathbf{j}}. \tag{2.24}$$

Averaging these equations over any possible fast fluctuations, we have as our fundamental kinetic equation:

$$\begin{aligned} \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} &= C[f_s] \\ - \frac{Z_s e}{m_s} \left\langle \left(\delta \mathbf{E} + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B} \right) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} &+ \langle C[\delta f_s] \rangle_{\text{turb}}, \end{aligned} \tag{2.25}$$

and as the field equations are linear, they apply separately to the averaged and fluctuating fields.

2.2.1. *The magnetic field*

The first problem we tackle is that of the structure of the magnetic field. As $\beta \sim L_{\perp} / L_{\parallel} \ll 1$, we naturally expect the time-dependent piece of the magnetic field to be small compared to the background, confining, field. To investigate this, we first take the $m_s \mathbf{v}$ moment of (2.25), and sum over all species to obtain

$$\frac{\partial}{\partial t} \sum_s n_s m_s \mathbf{u}_s + \nabla \cdot \sum_s \mathbf{P}_s = -\nabla \cdot \left(\frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B} + \nabla \cdot \left\langle \frac{\delta \mathbf{B} \delta \mathbf{B}}{4\pi} - I |\delta \mathbf{B}|^2 \right\rangle_{\text{turb}}, \tag{2.26}$$

where

$$\mathbf{P}_s = \int d^3 \mathbf{v} m_s \mathbf{v} \mathbf{v} f_s, \tag{2.27}$$

is the full pressure tensor of species s .

Estimating the size of each term in this equation, the magnetic pressure force is larger than all other terms by one power of ϵ , due to the small β . Thus, we conclude that the time-dependent piece of the magnetic field must be small compared to the background, confining, field, and so we make the split

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{2.28}$$

where $\mathbf{B}_1 \sim \epsilon \mathbf{B}_0$. Note that, despite this decomposition, we will define parallel and perpendicular components of vectors with respect to the total field \mathbf{B} .

With this ordering in hand, we can solve the lowest-order equation

$$\nabla_{\perp} B_0^2 = O(\epsilon), \tag{2.29}$$

by insisting that B_0 vary only on the long L_{\parallel} scale.

We note that the size estimate for \mathbf{B}_1 is the same as what one would obtain by assuming the field follows the fluid flow and so can be displaced at most L_{\perp} in the perpendicular direction, and that this displacement occurs after following the perturbed field for a distance L_{\parallel} :

$$\mathbf{B}_1 \sim \frac{L_{\perp}}{L_{\parallel}} \mathbf{B}_0. \tag{2.30}$$

This is also the estimate one would obtain by insisting that $\mathbf{B}_1 \cdot \nabla_{\perp} \sim \mathbf{B}_0 \cdot \nabla$.

Given that $\mathbf{b} \cdot \nabla \mathbf{b} \sim L_{\parallel}^{-1}$ by definition of the parallel scale, one might be tempted to conclude that \mathbf{B}_0 only varies on the L_{\parallel} scale. However, we can still admit a large magnetic shear. That is, the tensor $\nabla_{\perp} \mathbf{b}_0$ may have large components. As will be argued later, these cannot be as large as $O(L_{\perp}^{-1})$ because it would entail an infeasibly strong electron current. However we will retain the possibility that the shear length L_s of \mathbf{B}_0 is as short as $O(\sqrt{L_{\perp} L_{\parallel}})$.

Given this decomposition of the magnetic field into fixed and time-dependent pieces, we introduce the vector potential \mathbf{A} for \mathbf{B}_1 only:

$$\mathbf{B}_1 = \nabla \times \mathbf{A}, \tag{2.31}$$

and we will work in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.

The piece of \mathbf{B}_1 that will turn out to be important is the change in the direction of the magnetic field, thus we make the natural decomposition $\mathbf{A} = A_{\parallel} \mathbf{b} + \mathbf{A}_{\perp}$ to find

$$\mathbf{B}_1 = \nabla A_{\parallel} \times \mathbf{b} + \mathbf{B}_1 \cdot \mathbf{b} \mathbf{b}. \tag{2.32}$$

From here on in, we will define perpendicular and parallel components of vectors with respect to the total large-scale field \mathbf{B} , inclusive of \mathbf{B}_1 but not of any possible turbulent magnetic field. If we at any point we only wish to consider the static field \mathbf{B}_0 we will denote that explicitly with a subscript 0 on the appropriate terms.

2.2.2. Kinetic variables

It will be simplest, despite not retaining finite gyroradius effects, to follow the gyrokinetic literature and transform to guiding centre variables for our derivations. To simplify later derivations, we will also use a peculiar velocity relative to the mean $\mathbf{E} \times \mathbf{B}$ velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_E$. It is important to note that $\mathbf{u}_E = c \mathbf{E} \times \mathbf{b} / B$ is defined

using the total mean \mathbf{B} including \mathbf{B}_1 . Hence, $w_{\parallel} = v_{\parallel}$ exactly. Thus, we move from (\mathbf{r}, \mathbf{v}) to the Catto-transformed variables $(\mathbf{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$:

$$\mathbf{R}_s = \mathbf{r} - \frac{1}{\Omega_s} \mathbf{b} \times \mathbf{w}, \tag{2.33}$$

$$\varepsilon_s = \frac{1}{2} m_s w^2 \tag{2.34}$$

$$\mu_s = \frac{m_s w_{\perp}^2}{2B} + O(\epsilon), \tag{2.35}$$

where ϑ is the gyrophase, σ is the sign of the parallel velocity and perpendicular and parallel components are taken with respect to the total mean magnetic field \mathbf{B} . We take the magnetic moment μ_s to be exact adiabatic invariant of the Larmor motion that is conserved to all orders in ϵ (Kruskal 1958; Berkowitz & Gardner 1959).⁵

Finally, the gyrophase ϑ is defined by

$$\mathbf{w} = w_{\parallel} \mathbf{b} + w_{\perp} (\cos \vartheta \mathbf{e}_2 - \sin \vartheta \mathbf{e}_1), \tag{2.36}$$

where \mathbf{e}_1 and \mathbf{e}_2 are mutually orthogonal basis vectors that satisfy $\mathbf{b} = \mathbf{e}_2 \times \mathbf{e}_1$

We will find it convenient to average over this gyrophase at various points in this work. All of these gyroaverages are taken at constant \mathbf{R}_s , ε_s and μ_s ; we denote the gyroaverage by

$$\langle \dots \rangle_{\mathbf{R}} = \frac{1}{2\pi} \int_0^{2\pi} (\dots) d\vartheta. \tag{2.37}$$

We will also occasionally need a gyroaverage at fixed \mathbf{r} , w_{\parallel} , w_{\perp} , which is denoted by

$$\langle \dots \rangle_{\mathbf{r}} = \frac{1}{2\pi} \int_0^{2\pi} (\dots) d\vartheta |_{r, w_{\parallel}, w_{\perp}}. \tag{2.38}$$

The gyroaveraged (at fixed \mathbf{R}_s) time derivatives of the gyrokinetic variables are calculated in appendix A, and are found to be (cf. (A 3) and (A 5))

$$\langle \dot{\mathbf{R}}_s \rangle_{\mathbf{R}} = w_{\parallel} \mathbf{b} + \frac{c}{B} \mathbf{b} \times \nabla \varphi + O(\epsilon^{3/2} v_{\text{th}i}), \tag{2.39}$$

$$\langle \dot{\varepsilon}_s \rangle_{\mathbf{R}} = -Z_s e w_{\parallel} \left(\mathbf{b} \cdot \nabla \varphi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) + O \left(\sqrt{\epsilon} \frac{v_{\text{th}i}}{L_{\parallel}} T_e \right) \tag{2.40}$$

$$\langle \dot{\vartheta} \rangle_{\mathbf{R}} = \Omega_s + O(\epsilon^{1/2} \Omega_i). \tag{2.41}$$

⁵The excessively cautious reader may wonder if the fact that the ϵ of this paper and the small parameter of Kruskal (1958) differ will give rise to a problem in this argument. Thankfully, it is obvious that if a term is small in our ordering then it will also be small in Kruskal's – this follows from the fact that all of our time variation is slow compared to the cyclotron frequency, and all spatial variation (at this point) is long compared to the gyroradius. Extending these results to include a small component of the fields that has ρ_i -scale variation can be done order by order. The magnetic moment including first-order gyrokinetic corrections for electrostatic turbulence can be found in appendix A of Parra & Catto (2008).

2.2.3. Kinetic equations

Rewriting (2.25) in the above variables, we obtain

$$\begin{aligned} \frac{\partial f_s}{\partial t} + \dot{\mathbf{R}}_s \cdot \frac{\partial f_s}{\partial \mathbf{R}_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] \\ - \frac{Z_s e}{m_s} \left\langle \left(\delta \mathbf{E} + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B} \right) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} + \langle C[\delta f_s] \rangle_{\text{turb}}. \end{aligned} \tag{2.42}$$

Applying our ordering for the ELM dynamics to this equation, we immediately see that the lowest-order equation is

$$\Omega_s \frac{\partial f_s}{\partial \vartheta} = 0. \tag{2.43}$$

Thus, both ions and electrons are immediately found to be gyrotropic at fixed \mathbf{R}_s . The size of the gyrophase-dependent corrections differs between species and so the accuracy of this statement is

$$\frac{\partial f_i}{\partial \vartheta} = O(\epsilon f_i) \quad \text{and} \quad \frac{\partial f_e}{\partial \vartheta} = O(\epsilon^2 f_e). \tag{2.44a,b}$$

Continuing to higher order in our expansion of (2.42), we find the kinetic equation for the electrons

$$\frac{\partial f_e}{\partial t} + (w_{\parallel} \mathbf{b} + \mathbf{u}_E) \cdot \frac{\partial f_e}{\partial \mathbf{R}_e} + e w_{\parallel} \left(\mathbf{b} \cdot \nabla \varphi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) \frac{\partial f_e}{\partial \varepsilon_e} = C[f_e], \tag{2.45}$$

where we note that the fluctuations from the pedestal turbulence are not strong enough to affect f_e on this time scale. This kinetic equation can, in principle, have solutions with a non-zero parallel flow. This would, naturally, be of order $n_e v_{\text{th}_e}$. Such a strong parallel flow is prohibited for two reasons. Firstly, the parallel current, estimated from Ampère’s law and using the scaling for the magnetic shear posited above, results in

$$j_{\parallel} \sim \frac{c}{4\pi} \frac{B}{L_s} \sim \frac{cm_e B^2}{e B n_e T_e} \frac{v_{\text{th}_e}}{L_s} n_e v_{\text{th}_e} \sim \epsilon n_e v_{\text{th}_e}, \tag{2.46}$$

which is inconsistent with a strong electron flow (because $j_{\parallel} \sim en_e u_{\parallel e}$). Secondly, electron and ion velocities which have a relative drift comparable to the electron thermal speed are strongly unstable to Debye-scale fluctuations (Jackson 1960). Thus, we limit ourselves to only considering solutions to (2.45) that have no parallel flow (as this equation is correct only to lowest order in $\sqrt{\epsilon}$ the vanishing of the flow to this order simply means that the electron flow is at most comparable to the ion thermal speed). The constraint equation is simply the w_{\parallel} moment of (2.45) and can be written as an equation for the parallel vector potential A_{\parallel} :

$$en_e \left(\mathbf{b} \cdot \nabla \varphi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) = \mathbf{b} \cdot \nabla p_{\parallel e} - (\mathbf{b} \cdot \nabla \ln B)(p_{\parallel e} - p_{\perp e}). \tag{2.47}$$

Note that we can neglect the time derivative of f_e because the electron momentum is negligible (due to both the small electron mass and the absence of sonic electron flows) and the integral of the collision operator can be estimated as

$$\int d^3 \mathbf{w} w_{\parallel} C[f_e] \sim v_{ei} n_e (u_{\parallel e} - u_{\parallel i}) \sim v_{ei} j_{\parallel} / e \sim \sqrt{\epsilon} n_e \frac{T_e}{m_i L_{\parallel}}, \tag{2.48}$$

which is $\sqrt{\epsilon}$ smaller than the terms in (2.47).

For the ions, we have the simpler kinetic equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_i}\right) f_i = 0, \tag{2.49}$$

so the ions are moved by the plasma flow, but do not spread thermally along field lines and neither accelerate nor collide in one Alfvén time. This is easily seen to be correct as all ion time scales (such as the ion streaming time, or ion–ion collision time) are small compared to the Alfvén time (or the electron–electron collision time).

Finally, as we know that the distribution functions are independent of gyrophase to leading order (cf. 2.44), we have that

$$\mathbf{P}_s = m_s \int d^3\mathbf{w} \langle \mathbf{w}\mathbf{w} \rangle_r f_s = p_{\perp s} \mathbf{I} + (p_{\parallel s} - p_{\perp s}) \mathbf{b}\mathbf{b} + O(\epsilon n_s T_s), \tag{2.50}$$

where the perpendicular and parallel pressures have their usual definitions

$$p_{\perp s} = \int d^3\mathbf{w} \frac{1}{2} m_s w_{\perp}^2 f_s \quad \text{and} \quad p_{\parallel s} = \int d^3\mathbf{w} m_s w_{\parallel}^2 f_s, \tag{2.51a,b}$$

respectively, and so the pressure tensors are also gyrotropic to leading order (the off-diagonal terms are computed later in appendix B). Hence we can now write the largest non-zero part of the momentum equation (2.26) as

$$\nabla_{\perp} \cdot \left(\sum_s p_{\perp s} + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi} \right) = 0, \tag{2.52}$$

which determines the time-dependent part of the total field strength.

We now have equations for the distribution functions, and the two fields that comprise \mathbf{B}_1 . All that remains is to derive an equation for φ .

2.2.4. The vorticity equation

As is common to many long-wavelength theories of a magnetised plasma, the electrostatic potential is determined from the plasma vorticity equation. This is derived in appendix B, with the final result that

$$\begin{aligned} & \nabla \cdot \left\{ \sum_s \frac{n_s m_s}{B} \left[\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right] \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right\} \\ &= \nabla \cdot \left[-\frac{j_{\parallel} \mathbf{b}}{c} \left(1 - 4\pi \frac{p_{\parallel} - p_{\perp}}{B^2} \right) - p_{\perp} \nabla \times \left(\frac{\mathbf{b}_0}{B} \right) - \frac{\nabla \times \mathbf{b}_0}{B} (p_{\parallel} - p_{\perp}) \right] + \varpi, \end{aligned} \tag{2.53}$$

where ϖ is the possible contribution from turbulence, whose explicit expression is

$$\varpi = - \sum_{s=i} \frac{Z_i e}{2\Omega_i} \left\{ \nabla \nabla : \left\langle \int d^3\mathbf{w} [(\nabla \delta \varphi) \mathbf{w}_{\perp} + \mathbf{w}_{\perp} (\nabla \delta \varphi)] \delta f_i \right\rangle_{\text{turb}} \right\}. \tag{2.54}$$

The parallel current in (2.53) is found from Ampère’s law:

$$j_{\parallel} = \frac{c}{4\pi} \mathbf{b} \cdot \nabla \times \mathbf{B} = \frac{c}{4\pi} (\mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0 - \nabla_{\perp}^2 A_{\parallel}). \tag{2.55}$$

The first term of this expression is due to the shear of the confining field. This first term can be large when compared to the second, and thus the lowest order of (2.53) is

$$\nabla \cdot (\mathbf{b}j_{\parallel}) = \frac{c}{4\pi} (\mathbf{b}_0 + \mathbf{b}_1) \cdot \nabla (\mathbf{b}_0 \cdot \nabla \times \mathbf{b}_0) = 0. \tag{2.56}$$

Taking the time-dependent piece of this equation, we have that

$$\mathbf{b}_0 \cdot \nabla (\mathbf{b}_0 \cdot \nabla \times \mathbf{b}_0) = 0. \tag{2.57}$$

If we assume that \mathbf{B}_0 has magnetic surfaces labelled by ψ_0 then this becomes a constraint on the shear of \mathbf{B}_0 :

$$(\mathbf{b}_0 \cdot \nabla \times \mathbf{b}_0) = K(\psi_0), \tag{2.58}$$

and, from the time-independent piece of (2.56), we discover that

$$\mathbf{B}_1 \cdot \nabla \psi_0 \frac{\partial K}{\partial \psi_0} = 0, \tag{2.59}$$

so either $K(\psi_0)$ is constant on the spatial scale L_{\perp} or there are exact surfaces and $K = K(\psi)$. If K is approximately constant, then we have a strong, but unvarying shear across the pedestal. If the field \mathbf{B}_0 does not have surfaces then naturally such a large shear is forbidden and the solution of these constraints is $K = 0$.

Let us discuss the physics contained in this vorticity equation. The motion of the filamentary structures, related to the inertial term on the left-hand side of (2.53) is driven by gradients of the parallel current in \mathbf{B}_0 (kink-mode drive) and gradients in the plasma pressure (interchange-mode drive). The part of j_{\parallel} from \mathbf{B}_1 represents the opposition of such motion due to field-line bending. The combination of interchange drive and field-line bending gives rise to the usual ballooning-mode physics. Thus we have all the necessary ingredients for nonlinear peeling-ballooning modes.

We also wish to note that this vorticity equation, although in a different form, is consistent with the collisional one derived in Simakov & Catto (2003). This can be seen most clearly by using (D2) of Simakov & Catto (2003) in (69) and (70) of the same paper. This gives the following vorticity equation

$$\begin{aligned} & \nabla \cdot \left\{ \sum_s \frac{n_s m_s}{B} \left[\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right] \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right\} \\ & = \nabla \cdot \left[-\frac{j_{\parallel} \mathbf{b}}{c} \left(1 - \frac{4\pi \pi_{ci}}{B^2} \right) - \left(p_i + p_e - \frac{1}{3} \pi_{ci} \right) \nabla \times \left(\frac{\mathbf{b}}{B} \right) - \frac{\nabla \times \mathbf{b}}{B} \pi_{ci} \right], \tag{2.60} \end{aligned}$$

where we have used the identity (B 14) from appendix B to rewrite the term involving the ion parallel viscosity. All notation is ours, except for π_{ci} which is defined in (14) of Simakov & Catto (2003). Using the orderings of Simakov & Catto (2003), the terms involving π_{ci} are $O(p_i \delta_i \Delta_i / L_{\parallel} B \epsilon)$ and so can be neglected compared to the terms only involving the pressure (by using the second inequality of (7) of Simakov & Catto (2003)).

Comparing this to our vorticity equation, we see that they match if, as well as taking the collisional limit of our equations (which removes the electron pressure

anisotropy from our vorticity equation), one also assumes that the initial ion pressure is isotropic. Thus we include the same Alfvénic physics as Catto–Simakov.

We also discuss the difference of our vorticity equation with those found in reduced Braginskii models (Zeiler *et al.* 1997; Ricci *et al.* 2012). Taking the vorticity equation in Ricci *et al.* (2012) as the most modern and complete model currently in use, we see that we retain the diamagnetic contributions to the vorticity, but not the ion-sound contributions, which are small in our orderings. We anticipate that this is the correct trade-off for ELM filaments as they are often observed to be both narrow and extremely rapid.

Finally, we should discuss the potential contribution from small-scale turbulence. This contribution is exactly analogous to the Reynold’s stress generated by electrostatic gyrokinetic turbulence. Indeed, we will find later that the equations this turbulence obeys are reminiscent of gyrokinetics. Because of this, absent symmetry breaking effects, this stress may turn out to be small (Parra, Barnes & Peeters 2011). However, until calculations and simulations confirm this, we retain this term as it is formally of the required size.

2.3. Summary of the ELM model

The complete kinetic model for ELM-like behaviour on the fast, Alfvénic time scale is as follows

- (i) Electron and ion distribution functions are determined from (2.45) and (2.49), respectively.
- (ii) The vector potential for \mathbf{B}_1 , A_{\parallel} , is determined from (2.47).
- (iii) The electrostatic potential φ is determined from the vorticity equation (2.53), in which the pressure tensors are given by moments of the distribution functions we have already calculated.
- (iv) The turbulent contribution to the vorticity equation ϖ is defined by (2.54), where the fluctuations are calculated from the equations given in § 3.3.

These equations should be solved in an annular domain of closed field lines, with periodic parallel boundary conditions and static Dirichlet boundary conditions for φ and A_{\parallel} on the boundaries, such that \mathbf{u}_E and \mathbf{b} are tangential to the boundaries. This is achieved by taking the boundaries sufficiently far from the ELM that no plasma seeks to erupt out of the domain. Static boundary conditions are naturally appropriate as the time variation of fields outside of the ELM region is small compared to the time scale of the ELM.

3. Building up: inter-ELM pedestal evolution

The equations presented in the preceding section describe the rapid evolution of the pedestal region during an ELM. This rapid evolution will rearrange field lines and the plasma pressure profiles but does not itself contain the physics that would relax these profiles after an ELM. Nor do they contain the physics that builds up the density and temperature profiles between ELMs. For this, we need to consider what happens when the only solutions to the ELM equations are stationary.

To achieve this, we will first, in § 3.1 build up a set of orderings for the slow behaviour of the plasma. Then we will derive equations embodying such orderings. We will show that these equations naturally require stationary solutions to our ELM equations, which were summarised above in § 2.3. This proves one major part of self-consistency. The final piece of the puzzle will arrive in § 3.3 where we show that

small-scale fluctuations, consistent with slab-like electron temperature gradient-driven (ETG) or microtearing (MTM) turbulence, drive the inter-ELM transport through our pedestal. We will also confirm the result asserted above, that this turbulence is strong enough to provide the required transport but only affects ELM dynamics in a limited way.

3.1. Inter-ELM orderings

We must now re-evaluate our orderings to accommodate the slow inter-ELM behaviour. We retain the orderings on L_{\perp}/L_{\parallel} , ρ_i/L_{\perp} , β , m_e/m_i , and collisionalities from § 2.1. However, we now wish to accommodate frequencies on the ion-sound time scale, $\omega \sim v_{th_i}/L_{\parallel}$.

In order to do this, we need to slow down the nonlinear time scales due to the $\mathbf{E} \times \mathbf{B}$ motion ($\omega_{NL} \sim \mathbf{u}_E \cdot \nabla$ and due to diamagnetic effects $\omega_{NL} \sim \mathbf{b} \times \nabla p \cdot \nabla$). We manage this by introducing a new length scale $L_{\wedge} \sim \sqrt{L_{\perp}L_{\parallel}} \sim \sqrt{\epsilon}L_{\parallel}$. We assume that in the plane perpendicular to the magnetic field we can identify a fast direction, where the typical length scales are L_{\perp} , and a slow direction, where the typical length scales are L_{\wedge} . In a pedestal this is naturally given by the radial and the in-surface perpendicular directions respectively (if one views the poloidal coordinate as the distance along a field line, then this slow direction is the toroidal direction).

Under this assumption, we see that all the nonlinearities can be written in the form

$$\mathbf{b} \times \nabla g \cdot \nabla f \sim \frac{gf}{L_{\perp}L_{\wedge}}, \tag{3.1}$$

where we have used the fact that the derivatives cannot both be in the fast direction.

With this estimate in hand, we see that

$$\mathbf{u}_E \cdot \nabla \sim \frac{cT_e}{eB} \frac{e\varphi}{T_e} \frac{1}{L_{\perp}L_{\wedge}} \sim \sqrt{\epsilon} \frac{cT_e}{eB} \frac{1}{L_{\perp}^2} \sim \frac{v_{th_i}}{L_{\parallel}}, \tag{3.2}$$

as required.

No other ordering changes are required. We expect this set of equations to model the equilibrium and any possible coherent oscillation of the edge pedestal. Such coherent oscillations, with frequencies comparable to the local acoustic frequency, are thought to be important in novel modes of tokamak operation (Snipes *et al.* 2001; Burrell *et al.* 2002).

We now revisit and justify the turbulence orderings introduced earlier. To distinguish scales relating to the fluctuations from the scales of the pedestal itself, we will use k_{\perp} to denote typical perpendicular wavenumbers, and k_{\parallel} typical parallel wavenumbers. The only length scales smaller than L_{\perp} available to us are the gyroradii, and so we assume that the turbulence is predominantly at scales comparable to the ion gyroradius

$$k_{\perp}\rho_i \sim 1, \tag{3.3}$$

and because of our ordering for β and the mass ratio, this means that turbulent scales are also comparable to the electron skin depth $d_e = c/\omega_{pe}$:

$$k_{\perp}d_e \sim 1. \tag{3.4}$$

Making a mixing length estimate for the amplitude of this turbulence we see that

$$\frac{\delta f_s}{f_s} \sim \frac{e\delta\varphi}{T_e} \sim \frac{1}{k_\perp L_\perp} \sim \sqrt{\epsilon}. \tag{3.5}$$

This is also exactly the ordering required to balance the nonlinear turnover rate with the ω_* -drive:

$$\mathbf{u}_E \cdot \nabla \sim \frac{cT_e}{eB} \frac{e\delta\varphi}{T_e} k_\perp^2 \sim \sqrt{\epsilon} k_\perp v_{th_i}, \tag{3.6}$$

and

$$\omega_* \sim \frac{cT_e}{eB} \frac{k_\perp}{L_\perp} \sim \frac{\rho_i}{L_\perp} k_\perp v_{th_i} \sim \mathbf{u}_E \cdot \nabla. \tag{3.7}$$

Both of these frequencies are comparable to v_{th_i}/L_\perp . If this turbulence is mainly mediated by electrons, as in ETG or MTM turbulence, then the frequency sets a typical parallel length scale by

$$\omega \sim k_\parallel v_{th_e}. \tag{3.8}$$

This means that we have $k_\parallel L_\parallel \sim 1/\sqrt{\epsilon} \gg 1$. This is short parallel-wavelength, slab-like turbulence. We will also assume that turbulence is as electromagnetic as our low- β assumption allows:

$$\frac{\delta \mathbf{B}_\perp}{B} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon. \tag{3.9}$$

We can also make a random-walk estimate of the transport from this turbulence

$$\tau_E^{-1} \sim \frac{D}{L_\perp^2} \sim \frac{\omega}{(k_\perp L_\perp)^2} \sim \frac{v_{th_i}}{L_\parallel}, \tag{3.10}$$

where τ_E is the pedestal energy confinement time and D is a typical diffusion coefficient from our turbulence. We see that this is precisely in accord with our ordering of the inter-ELM evolution time scale.

Note that our exploration of these equations does not extend to the time scale on which the confining field evolves, \mathbf{B}_0 is fixed throughout. This is currently no great limitation, as a pedestal that is quiescent for so long that the diffusion of current becomes important is presently beyond experimental and theoretical reach. However, this must eventually be addressed in future work.

3.2. *The slow dynamics of a pedestal*

We use the same variables as in the prequel, and taking care to use the new ordering to estimate the size of their time derivatives we obtain

$$\langle \dot{\mathbf{R}}_s \rangle_R = w_\parallel \mathbf{b}(\mathbf{R}_s) + \frac{c}{B} \mathbf{b} \times \nabla \varphi + O(\epsilon^{3/2} v_{th_i}) \tag{3.11}$$

$$\langle \dot{\epsilon}_s \rangle_R = -Z_s e w_\parallel \left(\mathbf{b} \cdot \nabla \varphi + \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \right) + O\left(\epsilon \frac{v_{th_i}}{L_\parallel} T_e \right) \tag{3.12}$$

$$\langle \dot{\vartheta} \rangle_{\mathbf{R}} = \Omega_s + O(\epsilon^{1/2} \Omega_i), \tag{3.13}$$

as before.

In appendix E we prove that a consistent solution to the lowest-order electron kinetic equation, in a closed-field-line region, is a Maxwellian. If this is the chosen solution, then the magnetic field is frozen into a fluid flow and the magnetic topology is fixed on the time scales of interest. Then we can assume that the field has topologically toroidal flux surfaces (denoted by ψ) for all time, and we will average over them when needed.

The electrons are an isothermal (i.e. T_e is constant on a flux surface) fluid, with a continuity equation for the density (cf. (E 7)):

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e + \nabla \cdot (n_e u_{\parallel e} \mathbf{b}) + \nabla \cdot \left\langle \int d^3 \mathbf{v} \mathbf{V}_\chi h_e \right\rangle_{\text{turb}} = S_e^{(n)}, \tag{3.14}$$

where the fluctuating velocity is given by

$$\mathbf{V}_\chi = \frac{c}{B} \mathbf{b} \times \nabla \left\langle \varphi - \frac{w_{\parallel}}{c} \delta A_{\parallel} \right\rangle_{\mathbf{R}}, \tag{3.15}$$

and an evolution equation for the temperature (see (E 10))

$$\begin{aligned} \frac{3}{2} \left\langle \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e T_e \right\rangle_{\psi} + \left\langle \nabla \cdot \left\langle \int d^3 \mathbf{v} \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) \langle \mathbf{V}_\chi \rangle_{\mathbf{R}} \delta f_e \right\rangle_{\text{turb}} \right\rangle_{\psi} \\ = \langle e n_e u_{\parallel e} \mathbf{b} \cdot \nabla \varphi \rangle_{\psi} + \langle S_e^{(E)} \rangle_{\psi}, \end{aligned} \tag{3.16}$$

where $T_e = T_e(\psi)$. The terms on the right-hand side are easily interpreted as parallel compressional heating, turbulent ohmic heating and a possible energy source, respectively. The fluctuating distribution function δf_e is defined by (2.20), and given by the solution of (3.25) as explained in the next section.

The ions obey a drift-kinetic equation:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + w_{\parallel} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right) f_i - Z_i e w_{\parallel} \mathbf{b} \cdot \nabla \varphi \frac{\partial f_i}{\partial \varepsilon_i} + \nabla \cdot \left\langle \frac{c}{B} \mathbf{b} \times \nabla (\delta \varphi)_{\mathbf{R}} h_i \right\rangle_{\text{turb}} \\ = C[f_i] + S_i, \end{aligned} \tag{3.17}$$

in which the fluctuating ion distribution function h_i is defined by (3.24) and given by the solution of (3.23) as described in § 3.3.

The fluctuating magnetic field is given by the lowest-order vorticity equation

$$\begin{aligned} (\mathbf{B}_0 + \mathbf{b}_0 \times \nabla A_{\parallel}) \cdot \nabla \left(\frac{1}{B} \nabla_{\perp}^2 A_{\parallel} - \mathbf{b}_0 \cdot \nabla \times \mathbf{b}_0 \right) \\ = 4\pi \nabla \times \left(\frac{\mathbf{b}_0}{B} \right) \cdot \nabla p_{\perp} + \frac{4\pi}{B} \nabla \times \mathbf{b}_0 \cdot \nabla (p_{\parallel} - p_{\perp}). \end{aligned} \tag{3.18}$$

Equivalently, (3.18) is simply the constraint that the ELM time scale vorticity equation (2.53) has zero-vorticity solutions. The fluctuating A_{\parallel} then determines the parallel electron flow via Ampère’s law

$$u_{\parallel e} = -\frac{c}{4\pi n_e} \mathbf{b} \cdot \nabla \times \mathbf{B} + \frac{1}{n_e} \sum_{s=i} Z_s \int d^3 \mathbf{v} w_{\parallel} f_s, \tag{3.19}$$

with the sum on the right-hand side being over all ion species. We can then use (E5), which is

$$\mathbf{b} \cdot \nabla \left(\frac{T_e}{e} \ln n_e - \varphi \right) = 0, \tag{3.20}$$

to find the electrostatic potential up to a flux function $\bar{\varphi}(\psi)$.

Finally, the flux-surface average of the vorticity equation gives us the equation that determines $\bar{\varphi}$:

$$\begin{aligned} & \left\langle \nabla \cdot \left\{ \sum_s \frac{n_s m_s}{B} \left[\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right] \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right\} \right\rangle_{\psi} \\ &= \left\langle \nabla \cdot \left[\sum_s \mathbf{b} \cdot \nabla \left(\frac{u_{\parallel s} \nabla p_{\perp s}}{B \Omega_s} \right) \right] \right\rangle_{\psi} + \left\langle \sum_s \frac{1}{2B \Omega_s} [(\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla] q_{\parallel s} \right\rangle_{\psi} \\ &+ \langle \bar{\omega} \rangle_{\psi}, \end{aligned} \tag{3.21}$$

where

$$q_{\parallel s} = \int d^3 \mathbf{w} \frac{m_s w_{\perp}^2}{2} w_{\parallel} f_s. \tag{3.22}$$

with the definition of turbulent vorticity $\bar{\omega}$ from (2.54).

Let us pause and comment on these equations for a moment. These equations live a double life. Firstly, they are transport equations, with fluxes given by averages over the pedestal turbulence. Secondly, they support sound waves and their relatives like the geodesic acoustic mode (GAM). As these are on the same time scale, the interplay between poloidally inhomogeneous turbulence and the sound waves that such inhomogeneity can excite may result in interesting physics. Equation (3.18) gives the response of the magnetic field to this transport, and is in effect the constraint that the field-line-bending forces always balance pressure forces so no Alfvén waves can be excited.

In this system, transport occurs on the same time scale that poloidal flows are damped by parallel viscosity. Thus, this system can potentially support Stringer spin-up (Stringer 1969; Rosenbluth & Taylor 1969). This mechanism has been thought to play a role in the L–H transition (Hassam *et al.* 1991). Indeed, investigating the behaviour of the large poloidal flows that our equations can support will be a key point of future work.

3.3. Pedestal turbulence

Finally, we come to the equations governing the turbulence in the pedestal. These are derived in §E.2, by systematically expanding the fluctuating part of (2.22) in accordance with the orderings of §2.1.1. This derivation is standard, and closely resembles that of gyrokinetics. Thus, we leave the derivations to §E.2, and only present the results here.

The ions obey a simple kinetic equation:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_i} \right) h_i + \frac{c}{B} \{ \langle \delta \varphi \rangle_R, h_i \} \\ &= - \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_i} \right) Z_i e \langle \delta \varphi \rangle_R \frac{\partial f_i}{\partial \varepsilon_i} - \frac{c}{B} \mathbf{b} \times \nabla \langle \delta \varphi \rangle_R \cdot \nabla f_i, \end{aligned} \tag{3.23}$$

where \mathbf{u}_E is the $\mathbf{E} \times \mathbf{B}$ velocity defined with the large-scale potential φ , obtained from the equations in the previous section. Similarly, f_i is the large-scale distribution function. In this context these fields are static, and they can be handled exactly the same as the Maxwellian backgrounds in local gyrokinetic flux-tube codes. Finally, h_i is the non-adiabatic part of the ion distribution function

$$\delta f_i = h_i + Z_i e \delta \varphi \frac{\partial f_i}{\partial \varepsilon_i} + \frac{Z_i e}{B} (\delta \varphi - \langle \delta \varphi \rangle_R) \frac{\partial f_i}{\partial \mu_i}. \quad (3.24)$$

This equation is easily seen to be the same as the slab-geometry gyrokinetic equation around a non-Maxwellian background, if one were to neglect ion-sound effects (cf. (Frieman & Chen 1982; Abel *et al.* 2013)).

The electrons also obey a gyrokinetic equation

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_e} \right) h_e + w_{\parallel} \mathbf{b} \cdot \nabla h_e + \frac{c}{B} \{ \langle \delta \chi \rangle_R, h_e \} \\ & = - \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_e} \right) \frac{e \langle \delta \chi \rangle_R}{T_e} f_e - C [h_e] - \frac{c}{B} \mathbf{b} \times \nabla \langle \delta \chi \rangle_R \cdot \nabla f_e, \end{aligned} \quad (3.25)$$

where

$$\delta f_e = h_e + \frac{e \delta \varphi}{T_e} f_e, \quad (3.26)$$

as f_e is Maxwellian, and

$$\delta \chi = \delta \varphi - \frac{w_{\parallel}}{c} \delta A_{\parallel}. \quad (3.27)$$

These two equations are coupled with the following two field equations. Firstly, the quasineutrality condition,

$$-\frac{e \delta \varphi}{T_e} n_e + \sum_{s=i} Z_s e \int d^3 \mathbf{v} \left[\delta \varphi \frac{\partial f_s}{\partial \varepsilon_s} + (\delta \varphi - \langle \delta \varphi \rangle_R) \frac{\partial f_s}{\partial \mu_s} \right] = \sum_s Z_s e \int d^3 \mathbf{v} \langle h_s \rangle_r, \quad (3.28)$$

in which the sum on the left-hand side is taken over all ion species. Secondly, the parallel (to \mathbf{b} but not $\mathbf{b} + \delta \mathbf{B}$) component of Ampère's law

$$\frac{c}{4\pi e} \nabla_{\perp}^2 \delta A_{\parallel} = \int d^3 \mathbf{v} w_{\parallel} h_e, \quad (3.29)$$

where the ions do not participate due to the high frequency of the fluctuations. Perpendicular pressure balance obtains, and allows one to calculate turbulent fluctuations in the field strength, $\delta B_{\parallel} = \mathbf{b} \cdot \delta \mathbf{B}$, but these do not react back upon the turbulence, nor do they contribute to the ensuing transport. This is easy to see as the drift arising from δB_{\parallel} is $O(\epsilon v_{th_i})$ whereas the $\mathbf{E} \times \mathbf{B}$ drift due to $\delta \varphi$ is $O(\epsilon^{1/2} v_{th_i})$.

We should comment upon the geometry these equations should be solved in. The turbulence they describe occurs on spatial scales that are short compared to those of the pedestal profile. Hence, as in classic gyrokinetic simulations, flux-tubes with periodic perpendicular boundary conditions (Beer, Cowley & Hammett 1995) are the correct setting for solving these equations. However, as the turbulence is

such that the parallel correlation lengths are short compared even to the connection length, shear-periodic parallel boundary conditions are also required. Similarly, all geometric quantities that vary on the L_{\parallel} scale along the field line (poloidally) should be evaluated at one point and not vary through the simulation domain. Importantly, this means that trapped-particle effects are irrelevant for this turbulence, as only a small fraction of particles have a bounce point in any given $1/k_{\parallel}$ length of plasma. The only geometry of the field that is retained is the magnetic shear. This is easily seen by expanding the operator $\mathbf{b} \cdot \nabla$ around some radial point r_0

$$\mathbf{b} \cdot \nabla = \mathbf{b}(r=r_0) \cdot \nabla + (r-r_0) \frac{\partial \mathbf{b}}{\partial r} \cdot \nabla + \dots \approx \mathbf{b} \cdot \nabla + (r-r_0) \frac{\partial \mathbf{b}}{\partial r} \cdot \nabla_{\perp}, \quad (3.30)$$

where, as $r-r_0 \sim \nabla_{\perp}^{-1}$, the second term will be comparable to the first if $k_{\parallel} L_s \sim 1$. Referring back to our maximal estimates for the amount of magnetic shear in the pedestal, we see that this ordering is marginally satisfied. Thus, magnetic shear must be retained. These simplifications should render simulation of these equations simpler than current methods for simulating pedestal-relevant turbulence.

This set of equations is also interesting for several reasons. It is a strict superset of the equations of Zocco & Schekochihin (2011) which comprise a popular reduced model for studying reconnection. The equations also contain equations used to study microtearing physics in Zocco *et al.* (2015). In the limit of weak magnetic shear, and weak background ion gradients, our equations reduce to a model that has been shown to support strong electromagnetic ETG turbulence, due to an inverse cascade process (L. Milanese *et al.* 2017, Private communication). This strongly suggests our equations support the turbulent processes believed to be most important in the pedestal (Saarelma *et al.* 2013; Hatch *et al.* 2016; Hillesheim *et al.* 2016; Hatch *et al.* 2017).

3.4. Summary

Our inter-ELM equations consist of two time scales. Firstly the slow time scale equations, which comprise the following:

- (i) (3.14) and (3.16) to determine the electron density n_e and temperature T_e respectively,
- (ii) (3.17) to determine f_i ,
- (iii) (3.18) which determines A_{\parallel} ,
- (iv) (3.20), which determines φ up to a flux function
- (v) and (3.21) that determines that flux function.

All of these equations are meant to be solved on a global annular domain containing the entire pedestal. The detailed boundary conditions will be discussed in §4.

In some of these equations, there are terms that arise from averages over the turbulence. The turbulence is governed by the following equations:

- (i) the kinetic equations, (3.23) for h_i and (3.25) for h_e ,
- (ii) the quasineutrality condition (3.28) for $\delta\varphi$
- (iii) and parallel Ampère's law (3.29) for δA_{\parallel} .

There are also the relations (3.24), (3.26) and (3.27) that relate these fields to the fields that arise in the flux terms in the inter-ELM equations. As usual for coupling turbulent fluxes to a transport code, these equations should be run to a steady state and appropriately averaged to give the fluxes.

4. Betwixt and between: boundary conditions, matching and existing models

In this section, we tackle two remaining points regarding the systems of equations derived above. Firstly, how they interact, and how they couple both to core and scrape-off layer models. Then, secondly, how they compare to existing strategies for modelling this region of the plasma.

4.1. *The integrated approach: coupling and boundary conditions*

Having derived a full multiscale approach to the pedestal, the question naturally arises of how these equations should be coupled to each other, and to other parts of the plasma. We have a framework for physics on three disparate time scales. The fastest is the time scale of the individual turbulent fluctuations. Then comes the time scale on which ELM filaments erupt. Finally, the time scale on which transport occurs, which is the same as the acoustic time scale.

The outline of the envisioned implementation, reminiscent of the operation of multiscale core transport codes, is as follows. Given an initial pedestal profile, one tests for stability in the ELM equations. Due to the meta-stable nature of erupting ELM filaments (Cowley *et al.* 2015; Ham *et al.* 2016), a purely linear solver may be insufficient to evaluate the stability of the pedestal to ELMs.⁶ A more robust approach is to initialise the ELM equations with the initial profile and, in addition, some specified amount of noise. One then evolves the fully nonlinear equations for a short time. Either nothing happens, if the pedestal is suitably stable, or the filaments erupt and a new equilibrium is rapidly reached. Once a situation is reached where no more evolution occurs on the fast time scale, the state is such that it is a valid initial condition for the inter-ELM equations.

The inter-ELM equations are then used to evolve the profiles on the long time scale, with fluxes given by the pedestal turbulence. We envision these equations being solved in a (topologically) annular region, with inner and outer surfaces corresponding to flux surfaces. The boundary conditions at the inner radial location is given by matching to multiscale gyrokinetics. The core is solved as in Barnes *et al.* (2010), with the boundary values for the transport equations being taken from our inter-ELM pedestal profile. That profile is itself solved for with a fixed-flux boundary condition obtained from the outermost radial grid point of the gyrokinetic transport calculation. In general, this indicates that the core transport time step and the Inter-ELM time step must be taken together. However, for a steady-state situation one could envision solving the inter-ELM equations with a fixed flux given by the total heating and fuelling sources before then solving the core with a fixed value for the pedestal. The outer radial boundary condition is given, in principle, by matching to a scrape-off-layer model just inside the separatrix. We discuss the difficulties and possibilities of such matching in §4.3. Absent an exact model to match to, we take the same *ad hoc* approach as Barnes *et al.* (2010) and suggest fixing values for top of the scrape-off layer at the separatrix. The pedestal turbulence, which gives rise to the fluxes, is solved locally in slab-like flux tubes, as described in detail at the end of §3.3.

Periodically during this evolution, stability in the ELM equations is checked for. If at any point the ELM equations are unstable, then one returns to the start of this procedure and evolves on the short time scale until a new equilibrium is reached. In principle this is checked every time step of the inter-ELM equations (just as MHD

⁶In addition, on account of the inclusion of electron Landau resonances and diamagnetic effects, there will not be an energy principle for our ELM equations. Thus, any linear stability calculation would have to perform a full initial-value solution for a finite time anyway.

stability should be checked every time step of a core transport code), but hopefully heuristics can be developed to ameliorate this cost.⁷ It is easy to see how this cyclical procedure would produce an ELM cycle.

Ideally, this framework would resolve the problematic issue of boundary conditions in core multiscale codes – the solution to the core transport equations is sensitive to the fixed boundary condition at the top of the pedestal. One might hope that sensitivity studies of the equations in this paper would demonstrate an insensitivity of the pedestal top temperature to conditions at the separatrix. Of course, this is mere speculation, and one may wonder, pessimistically, if sensitivity to boundary values continues all the way to a material surface.

Now, the fact that multiscale gyrokinetics has a subsidiary limit that provides a large enough heat flux to appropriately source the inter-ELM equations is not obvious. Indeed, it is not obvious that our inner boundary is even consistent with multiscale gyrokinetics. To this end we will show explicitly in the next section that these two theories can indeed be matched on to each other at the pedestal top.

4.2. Matching at the pedestal top

In this section we show that our pedestal model can be smoothly matched on to multiscale gyrokinetics. More precisely, we mean that there is a subsidiary expansion of each of these systems that results in the same set of equations. This implies that there is potentially a region in the plasma that would be well described by both multiscale gyrokinetics and the model contained in this paper. Suggestively, we will designate this location as the top of the pedestal.

In this section, we will only detail the orderings of the subsidiary expansions. Detailed calculations showing that these orderings do indeed result in the same set of equations are performed in appendix F. We begin with the expansion of multiscale gyrokinetics. This should be seen as an expansion in nearness to the pedestal. As the pedestal top is approached from the core, steepening density gradients and decreasing η_i conspire to stabilise the ion-temperature-gradient (ITG) mode. Thus, we are looking at an expansion that should result in strongly driven microtearing and ETG turbulence.

There are two main expansions we will need to do. Firstly, we will need an expansion in small Mach number to match the rapidly rotating core onto the subsonic pedestal. Secondly, in order to begin to match the pedestal, we will need a subsidiary expansion in both the mass ratio and in β . This will mainly affect the equations for the fluctuations, and closely parallels the derivation in Zocco & Schekochihin (2011).

We do this as two subsidiary expansions, so the first expansion is that of small Mach number M . We will use a naïve low-Mach-number expansion, where $M \ll 1$ but the scale length of the flow is the same as the perpendicular scale length of all other mean quantities. Such an expansion simply results in the zero-flow limit of gyrokinetics. The resulting equations are simply the equations of section 11 of Abel *et al.* (2013) with any remaining terms involving the equilibrium flow neglected. This is the same as the system of equations presented in Sugama *et al.* (1996), except in a notation that is closer to the one used in this paper. It is this set of equations we will now expand in β and m_e/m_i .

⁷As an example, one simple approach would be to require a certain change in global pedestal parameters (height and width) before initialising a stability calculation. Speculatively executing both the next slow time step and the stability calculation in parallel, throwing away whichever turns out to be irrelevant, might also save wall-clock time.

To formalise our second subsidiary expansion, we introduce a small parameter ξ , defined by

$$\xi \sim \sqrt{\beta} \sim \sqrt{\frac{m_e}{m_i}} \ll 1. \tag{4.1}$$

We then order

$$\frac{\delta f_s}{f_s} \sim \frac{e\delta\varphi}{T_e} \sim \frac{\epsilon_{\text{GK}}}{\xi}, \tag{4.2}$$

where $\epsilon_{\text{GK}} = k_{\parallel}/k_{\perp}$ is the small parameter of gyrokinetics. Other than this rebranding, we will use the notation of Abel *et al.* (2013) when discussing multiscale gyrokinetics. We retain

$$k_{\perp}\rho_i \sim 1, \tag{4.3}$$

from gyrokinetics and due to our ordering for β_e we keep the electron skin depth scale,

$$k_{\perp}d_e \sim 1, \tag{4.4}$$

as well. With this ordering for k_{\perp} , the frequencies become

$$\omega \sim \frac{cT_e}{eB} \mathbf{b} \times \nabla \left(\frac{e\delta\varphi}{T_e} \right) \cdot \nabla \delta f_s \sim \frac{\epsilon_{\text{GK}}}{\xi} \Omega_s \sim k_{\parallel} v_{\text{the}}. \tag{4.5}$$

As in (Zocco & Schekochihin 2011) this expansion in β reduces the relative importance of some electromagnetic effects. Thus, we have

$$\frac{\delta B_{\perp}}{B} \sim \epsilon \quad \text{and} \quad \frac{\delta B_{\parallel}}{B} \sim \xi \epsilon. \tag{4.6a,b}$$

To drive this increase in turbulent amplitude, we assume that the gradients in F_{0s} are increased

$$\frac{\nabla F_{0s}}{f_s} \sim \frac{1}{a\xi^2}, \tag{4.7}$$

with a the long equilibrium length scale, assumed to be comparable to the parallel connection length qR . However, the parallel length scale of the fluctuations is now

$$k_{\parallel}a \sim \frac{\omega a}{v_{\text{the}}} \sim \frac{1}{\xi}. \tag{4.8}$$

Note that this scaling is consistent with turbulence that has its outer scale fixed (independent of drive) around ρ_i such as might follow from an inverse cascade from electron scales that is broken by ion gyroscale physics (L. Milanese *et al.* 2017, Private communication). As the drive is increased, instead of the outer scale increasing, the frequency increases and the parallel length scale decreases due to causality. Thus, this ordering is not consistent with a naïve scaling theory of critically

balanced ETG turbulence which might be expected to follow the orderings of Barnes, Parra & Schekochihin (2011).

With our ordering, the transport time scale of such turbulence is now

$$\tau_E^{-1} \sim \frac{\omega}{k_{\perp}^2} \frac{\nabla F_{0s}}{F_{0s}} \sim \frac{\omega}{\xi^2} \epsilon_{\text{GK}}^2 \sim \frac{v_{\text{th}i}}{qR} \frac{\epsilon_{\text{GK}}^2}{\xi^4} \sim \frac{\epsilon_{\text{GK}}^3 \Omega_i}{\xi^4}. \tag{4.9}$$

As we are allowing the various scales of the equilibrium to be ordered differently (unlike normal gyrokinetics), we must consider how to handle the magnetic shear length scale L_s . As our pedestal allows for strong shear, we take the ordering

$$L_s \sim \xi a, \tag{4.10}$$

such that $k_{\parallel} L_s \sim 1$, which is consistent with electromagnetic microtearing turbulence.

Revisiting the collisionality estimate from above, $v_{ee} \sim v_{\text{th}e}/qR$, we see that our turbulence should be collisionless if our pedestal is trans-collisional. Hence, we will assume that the turbulence self-generates phase-space structure sufficient that the collision operator is retained in the resultant gyrokinetic equation.⁸ This assumption of steep gradients in velocity space is needed in order that dissipation can continue to balance the turbulent fluxes in the free-energy conservation equation of Abel *et al.* (2013).

Applying this ordering to the low-Mach-number equations of gyrokinetics results in the set of equations detailed in appendix F. We thus turn to the subsidiary ordering of our inter-ELM equations. We do not need to revisit our ELM equations as gyrokinetics assumes that the plasma is MHD stable, and so to match on to it we make the same assumption.

The subsidiary expansion we engage in should be thought of as approaching the top of the pedestal from inside the pedestal region. Perpendicular gradients are shallower than they are in the pedestal. ITG-driven turbulence is still suppressed, but the microtearing and ETG turbulence is not as strong as it is in the pedestal proper. To quantify this, we introduce a second subsidiary parameter ζ to quantify our expansion of the inter-ELM equations. It is defined by

$$\frac{\delta f_s}{f} \sim \frac{e\delta\varphi}{T_e} \sim \zeta \sqrt{\epsilon}. \tag{4.11}$$

This lower-amplitude turbulence naturally leads to a longer k_{\parallel} for the fluctuations, obtained from a critical-balance estimate

$$k_{\parallel} v_{\text{th}e} \sim \frac{c}{B} \mathbf{b} \times \nabla \delta\varphi \cdot \nabla \sim \zeta \sqrt{\epsilon} \Omega_i, \tag{4.12}$$

giving

$$k_{\parallel} \sim \frac{\zeta}{\sqrt{L_{\perp} L_{\parallel}}}. \tag{4.13}$$

⁸If semi-collisional or truly collisional instabilities are to be studied, they must be investigated with a collisional pedestal as the background. This is a trivial large-collisionality limit of the equations presented above.

Consistent with this reduction in turbulent amplitude is a reduction in the driving gradients (a linear scaling of $\delta f_s/f_s$ with driving gradient is in accord with the results of Barnes *et al.* (2011))

$$\nabla_{\perp} f_s \sim \zeta \frac{f_s}{L_{\perp}}. \quad (4.14)$$

We will order the time derivatives of the mean quantities with the modified turbulent transport time,

$$\frac{\partial f_s}{\partial t} \sim \zeta^3 \frac{v_{\text{th}i}}{L_{\parallel}} f_s, \quad (4.15)$$

but we shall keep the collision time comparable to the parallel sound time.

The mean electrostatic potential is ordered so as to allow us to match up with a low-Mach-number expansion of gyrokinetics. We fix the amplitude of the potential at T_e/e as we head towards the interior of the tokamak, but keep the scale length that of the other mean quantities:

$$\nabla_{\perp} \varphi \sim \zeta \frac{\varphi}{L_{\perp}} \quad \text{and} \quad \frac{e\varphi}{T_e} \sim 1. \quad (4.16a,b)$$

This means that the shearing rate drops with decreasing ζ and so the mean flow is eliminated from our equations.

This set of orderings reproduces exactly the same set of equations as the expansion of gyrokinetics detailed above. Hence, we can confidently say that there is a physically plausible regime which is described both by gyrokinetics and by our pedestal model, and acts as a bridge between them. Operationally, we expect this region to be located at the top of a pedestal, but one could also envision using our model to embed a transport barrier region inside core gyrokinetics.

Having shown that we match on to the best tested model of core turbulence, in the next section we will discuss how our model matches on to scrape-off-layer models which are much less developed.

4.3. Matching on to the scrape-off layer

To leave the plasma and reach such a surface, one can imagine coupling the above pedestal to a collisional scrape-off-layer (SOL) model such as Catto–Simakov or GBS, where the boundary between closed and open field lines is completely contained in the SOL model. If the plasma is still hot enough to be weakly collisional at the separatrix then such a coupling may not be consistent, and further work is required. Exciting models for arbitrary collisionality scrape-off layers have recently been proposed (Jorge, Ricci & Loureiro 2017), and matching to such models should be addressed in the future.

As yet, such an explicit matching (either collisional or collisionless) has not been performed. However, even without the explicit forms of the equations that are to be matched, we know that they must obey certain constraints. We know that they must conserve energy and particles, and as such we can envisage using the fluxes leaving the pedestal (as simulated by our equations) as boundary conditions for a scrape-off-layer model. One could also, perhaps, use the general spectrum of fluctuations in our model to initialise some noise in a scrape-off-layer code. This

procedure would, of course, be fundamentally inconsistent. Consistency requires not only a smooth matching of the fluxes at the boundary, but also a matching of the mechanisms that carry that flux. Rigorous subsidiary expansions would provide a finite-width region in which both models were valid, and we could be confident in matching across this region. We leave the detailed derivation of such expansions to future work.

4.4. Existing models

Individual parts of our model are directly comparable to some parts of existing models, and we have highlighted these both in the discussion of the ELM vorticity equation in § 2.2 and in § 3.3 where we discuss pedestal turbulence. Thus, in this section, we focus on comparing our model as a whole to other approaches for simulation and modelling of the pedestal region of a tokamak.

The most successful empirical model of the pedestal is the EPED model (Snyder *et al.* 2002, 2011). This combines a collisional, long-wavelength model for peeling-ballooning modes (Wilson *et al.* 2002), with an assumption that the inter-ELM turbulence is given by kinetic-ballooning modes (KBM), modelled heuristically by the infinite- n ballooning stability limit. Our model improves upon this by including weakly collisional kinetic effects including electron Landau damping, trapped electron physics and self-consistent diamagnetic stabilisation into the time-dependent ELM dynamics. We also have a self-consistent model of the underlying pedestal turbulence. Indeed, the assumption that the turbulence is strictly KBM in nature has recently come under scrutiny (Hatch *et al.* 2017) as detailed gyrokinetic analysis lends more support to the turbulence being MTM or ETG in nature. Two-fluid simulations of ELMs with BOUT++ (Xu *et al.* 2011), which have been included in EPED, can capture some of the non-ideal effects of our model, but not the electron kinetic effects. In addition, our model makes precise the potential sources of the effects of turbulence upon the ELM dynamics.

More generally, two-fluid or reduced Braginskii simulations will not capture the weakly collisional physics that may be important in a hot pedestal. The most complete fluid equations are those of GBS (Ricci *et al.* 2012) or Catto–Simakov (Simakov & Catto 2003). The former of these has almost exclusively been used to study open-field-line regions, but the latter was formulated with the explicit intent of handling closed-field-line regions in pedestals. The fluid nature of the models means that they cannot capture kinetic effects that may affect ELM stability. In addition, these models do not consistently separate the sound and Alfvénic time scales, despite the low- β nature of their models. Their ordering schemes do not touch on the possibility of small-scale fluctuations existing within the pedestal, but could be extended to include the turbulence model we have presented above. Indeed, the Catto–Simakov model, with turbulence and separating the two equilibrium time scales is the collisional limit of the equations we have presented.

The final approach that is considered for pedestal modelling is global full- f gyrokinetics (Chang *et al.* 2006; Churchill *et al.* 2017). This approach promises to include all the physics, at all scales, consistently. The potential pitfalls of this approach are twofold. Firstly, the numerical requirements are extraordinary, requiring all parts of the simulation to resolve the fastest and smallest fluctuations. This may render predictive parameter scans prohibitively expensive. Secondly, by incorporating all temporal and spatial scales on the same footing, it becomes incredibly hard to extract the physics of the interaction of these scales. Our model should be much

less numerically challenging, especially as the turbulence can be parallelised over the background parameters in an approach reminiscent of current core transport codes (Barnes *et al.* 2010). Again, because we have decoupled the filamentary dynamics from the slow sound-time dynamics and the fluctuations, we can examine each piece of the puzzle separately.

5. Summary and conclusions

In this paper we have derived a self-consistent first-principles model for a pedestal.

The first part of the model is a trans-collisional ELM model, given by (2.45), (2.47), (2.49), (2.53) and closed with (2.31) and (2.55). This model contains the eruption of filamentary structures driven both by magnetic buoyancy forces and launched by current-driven instabilities. It is fully nonlinear and capable of dynamically evolving a filamentary structure from one equilibrium to another.

These equilibria provide the backdrop for the next part of the model. The evolution of the inter-ELM profiles. This is given by (3.14), (3.16), (3.17), (3.18), (3.20) and (3.21). This component of the framework contains the physics of sound waves, including GAMs which are often seen in the edge, poloidal spin-up, and even systematically includes turbulent transport.

These equations are, in turn, closed by a set of equations governing the turbulence that generates that transport. These are, explicitly, the set (3.23)–(3.29). This set contains the physics of strongly driven microtearing and ETG turbulence.

Future work must include exploring the equilibria of this system, with special focus on the stability of zero-flow solutions to spontaneous spin-up. In addition, the scaling of pedestal turbulence with drive parameters is a high priority – if pedestal turbulence exhibits the $(R/L_T)^3$ scaling of core turbulence, then this may permit the formation of boundary layers that govern the size of ELM filaments.

Acknowledgements

Support for I.G.A. was provided by the Princeton Center for Theoretical Science, for the initial work, and later the Framework grant for Strategic Energy Research (Dnr. 2014-5392) from Vetenskapsrådet. Discussions with G. Hammett of Princeton Plasma Physics Lab led to many of the key insights that made this work possible. The authors also thank P. Catto, F. Parra, P. Ricci, F. Militello, J. Connor, P. Helander and T. Fülöp for fascinating and productive discussions. These discussions were facilitated by the generous hospitality and material support provided by the Wolfgang Pauli Institute in Vienna and Chalmers University of Technology. This work was conducted in part within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014–2018 under grant agreement no. 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

Appendix A. Time derivatives of the gyrokinetic variables

In this appendix we take the time derivatives of the variables used in the body of the paper. We start with the gyrocentre position,

$$\dot{\mathbf{R}}_s = \frac{d}{dt} \left(\mathbf{r} + \frac{\mathbf{w} \times \mathbf{b}}{\Omega_s} \right), \quad (\text{A } 1)$$

where the derivative is taken along unperturbed orbits. Noting that all effects of magnetic inhomogeneity result in terms of order $v_{\text{th}s}\rho_s/L_{\parallel}$, we can neglect them. Thus this becomes

$$\dot{\mathbf{R}}_s = w_{\parallel}\mathbf{b} + \mathbf{u}_E + O\left(\frac{\rho_s}{L_{\parallel}}v_{\text{th}s}\right). \tag{A 2}$$

In the fast ELM ordering, the component of \mathbf{u}_E due to A_{\parallel} is $O(\epsilon^{3/2}v_{\text{th}i})$. In the inter-ELM ordering it is suppressed by another factor of $\epsilon^{1/2}$. Hence we can neglect it. Gyroaveraging our formula is trivial, and results in

$$\dot{\mathbf{R}}_s = w_{\parallel}\mathbf{b}(\mathbf{R}_s) + \frac{c}{B}\mathbf{b} \times \nabla\varphi + O(\epsilon^{3/2}v_{\text{th}i}), \tag{A 3}$$

as required.

The energy is also simple to handle. Working from the definition we immediately see that

$$\dot{\epsilon}_s = Z_s e \mathbf{E} \cdot \mathbf{b} - m_s \mathbf{w} \cdot \frac{d}{dt} \mathbf{u}_E. \tag{A 4}$$

Gyroaveraging gives

$$\langle \dot{\epsilon}_s \rangle_{\mathbf{R}} = -Z_s e \left(\mathbf{b} \cdot \nabla\varphi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) + O\left(\sqrt{\epsilon} \frac{v_{\text{th}i}}{L_{\parallel}} T_e\right). \tag{A 5}$$

The magnetic moment is defined to be the exact adiabatic invariant

$$\begin{aligned} \mu_s = & \frac{m_s w_{\perp}^2}{2B} - \frac{m_s}{B} \left[\mathbf{w}_{\perp} \cdot \mathbf{V}_{D_s} - \frac{1}{\Omega_s} \mathbf{W} : (w_{\parallel} \nabla \mathbf{b} + \nabla \mathbf{u}_E) \right] \\ & + \frac{w_{\perp}^2}{2B\Omega_s} (w_{\parallel} \mathbf{b} \cdot \nabla \times \mathbf{b} + \nabla \times \mathbf{u}_E) + O\left(\epsilon^2 \frac{m_s v_{\text{th}s}^2}{2B}\right), \end{aligned} \tag{A 6}$$

where \mathbf{W} is given by

$$\mathbf{W} = -\frac{1}{4} (\mathbf{w}_{\perp} \mathbf{w}_{\perp} \times \mathbf{b} + \mathbf{w}_{\perp} \times \mathbf{b} \mathbf{w}_{\perp}). \tag{A 7}$$

This tensor is useful because it satisfies

$$\mathbf{w}_{\perp} \mathbf{w}_{\perp} = \frac{w_{\perp}^2}{2} (\mathbf{I} - \mathbf{b}\mathbf{b}) + \frac{\partial \mathbf{W}}{\partial \vartheta}. \tag{A 8}$$

Finally, for $\dot{\vartheta}$ we take the time derivative of

$$\mathbf{w} = w_{\parallel}\mathbf{b} + w_{\perp} (\cos \vartheta \mathbf{e}_2 - \sin \vartheta \mathbf{e}_1), \tag{A 9}$$

and find

$$\begin{aligned} \frac{Z_s e}{m_s} \left(E_{\parallel} \mathbf{b} + \frac{1}{c} \mathbf{w} \times \mathbf{B} \right) = & \frac{dw_{\parallel}}{dt} \mathbf{b} + w_{\parallel} \left(\frac{\partial \mathbf{b}}{\partial t} + \mathbf{w} \cdot \nabla \mathbf{b} \right) + \frac{dw_{\perp}}{dt} \frac{\mathbf{w}_{\perp}}{w_{\perp}} \\ & - \dot{\vartheta} w_{\perp} (\sin \vartheta \mathbf{e}_2 + \cos \vartheta \mathbf{e}_1) + w_{\perp} (\cos \vartheta \dot{\mathbf{e}}_2 - \sin \vartheta \dot{\mathbf{e}}_1). \end{aligned} \tag{A 10}$$

Taking the inner product with $\mathbf{w} \times \mathbf{b} = -w_\perp (\sin \vartheta \mathbf{e}_2 + \cos \vartheta \mathbf{e}_1)$ gives

$$w_\perp^2 \Omega_s = w_\perp^2 \dot{\vartheta} + O\left(v_{\text{th}_s}^2 \frac{L_\perp}{L_s} \epsilon^{1/2} \Omega_s\right), \tag{A 11}$$

where L_s is the magnetic shear length. Gyroaveraging gives the required answer

$$\dot{\vartheta} = \Omega_s(\mathbf{R}_s) + O(\epsilon \Omega_s) = \Omega_s + O(\epsilon^{1/2} \Omega_s). \tag{A 12}$$

In these variables, gyrophase derivatives have the property that

$$\frac{\partial}{\partial \vartheta} \Big|_{\mathbf{R}_s, \varepsilon_s, \mu_s} = \frac{\partial}{\partial \vartheta} \Big|_{\mathbf{R}_s, w_\parallel, w_\perp} + \frac{\partial \mu_s}{\partial \vartheta} \Big|_{\mathbf{R}_s, w_\parallel, w_\perp} \frac{\partial}{\partial \mu_s} \Big|_{\mathbf{R}_s, \varepsilon_s, \vartheta}. \tag{A 13}$$

We will also need the leading-order gyrophase dependent pieces of the time derivatives of \mathbf{R}_s and ε_s , which are

$$\dot{\mathbf{R}}_s - \langle \dot{\mathbf{R}}_s \rangle_{\mathbf{R}} = \frac{c}{B} \mathbf{b} \times \nabla \left(\frac{\mathbf{b} \times \mathbf{w}}{\Omega_s} \cdot \nabla \varphi \right) + O(\epsilon^{3/2} v_{\text{th}_s}), \tag{A 14}$$

and

$$\dot{\varepsilon}_s - \langle \dot{\varepsilon}_s \rangle_{\mathbf{R}} = -m_s \mathbf{w} \cdot \left[\frac{\partial}{\partial t} + (\mathbf{u}_E + \mathbf{w}) \cdot \nabla \right] \mathbf{u}_E + O\left(\sqrt{\epsilon} \frac{v_{\text{th}_i}}{L_\parallel} T_s\right). \tag{A 15}$$

respectively.

In these variables the ion distribution function is given by

$$f_i = f_i(\mathbf{R}_s, \varepsilon_i, \mu_i, t) + \tilde{f}_i, \tag{A 16}$$

with $\tilde{f}_i \sim \epsilon^{3/2} f_i$.

A.1. Kinetic equations

In the $(\mathbf{R}_s, \varepsilon_s, \mu_s, \vartheta, t)$ variables we can write the averaged kinetic equation as

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{R}}_s \cdot \frac{\partial}{\partial \mathbf{R}_s} + \dot{\varepsilon}_s \frac{\partial}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial}{\partial \vartheta} \right) f_s + \left\langle \delta \mathbf{a}_s \cdot \frac{\partial f_s}{\partial \mathbf{w}} \right\rangle_{\text{turb}} = C[f_s] + S_s, \tag{A 17}$$

in which the fluctuating acceleration is

$$\delta \mathbf{a}_s = \frac{Z_s}{m_s} \left(\delta E + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B} \right). \tag{A 18}$$

In all situations, the leading-order term in this equation is the gyrophase derivative, from which we learn that the distribution functions are gyrophase independent to some order in ϵ . Hence, all the kinetic equations for the mean quantities in this paper are derived by gyroaveraging (A 17) and retaining the leading-order terms.

We will also need an equation for the gyrophase dependence of the ion distribution function (the gyrophase dependence of the electrons is too small to matter). Writing the ion distribution function as

$$f_i = f_i(\mathbf{R}_s, \varepsilon_s, \mu_s, t) + \tilde{f}_i, \tag{A 19}$$

where all the gyrophase dependence is in \tilde{f}_i we have

$$\frac{\partial f_i}{\partial \vartheta} \Big|_{\mathbf{r}, w_{\parallel}, w_{\perp}} = -\frac{\mathbf{w}_{\perp}}{\Omega_i} \cdot \frac{\partial f_i}{\partial \mathbf{R}_i} \Big|_{\varepsilon_s, \mu_s} + \frac{\partial \mu_s}{\partial \vartheta} \frac{\partial f_i}{\partial \mu_s} \Big|_{\mathbf{R}_i, \varepsilon_i} + \frac{\partial \tilde{f}_i}{\partial \vartheta} \Big|_{\mathbf{R}_i, w_{\parallel}, w_{\perp}} + O(\epsilon^2 f_i), \quad (\text{A } 20)$$

with

$$\Omega_i \frac{\partial \tilde{f}_i}{\partial \vartheta} = -(\dot{\mathbf{R}}_i - \langle \dot{\mathbf{R}}_i \rangle_{\mathbf{R}}) \cdot \frac{\partial f_i}{\partial \mathbf{R}_i} - (\dot{\varepsilon}_i - \langle \dot{\varepsilon}_i \rangle_{\mathbf{R}}) \frac{\partial f_i}{\partial \varepsilon_i} - \left\langle \delta \mathbf{a}_i \cdot \frac{\partial \delta f_i}{\partial \mathbf{w}} \right\rangle_{\text{turb}}. \quad (\text{A } 21)$$

In addition, using our orderings for the inter-ELM turbulence, we can write the fluctuating acceleration as

$$\delta \mathbf{a}_s = -\frac{Z_s}{m_s} \nabla \left(\delta \varphi - \frac{w_{\parallel}}{c} \delta A_{\parallel} \right) - \frac{Z_s}{m_s c} \mathbf{b} \mathbf{w}_{\perp} \cdot \nabla \delta A_{\parallel} + O\left(\epsilon v_{\text{th}i} \Omega_i \frac{m_i}{m_s}\right), \quad (\text{A } 22)$$

which will be used in later appendices.

Appendix B. Derivation of (2.53)

In this appendix we derive the vorticity equation to a high enough order to serve for both the ELM and inter-ELM equations. To begin our derivation we multiply the averaged kinetic equation by $m_s \mathbf{v}$ and integrate over \mathbf{v} to obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \sum_s m_s n_s \mathbf{u}_s + \nabla \cdot \left(\sum_s \int d^3 \mathbf{v} m_s \mathbf{v} \mathbf{v} f_s \right) \\ &= \frac{1}{c} \mathbf{j} \times \mathbf{B} + \frac{1}{4\pi} \nabla \cdot \left\langle \delta \mathbf{B} \delta \mathbf{B} - \frac{1}{2} \delta \mathbf{B}^2 \mathbf{I} \right\rangle_{\text{turb}}, \end{aligned} \quad (\text{B } 1)$$

where $n_s \mathbf{u}_s = \int d^3 \mathbf{v} \mathbf{v} f_s$. In deriving this equation, we have used quasineutrality to eliminate all terms involving electric fields and the conservation properties of the collision operator. Our next step is to write this in terms of \mathbf{w} . Defining

$$n_s \mathbf{U}_s = n_s \mathbf{u}_s - n_s \mathbf{u}_E = \int d^3 \mathbf{w} \mathbf{w} f_s, \quad (\text{B } 2)$$

we have

$$\begin{aligned} & \frac{\partial}{\partial t} (n_s m_s \mathbf{u}_s) + \nabla \cdot \left(\int d^3 \mathbf{v} m_s \mathbf{v} \mathbf{v} f_s \right) \\ &= n_s m_s \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) \mathbf{u}_s + \nabla p_{\perp s} + \nabla \cdot [(p_{\parallel s} - p_{\perp s}) \mathbf{b} \mathbf{b}] \\ &+ \nabla \cdot \boldsymbol{\pi}_s - \mathbf{U}_s \nabla \cdot (n_s m_s \mathbf{U}_s) + n_s m_s \mathbf{U}_s \cdot \nabla \mathbf{u}_E, \end{aligned} \quad (\text{B } 3)$$

where the new viscous stress tensor is

$$\nabla \cdot \boldsymbol{\pi}_s = \int d^3 \mathbf{w} m_s \left[\mathbf{w} \mathbf{w} - \left(w_{\parallel}^2 - \frac{1}{2} w_{\perp}^2 \right) \mathbf{b} \mathbf{b} - \frac{1}{2} w_{\perp}^2 \mathbf{I} \right] f_s. \quad (\text{B } 4)$$

It will be convenient to rearrange (B 3) slightly into

$$\begin{aligned} & \frac{\partial}{\partial t} (n_s m_s \mathbf{u}_s) + \nabla \cdot \left(\int d^3 \mathbf{v} m_s \mathbf{v} \mathbf{v} f_s \right) \\ & = n_s m_s \left(\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right) \mathbf{u}_s + \nabla p_{\perp s} + \nabla \cdot [(p_{\parallel s} - p_{\perp s}) \mathbf{b} \mathbf{b}] + \mathbf{X}_s \end{aligned} \quad (\text{B } 5)$$

with the abbreviation that

$$\mathbf{X}_s = \nabla \cdot \boldsymbol{\pi}_s - n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla \mathbf{u}_s - \mathbf{U}_s \nabla \cdot (n_s m_s \mathbf{U}_s) + n_s m_s \mathbf{U}_s \cdot \nabla \mathbf{u}_E. \quad (\text{B } 6)$$

We now proceed to partially derive our vorticity equation. Substituting (B 5) into (B 1) we have

$$\begin{aligned} & \sum_s n_s m_s \left(\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right) \mathbf{u}_s + \nabla p_{\perp s} + \nabla \cdot [(p_{\parallel s} - p_{\perp s}) \mathbf{b} \mathbf{b}] + \mathbf{X} \\ & = \frac{1}{c} \mathbf{j} \times \mathbf{B} + \frac{1}{4\pi} \nabla \cdot \left\langle \delta \mathbf{B} \delta \mathbf{B} - \frac{1}{2} \delta \mathbf{B}^2 \mathbf{I} \right\rangle_{\text{turb}}, \end{aligned} \quad (\text{B } 7)$$

again using the natural notation that $p_{\parallel} = \sum_s p_{\parallel s}$, etc. Now, taking $\nabla \cdot [\mathbf{b} \times (\text{B } 7) / B]$, we obtain

$$\begin{aligned} & \nabla \cdot \left[\sum_s \frac{n_s m_s}{B} \left(\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right) \mathbf{b} \times \mathbf{u}_s - \sum_s \frac{n_s m_s}{B} u_{\parallel s} \mathbf{b} \cdot \nabla \mathbf{b} \times \mathbf{u}_s \right] \\ & = \nabla \cdot \left[-\frac{j_{\parallel} \mathbf{b}}{c} \left(1 - 4\pi \frac{p_{\parallel} - p_{\perp}}{B^2} \right) - p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) - \frac{\nabla \times \mathbf{b}}{B} (p_{\parallel} - p_{\perp}) \right] \\ & + \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \mathbf{X} \right), \end{aligned} \quad (\text{B } 8)$$

in which we have used (B 12), (B 13) and (B 15) to rewrite the first line of the right-hand side. The unadorned vector \mathbf{X} is given by

$$\mathbf{X} = \sum_s \mathbf{X}_s + \frac{1}{4\pi} \nabla \cdot \left\langle \delta \mathbf{B} \delta \mathbf{B} - \frac{1}{2} \delta \mathbf{B}^2 \mathbf{I} \right\rangle_{\text{turb}}. \quad (\text{B } 9)$$

We now use the result (B 19) to find \mathbf{U}_s :

$$n_s m_s \mathbf{U}_s = n_s m_s u_{\parallel s} \mathbf{b} + \frac{1}{\Omega_s} \mathbf{b} \times \nabla p_{\perp s}, \quad (\text{B } 10)$$

which allows us to simplify \mathbf{X}_s :

$$\begin{aligned} \mathbf{X}_s & = \nabla \cdot \boldsymbol{\pi}_s - n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla (u_{\parallel s} \mathbf{b} + \mathbf{u}_{\text{ds}}) \\ & - (u_{\parallel s} \mathbf{b} + \mathbf{u}_{\text{ds}}) \nabla \cdot (n_s m_s u_{\parallel s} \mathbf{b} + n_s m_s \mathbf{u}_{\text{ds}}) + n_s m_s \mathbf{u}_{\text{ds}} \cdot \nabla \mathbf{u}_E, \end{aligned} \quad (\text{B } 11)$$

with the notation $\mathbf{u}_{\text{ds}} = \mathbf{b} \times \nabla p_{\perp s} / n_s m_s \Omega_s$ for the diamagnetic velocity of species s .

B.1. Manipulations leading to (B8)

We now provide the results alluded to above. Firstly,

$$\nabla \cdot \left[\frac{\mathbf{b}}{B} \times (\mathbf{j} \times \mathbf{B}) \right] = -\nabla \cdot (\mathbf{j}_{\parallel} \mathbf{b}), \quad (\text{B } 12)$$

in which we have used quasineutrality in the form $\nabla \cdot \mathbf{j} = 0$. For our next trick, we see that

$$\nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla p_{\perp} \right) = \nabla \times \left(\frac{\mathbf{b}}{B} \right) \cdot \nabla p_{\perp} = \nabla \cdot \left[p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right]. \quad (\text{B } 13)$$

Finally,

$$\nabla \cdot \left[\frac{\mathbf{b}}{B} \times \nabla \cdot (abb) \right] = \nabla \cdot \left[\frac{a}{B} \mathbf{b} \times \mathbf{b} \cdot \nabla \mathbf{b} \right] = \nabla \cdot \left[\frac{a}{B} (\nabla \times \mathbf{b} - \mathbf{b} \mathbf{b} \cdot \nabla \times \mathbf{b}) \right], \quad (\text{B } 14)$$

whereupon we can use $\mathbf{b} \cdot \nabla \times \mathbf{b} = \mathbf{b} \cdot \nabla \times \mathbf{B}/B$ and Ampère's law to obtain

$$\nabla \cdot \left[\frac{\mathbf{b}}{B} \times \nabla \cdot (abb) \right] = \nabla \cdot \left[\frac{a}{B} \left(\nabla \times \mathbf{b} - \frac{4\pi j_{\parallel}}{cB} \mathbf{b} \right) \right]. \quad (\text{B } 15)$$

Setting $a = p_{\parallel} - p_{\perp}$ gives exactly the result needed above.

B.2. Lowest-order flows

Let us first calculate the perpendicular velocity of a generic ion species:

$$n_s \mathbf{u}_{\perp s} = \int d^3 \mathbf{v} \mathbf{v}_{\perp} f_s = n_s \mathbf{u}_E + \int d^3 \mathbf{w} \mathbf{w}_{\perp} f_s. \quad (\text{B } 16)$$

We then integrate by parts in ϑ , using $\mathbf{w}_{\perp} = \partial \mathbf{b} \times \mathbf{w} / \partial \vartheta$, to obtain

$$n_s \mathbf{u}_{\perp s} = n_s \mathbf{u}_E - \int d^3 \mathbf{w} \mathbf{b} \times \mathbf{w}_{\perp} \left. \frac{\partial f_s}{\partial \vartheta} \right|_{r, w_{\parallel}, w_{\perp}}. \quad (\text{B } 17)$$

Next we substitute from (A20), noting that only the first term on the right-hand side of (A20) is large enough to contribute, and find

$$n_s \mathbf{u}_{\perp s} = n_s \mathbf{u}_E + \frac{1}{\Omega_s} \int d^3 \mathbf{w} \mathbf{b} \times \mathbf{w}_{\perp} \mathbf{w}_{\perp} \cdot \nabla f_s. \quad (\text{B } 18)$$

Using the gyrotropy of f_s and performing the gyroaverage, we obtain the final result

$$n_s \mathbf{u}_{\perp s} = \frac{c}{B} \mathbf{b} \times \left[\nabla \left(\frac{1}{Z_s e} \int d^3 \mathbf{v} \frac{m_s}{2} v_{\perp}^2 f_s \right) + n_s \nabla \varphi \right], \quad (\text{B } 19)$$

where we have expanded the $\mathbf{E} \times \mathbf{B}$ flow to lowest order and reverted to using \mathbf{v} instead of \mathbf{w} . Thus the perpendicular flow is simply the sum of the diamagnetic flow and the electrostatic $\mathbf{E} \times \mathbf{B}$ flow.

B.3. The ion stress tensor

In this section we calculate the stress tensor π_s . By using the identity (A 8) in the definition (B 4), we can write π_s as

$$\pi_s = \int d^3\mathbf{w} m_s \left(w_{\parallel} \mathbf{b} \mathbf{w}_{\perp} + w_{\parallel} \mathbf{w}_{\perp} \mathbf{b} + \frac{\partial \mathbf{W}}{\partial \vartheta} \right) f_s. \tag{B 20}$$

By the same manipulations as in the previous section we can show that

$$\int d^3\mathbf{w} m_s w_{\parallel} \mathbf{b} \mathbf{w}_{\perp} f_s = \mathbf{b} \left(\frac{m_s}{2\Omega_s} \int d^3\mathbf{w} w_{\parallel} w_{\perp}^2 \mathbf{b} \times \nabla f_s \right). \tag{B 21}$$

Using this result, we can rewrite π_s in the following way

$$\pi_s = \mathbf{b} \mathbf{h} + \mathbf{h} \mathbf{b} + \int d^3\mathbf{w} m_s \frac{\partial \mathbf{W}}{\partial \vartheta} f_s, \tag{B 22}$$

where we have abbreviated

$$\mathbf{h} = \frac{B^2}{\Omega_s} \mathbf{b} \times \nabla \int d^3\mathbf{w} w_{\parallel} \frac{m_s w_{\perp}^2}{2B^2} f_s. \tag{B 23}$$

We now proceed to the final term of (B 22). Using (A 20) we see that

$$m_s \int d^3\mathbf{w} \frac{\partial \mathbf{W}}{\partial \vartheta} f_s = -m_s \int d^3\mathbf{w} \mathbf{W} \frac{\partial f_s}{\partial \vartheta} = m_s \int d^3\mathbf{w} \mathbf{W} \left(\frac{\mathbf{w}_{\perp}}{\Omega_s} \cdot \nabla f_s - \frac{\partial f_s}{\partial \mu_s} \frac{\partial \mu_s}{\partial \vartheta} - \frac{\partial \tilde{f}_s}{\partial \vartheta} \right). \tag{B 24}$$

The first of these terms is easily handled,

$$\frac{m_s}{\Omega_s} \int d^3\mathbf{w} \mathbf{W} \mathbf{w} \cdot \nabla f_s(\mathbf{R}_s, \varepsilon_s, \mu_s, t) = \frac{m_s}{\Omega_s^2} \int d^3\mathbf{w} \mathbf{W} \mathbf{w} \mathbf{w} \times \mathbf{b} : \nabla_{\perp} \nabla_{\perp} f_s. \tag{B 25}$$

We then perform the gyroaverage, to obtain

$$\begin{aligned} \frac{m_s}{\Omega_s} \int d^3\mathbf{w} \mathbf{W} \mathbf{w} \cdot \nabla f_s &= -\frac{m_s}{\Omega_s^2} \int d^3\mathbf{w} \frac{w_{\perp}^4}{16} [\nabla_{\perp} \nabla_{\perp} - (\mathbf{b} \times \nabla)(\mathbf{b} \times \nabla)] f_s + O(\epsilon^2 p_{\perp s}) \\ &= [(\mathbf{b} \times \nabla)(\mathbf{b} \times \nabla) - \nabla_{\perp} \nabla_{\perp}] \frac{m_s}{\Omega_s^2} \int d^3\mathbf{w} \frac{w_{\perp}^4}{16} f_s + O(\epsilon^2 p_{\perp s}), \end{aligned} \tag{B 26}$$

where the operator $(\mathbf{b} \times \nabla)(\mathbf{b} \times \nabla)$ is given by its components as $\epsilon_{ikl} \epsilon_{jmn} b_k b_m \nabla_l \nabla_n$ with ϵ_{ijk} the Levi-Civita symbol. Writing the product $\epsilon_{ikl} \epsilon_{jmn}$ in terms of Kronecker deltas, we have that

$$\epsilon_{ikl} \epsilon_{jmn} b_k b_m \nabla_l \nabla_n = \delta_{ij} (\nabla^2 - \mathbf{b} \cdot \nabla^2) - b_i b_j \nabla^2 + b_i \mathbf{b} \cdot \nabla \nabla_j + b_j \mathbf{b} \cdot \nabla \nabla_i - \nabla_i \nabla_j, \tag{B 27}$$

whereupon (B 26) becomes

$$\frac{m_s}{\Omega_s} \int d^3\mathbf{w} \mathbf{W} \mathbf{w} \cdot \nabla f_s = [I \nabla^2 - 2 \nabla_{\perp} \nabla_{\perp}] \frac{m_s}{\Omega_s^2} \int d^3\mathbf{w} \frac{w_{\perp}^4}{16} f_s + O(\epsilon^2 p_{\perp s}). \tag{B 28}$$

Moving to the second term in (B 24), and substituting from (A 6), we have

$$\begin{aligned}
 & m_s \int d^3 \mathbf{w} \mathbf{W} \frac{\partial \mu_s}{\partial \vartheta} \frac{\partial f_s}{\partial \mu_s} \\
 &= m_s \int d^3 \mathbf{w} \left\langle \mathbf{W} \left[-\frac{m_s}{B} \mathbf{w} \times \mathbf{b} \cdot \mathbf{V}_{D_s} + \mathbf{w}_\perp \mathbf{w}_\perp : (w_\parallel \nabla \mathbf{b} + \nabla \mathbf{u}_E) \right] \right\rangle_r \frac{\partial f_s}{\partial \mu_s}, \quad (\text{B } 29)
 \end{aligned}$$

where we have used the fact that $(\mathbf{I} - \mathbf{b}\mathbf{b}) : (w_\parallel \nabla \mathbf{b} + \nabla \mathbf{u}_E)$ is small. Explicitly performing the gyroaverages, the first term vanishes (gyroaverage of an odd power of \mathbf{w}_\perp) and the second term gives

$$m_s \int d^3 \mathbf{w} \mathbf{W} \frac{\partial \mu_s}{\partial \vartheta} \frac{\partial f_s}{\partial \mu_s} = - \int d^3 \mathbf{w} \frac{w_\perp^4}{32} \frac{m_s^2}{B \Omega_s} [w_\parallel \mathbf{B} + \mathbf{U}] \frac{\partial f_s}{\partial \mu_s}, \quad (\text{B } 30)$$

where the symmetric tensors \mathbf{B} and \mathbf{U} are given by their components

$$B_{ij} = \epsilon_{jkl} b_l \nabla_i b_k + \epsilon_{jkl} b_l \nabla_k b_i + (i \leftrightarrow j) \quad (\text{B } 31)$$

and

$$U_{ij} = \frac{c}{B} \nabla_i \nabla_j \varphi - \epsilon_{ikl} b_k \nabla_l (\mathbf{u}_E)_j + (i \leftrightarrow j), \quad (\text{B } 32)$$

respectively.

Finally, the third term in (B 24) is

$$\begin{aligned}
 & m_s \int d^3 \mathbf{w} \mathbf{W} \frac{\partial \tilde{f}_s}{\partial \vartheta} = -m_s \int d^3 \mathbf{w} \frac{\mathbf{W}}{\Omega_s} \left[(\dot{\mathbf{R}}_s - \langle \dot{\mathbf{R}}_s \rangle_R) \cdot \frac{\partial f_s}{\partial \mathbf{R}_s} + (\dot{\epsilon}_s - \langle \dot{\epsilon}_s \rangle_R) \frac{\partial f_s}{\partial \epsilon_s} \right] \\
 & - \frac{m_s}{\Omega_s} \int d^3 \mathbf{w} \mathbf{W} \left\langle \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{w}} \right\rangle_{\text{turb}}. \quad (\text{B } 33)
 \end{aligned}$$

Using the expression (A 14) for $\dot{\mathbf{R}}_s$, we can see that the first term on the right-hand side of this equation vanishes upon gyroaveraging. For the second term, we have

$$- \frac{m_s}{\Omega_s} \int d^3 \mathbf{w} \mathbf{W} (\dot{\epsilon}_s - \langle \dot{\epsilon}_s \rangle_R) \frac{\partial f_s}{\partial \epsilon_s} = \frac{m_s^2}{\Omega_s} \int d^3 \mathbf{w} \mathbf{W} \mathbf{w}_\perp \mathbf{w}_\perp : \nabla \mathbf{u}_E \frac{\partial f_e}{\partial \epsilon_e}. \quad (\text{B } 34)$$

Whereupon we can perform the gyroaverage as above to obtain

$$- \frac{1}{\Omega_s} \int d^3 \mathbf{w} \mathbf{W} (\dot{\epsilon}_s - \langle \dot{\epsilon}_s \rangle_R) \frac{\partial f_s}{\partial \epsilon_s} = - \frac{m_s^2}{\Omega_s} \int d^3 \mathbf{w} \frac{w_\perp^4}{32} \mathbf{U} \frac{\partial f_e}{\partial \epsilon_e}. \quad (\text{B } 35)$$

Collecting these intermediate results, we have that

$$\begin{aligned}
 \pi_s &= \mathbf{b}\mathbf{h} + \mathbf{h}\mathbf{b} + [I\nabla^2 - 2\nabla_\perp \nabla_\perp] \int d^3 \mathbf{w} \frac{m_s w_\perp^4}{16\Omega_s^2} f_s \\
 &+ \frac{\mathbf{B}}{32} \int d^3 \mathbf{w} \frac{m_s^2 w_\perp^4}{B \Omega_s} w_\parallel \frac{\partial f_s}{\partial \mu_s} + \mathbf{U} \left[\int d^3 \mathbf{w} \frac{m_s^2 w_\perp^4}{32\Omega_s} \left(\frac{\partial f_s}{\partial \epsilon_s} + \frac{1}{B} \frac{\partial f_s}{\partial \mu_s} \right) \right] \\
 &+ \frac{m_s}{\Omega_s} \int d^3 \mathbf{w} \mathbf{W} \left\langle \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{w}} \right\rangle_{\text{turb}}. \quad (\text{B } 36)
 \end{aligned}$$

One final manipulation is to integrate by parts where possible to obtain

$$\begin{aligned}
 \pi_s &= \mathbf{b}\mathbf{h} + \mathbf{h}\mathbf{b} + [I\nabla^2 - 2\nabla_\perp \nabla_\perp] \int d^3 \mathbf{w} \frac{m_s w_\perp^4}{16\Omega_s^2} f_s \\
 &- \frac{\mathbf{B}}{4\Omega_s} \int d^3 \mathbf{w} \frac{m_s}{2} w_\perp^2 w_\parallel f_s - \frac{p_\perp s}{4\Omega_s} \mathbf{U} - \frac{m_s}{\Omega_s} \left\langle \int d^3 \mathbf{w} \delta \mathbf{a}_s \cdot \frac{\partial \mathbf{W}}{\partial \mathbf{w}} \delta f_s \right\rangle_{\text{turb}}. \quad (\text{B } 37)
 \end{aligned}$$

B.4. The ELM vorticity equation

Let us now simplify the quantity X_s in the vorticity equation for the ELM ordering. Estimating the size of other terms, we see that we need the final vorticity equation correct to order $O(p_{\perp}/L_{\parallel}L_{\perp}B)$. This means we need X to $O(p_{\perp}/L_{\parallel})$.

With this ordering, (B 11) becomes

$$X_s = \nabla \cdot \boldsymbol{\pi}_s - n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla (u_{\parallel s} \mathbf{b}) - u_{\parallel s} \mathbf{b} \nabla \cdot (n_s m_s u_{\parallel s} \mathbf{b}) + n_s m_s \mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E. \quad (\text{B } 38)$$

Examining the form of $\boldsymbol{\pi}_s$, it can easily be seen that

$$\nabla \cdot \left\{ \frac{\mathbf{b}}{B} \times \nabla \cdot [(I\nabla^2 - 2\nabla\nabla) \lambda] \right\} = -\nabla \times \left(\frac{\mathbf{b}}{B} \right) \cdot \nabla \nabla^2 \lambda, \quad (\text{B } 39)$$

and so the third term in (B 37) does not contribute to (B 38). Similarly terms involving \mathbf{h} or \mathbf{B} can be seen to be $O(\sqrt{\epsilon} p_{\perp}/L_{\parallel})$ and so too small to be kept.

Hence, we can substitute from (B 37) into (B 38) to obtain

$$X_s = -\nabla \cdot \left(\frac{p_{\perp s}}{4\Omega_s} \mathbf{U} \right) - n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla (u_{\parallel s} \mathbf{b}) - u_{\parallel s} \mathbf{b} \nabla \cdot (n_s m_s u_{\parallel s} \mathbf{b}) + n_s m_s \mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E - \frac{m_s}{\Omega_s} \left\langle \int d^3 \mathbf{w} \delta \mathbf{a}_s \cdot \frac{\partial \mathbf{W}}{\partial \mathbf{w}} \delta f_s \right\rangle_{\text{turb}} + O\left(\sqrt{\epsilon} \frac{p_{\parallel}}{L_{\parallel}}\right). \quad (\text{B } 40)$$

We now use this in the definition of X to obtain

$$\begin{aligned} \frac{\mathbf{b}}{B} \times X = & \frac{\mathbf{b}}{B} \times \left[n_s m_s \mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E - \nabla \cdot \left(\frac{p_{\perp s}}{4\Omega_s} \mathbf{U} \right) \right] - n_s m_s u_{\parallel s}^2 \mathbf{b} \times \mathbf{b} \cdot \nabla \mathbf{b} \\ & + \frac{\mathbf{b}}{B} \times \nabla \cdot \left(\mathbf{P}_2 + \frac{1}{4\pi} \langle \delta \mathbf{B} \delta \mathbf{B} \rangle_{\text{turb}} \right) - \frac{\mathbf{b}}{8\pi B} \mathbf{b} \times \nabla \langle \delta B^2 \rangle_{\text{turb}} + O\left(\sqrt{\epsilon} \frac{p_{\perp}}{L_{\parallel} B}\right), \end{aligned} \quad (\text{B } 41)$$

where we have used the explicit formula for \mathbf{W} to evaluate the term involving the fluctuations and then, as we need to keep only the leading order, substituted \mathbf{v} for \mathbf{w} in this expression, before using

$$\mathbf{P}_2 = \frac{m_s}{\Omega_s} \left\langle \int d^3 \mathbf{w} \mathbf{W} \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{w}} \right\rangle_{\text{turb}}, \quad (\text{B } 42)$$

to write the result concisely.

Taking the divergence of this result, the final term vanishes and the terms involving the parallel velocity will cancel those on the left-hand side of (B 8). This leaves us with the expected terms involving fluctuations, and also terms involving the diamagnetic and $\mathbf{E} \times \mathbf{B}$ velocities. However, it can be shown by direct evaluation in index notation that

$$\nabla \cdot \left[\frac{\mathbf{b}}{B} \times \nabla \cdot \left(-\frac{p_{\perp s}}{4\Omega_s} \mathbf{U} \right) \right] = -\nabla \cdot \left[\frac{\mathbf{b}}{B} \times (\mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E) \right] + O\left(\epsilon^2 \frac{p_{\perp}}{L_{\perp}^2 B}\right). \quad (\text{B } 43)$$

Thus, these terms cancel. This important result is the expression, in our formulation, of the well-known gyro-viscous cancellation (Hazeltine & Meiss 2003).

B.5. Turbulent contributions to the vorticity equation

Let us now massage the term involving the fluctuations. The tensor \mathbf{P}_2 is given explicitly by,

$$\mathbf{P}_2 = \frac{m_s}{\Omega_s} \left\langle \int d^3\mathbf{w} \mathbf{W} \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{w}} \right\rangle_{\text{turb}}, \tag{B 44}$$

where we have used the fact that $\int d^3\mathbf{v} = \int d^3\mathbf{w}$ and $\partial/\partial \mathbf{v} = \partial/\partial \mathbf{w}$. Integrating by parts and using the definition of \mathbf{W} we have

$$\mathbf{P}_2 = \frac{m_s}{4\Omega_s} \left\langle \int d^3\mathbf{w} [\delta \mathbf{a}_{s\perp} \mathbf{w} \times \mathbf{b} + \mathbf{w} \delta \mathbf{a}_s \times \mathbf{b} + (\leftrightarrow)] \delta f_s \right\rangle_{\text{turb}}. \tag{B 45}$$

Now, we calculate the contribution to the vorticity equation in index notation

$$\begin{aligned} & \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \mathbf{P}_2 \right) \\ &= \frac{m_s}{4\Omega_s} \partial_i \partial_l \left\langle \epsilon_{ijk} b_j \int d^3\mathbf{w} [\epsilon_{kmn} b_n (\delta a_{\perp l} w_{\perp m} + w_{\perp l} \delta a_{\perp m}) + (l \leftrightarrow k)] \delta f_s \right\rangle_{\text{turb}} \\ &= \frac{m_s}{4\Omega_s} \partial_i \partial_l \left\langle \int d^3\mathbf{w} [(\delta a_{\perp l} w_{\perp i} + w_{\perp l} \delta a_i) \right. \\ &\quad \left. + \epsilon_{ijk} \epsilon_{lmn} b_j b_n (\delta a_{\perp k} w_{\perp m} + w_{\perp k} \delta a_m)] \delta f_s \right\rangle_{\text{turb}} \\ &= \frac{m_s}{4\Omega_s} \partial_i \partial_l \left\langle \int d^3\mathbf{w} [2(\delta a_{\perp l} w_{\perp i} + w_{\perp l} \delta a_i) - (\delta_{il} - b_i b_l) \delta \mathbf{a}_s \cdot \mathbf{w}_{\perp}] \delta f_s \right\rangle_{\text{turb}}. \end{aligned} \tag{B 46}$$

We now attack the second term in the brackets here, which contains at its core the following gyroaverage

$$\langle \langle \delta \mathbf{a}_s \cdot \mathbf{w}_{\perp} \delta f_s \rangle_r \rangle_{\text{turb}}. \tag{B 47}$$

The first identity we will need is

$$\delta \mathbf{a}_i = -\frac{Z_i e}{m_i} \nabla \left(\delta \varphi - \frac{w_{\parallel}}{c} \delta A_{\parallel} \right) - \frac{\Omega_i}{B} \mathbf{v}_{\perp} \cdot \nabla \delta A_{\parallel} \mathbf{b} + O(\epsilon^{3/2} v_{\text{th},i} \Omega_s), \tag{B 48}$$

which can be derived from the definition of $\delta \mathbf{a}_s$ by direct substitution. We have specialised to the ions as the electrons do not contribute to the momentum (and thus vorticity) equation. Inserting this result into our gyroaverage gives

$$\begin{aligned} \langle \langle \delta \mathbf{a}_s \cdot \mathbf{w}_{\perp} \delta f_s \rangle_r \rangle_{\text{turb}} &= \langle \langle \delta f_s \mathbf{w}_{\perp} \cdot \nabla \delta \chi \rangle_{\text{turb}} \rangle_r \\ &= \nabla \cdot \langle \delta \chi \langle \mathbf{w}_{\perp} \delta f_s \rangle_r \rangle_{\text{turb}} - \langle \delta \chi \langle \mathbf{w}_{\perp} \cdot \nabla \delta f_s \rangle_r \rangle_{\text{turb}}. \end{aligned} \tag{B 49}$$

Now

$$\mathbf{w}_{\perp} \cdot \nabla \delta f_s = \Omega_s \left(\frac{\partial}{\partial \vartheta} \Big|_{\mathbf{R}_s, \epsilon_s, \mu_s} - \frac{\partial}{\partial \vartheta} \Big|_{\mathbf{r}, w_{\parallel}, w_{\perp}} \right) \delta f_s. \tag{B 50}$$

Thus we have

$$\langle \langle \delta \mathbf{a}_s \cdot \mathbf{w}_\perp \delta f_s \rangle_r \rangle_{\text{turb}} = \frac{Z_s e}{m_s} \nabla \cdot \langle \delta \chi \langle \mathbf{w}_\perp \delta f_s \rangle_r \rangle_{\text{turb}} - \frac{Z_s e}{m_s} \left\langle \delta \varphi \left\langle \Omega_s \frac{\partial \delta f_s^{(2)}}{\partial \vartheta} \Big|_{\mathbf{R}_s} \right\rangle_r \right\rangle_{\text{turb}}, \quad (\text{B } 51)$$

where we have defined the higher-order piece of δf_s by

$$\begin{aligned} \delta f_s &= Z_s e \delta \varphi \frac{\partial f_s}{\partial \varepsilon_s} + Z_s e (\delta \chi - \langle \delta \chi \rangle_R) \frac{\partial f_s}{\partial \mu_s} + h_s + \delta f_s^{(2)}. \quad (\text{B } 52) \\ \Omega_s \frac{\partial \delta f_s^{(2)}}{\partial \vartheta} &= \left\langle \frac{\partial \delta f_s}{\partial t} \right\rangle_R - \frac{\partial \delta f_s}{\partial t} - \delta \mathbf{a}_s \cdot \frac{\partial}{\partial \mathbf{v}} (f_s + \delta f_s) + \left\langle \delta \mathbf{a}_s \cdot \frac{\partial}{\partial \mathbf{v}} (f_s + \delta f_s) \right\rangle_R \\ &= Z_i e \frac{\partial}{\partial t} (\langle \delta \varphi \rangle_R - \delta \varphi) + \frac{c}{B} \mathbf{b} \times \frac{\partial}{\partial \mathbf{R}_i} (\langle \delta \varphi \rangle_R - \delta \varphi) \cdot \frac{\partial}{\partial \mathbf{R}_i} (f_i + h_i) \\ &\quad + O(\varepsilon^{3/2} \Omega_s f_i), \quad (\text{B } 53) \end{aligned}$$

where in the second line we have specialised to the ion species. Substituting back into (B 51), we see that

$$\langle \langle \delta \mathbf{a}_i \cdot \mathbf{w}_\perp \delta f_i \rangle_r \rangle_{\text{turb}} = \frac{Z_s e}{m_s} \nabla \cdot \langle \delta \varphi \langle \mathbf{w}_\perp \delta f_i \rangle_r \rangle_{\text{turb}} + O(\varepsilon^2 v_{\text{th}}^2 \Omega_i). \quad (\text{B } 54)$$

Combining these results together, the leading-order part of (B 46) is given by

$$\begin{aligned} \varpi &\equiv \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \mathbf{P}_2 \right) \\ &= - \sum_{s=i} \frac{Z_i e}{2 \Omega_i} \left\{ \nabla \nabla : \left\langle \int d^3 \mathbf{w} [(\nabla \delta \varphi) \mathbf{w}_\perp + \mathbf{w}_\perp (\nabla \delta \varphi)] \delta f_i \right\rangle_{\text{turb}} \right\}. \quad (\text{B } 55) \end{aligned}$$

Finally, we show that the fluctuating Maxwell stress is negligible.

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \langle \delta \mathbf{B} \delta \mathbf{B} \rangle_{\text{turb}} \right) &= \partial_i \epsilon_{ijk} b_j \partial_l \langle \delta B_l \delta B_k \rangle_{\text{turb}} \\ &= \epsilon_{ijk} b_j \partial_i \partial_l \langle \epsilon_{lmn} b_m \partial_n \delta A_{\parallel} \epsilon_{krs} b_r \partial_s \delta A_{\parallel} \rangle_{\text{turb}} \\ &= \epsilon_{lmn} (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) b_j b_m b_r \partial_i \partial_l \langle \partial_n \delta A_{\parallel} \partial_s \delta A_{\parallel} \rangle_{\text{turb}}, \quad (\text{B } 56) \end{aligned}$$

where we have used the fact that terms involving $\delta \mathbf{B} \cdot \mathbf{b}$ and those involving gradients of \mathbf{b} are too small to contribute (these terms are $O(\varepsilon^2 p_\perp / L_\perp^2 B)$). Now, using the fact that $b_r \partial_r = \mathbf{b} \cdot \nabla \ll \nabla_\perp$, we have

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \langle \delta \mathbf{B} \delta \mathbf{B} \rangle_{\text{turb}} \right) &= -\epsilon_{lmn} b_m \partial_i \partial_l \langle \partial_n \delta A_{\parallel} \partial_s \delta A_{\parallel} \rangle_{\text{turb}} \\ &= -\epsilon_{lmn} b_m \partial_i \partial_l (\partial_n \langle \delta A_{\parallel} \partial_s \delta A_{\parallel} \rangle_{\text{turb}} - \langle \delta A_{\parallel} \partial_n \partial_s \delta A_{\parallel} \rangle_{\text{turb}}) \\ &= \epsilon_{lmn} \partial_i \partial_l \left\langle \partial_n \partial_s \frac{1}{2} \delta A_{\parallel}^2 \right\rangle_{\text{turb}} = 0. \quad (\text{B } 57) \end{aligned}$$

With these results, we see that we do indeed obtain the required vorticity equation (2.53).

B.6. The inter-ELM vorticity equation

We now need terms in the vorticity equation to $O(\epsilon^{3/2}p_{\perp}/L_{\perp}^2)$. Firstly, we note that the gyroviscous calculation, and the calculation of the fluctuating Maxwell stress are already accurate to a high enough order; we do not need to revisit them. However, we now need to go back and handle some extra terms in both X_s and our formula for π_s . Firstly, the terms on the left-hand side of (B 8) involving $\mathbf{b} \cdot \nabla \mathbf{b}$ are

$$\begin{aligned} \nabla \cdot (n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla \mathbf{b} \times \mathbf{u}_s) &= \nabla \cdot [n_s m_s u_{\parallel s} \mathbf{b} \cdot \nabla \mathbf{b} \times (\mathbf{u}_{\parallel s} \mathbf{b} + \mathbf{u}_{ds} + \mathbf{u}_E)] \\ &= \nabla \cdot \left[n_s m_s u_{\parallel s}^2 \mathbf{b} \cdot \nabla \mathbf{b} \times \mathbf{b} + n_s m_s u_{\parallel s} (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \mathbf{b} \right]. \end{aligned} \tag{B 58}$$

We can see that, because of our ordering on $\mathbf{b} \cdot \nabla$, only the centrifugal term is large enough to matter. This will, of course, cancel with terms in X_s as in the previous calculation.

Returning to (B 11) and retaining all terms, we see that

$$\begin{aligned} \frac{\mathbf{b}}{B} \times X_s &= \frac{\mathbf{b}}{B} \times (\nabla \cdot \pi_s) - \frac{n_s m_s}{B} u_{\parallel s}^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) - \frac{n_s m_s}{B} u_{\parallel s} \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{u}_{ds}) \\ &\quad + \frac{\nabla_{\perp} p_{\perp s}}{n_s m_s B \Omega_s} \nabla \cdot (n_s m_s u_{\parallel s} \mathbf{b} + n_s m_s \mathbf{u}_{ds}) + \frac{n_s m_s}{B} \mathbf{b} \times (\mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E), \end{aligned} \tag{B 59}$$

in which we have used the definition of the diamagnetic velocity. Expanding the third and fourth terms in this expression it becomes

$$\begin{aligned} \frac{\mathbf{b}}{B} \times X_s &= \frac{\mathbf{b}}{B} \times (\nabla \cdot \pi_s) - \frac{n_s m_s}{B} u_{\parallel s}^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) \\ &\quad - \frac{n_s m_s}{B} u_{\parallel s} \left[\frac{(\mathbf{b} \cdot \nabla \mathbf{b}) \mathbf{b} \cdot \nabla p_{\perp s}}{n_s m_s \Omega_s} + \mathbf{b} \cdot \nabla \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} \right) \right] \\ &\quad + \frac{\nabla_{\perp} p_{\perp s}}{n_s m_s B \Omega_s} \left[n_s m_s u_{\parallel s} \nabla \cdot \mathbf{b} + \mathbf{b} \cdot \nabla (n_s m_s u_{\parallel s}) + \nabla \cdot \left(\frac{\mathbf{b} \times \nabla p_{\perp s}}{\Omega_s} \right) \right] \\ &\quad + \frac{n_s m_s}{B} \mathbf{b} \times (\mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E). \end{aligned} \tag{B 60}$$

Eliminating terms that are smaller than $O(\epsilon^{3/2}p_{\perp s}/BL_{\perp}^2)$ and collecting terms together we obtain

$$\begin{aligned} \frac{\mathbf{b}}{B} \times X_s &= \frac{\mathbf{b}}{B} \times (\nabla \cdot \pi_s) - \frac{n_s m_s}{B} u_{\parallel s}^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) \\ &\quad + \mathbf{b} \cdot \nabla \left(\frac{u_{\parallel s} \nabla p_{\perp s}}{B \Omega_s} \right) + \frac{n_s m_s}{B} \mathbf{b} \times (\mathbf{u}_{ds} \cdot \nabla \mathbf{u}_E). \end{aligned} \tag{B 61}$$

Ultimately we must handle the remaining terms in $\nabla \cdot \pi_s$. It can be shown that:

$$\begin{aligned} \nabla \cdot \left[\frac{\mathbf{b}}{B} \times \nabla \cdot (\mathbf{h}\mathbf{b} + \mathbf{b}\mathbf{h}) \right] &= -\mathbf{B} \cdot \nabla \left[\frac{1}{\Omega_s} \nabla^2 \left(\frac{1}{2B^2} \int d^3 \mathbf{w} m_s w_{\parallel} w_{\perp}^2 f_s \right) \right] \\ &\quad + O\left(\epsilon^2 \frac{p_{\perp s}}{BL_{\perp}^2} \right). \end{aligned} \tag{B 62}$$

Similarly,

$$\begin{aligned} & \nabla \cdot \left[\frac{\mathbf{b}}{B} \times \nabla \cdot \left(-\frac{\mathbf{B}}{4\Omega_s} \int d^3\mathbf{w} \frac{m_s}{2} w_{\parallel} w_{\perp}^2 f_s \right) \right] \\ &= \frac{m_s c}{2Z_s e} [(\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla] \int d^3\mathbf{w} \frac{m_s}{2B^2} w_{\parallel} w_{\perp}^2 f_s. \end{aligned} \quad (\text{B } 63)$$

Thus, the vorticity equation in the inter-ELM ordering, accurate to $O(\epsilon^{3/2} p_{\perp} / L_{\perp}^2 B)$ is

$$\begin{aligned} & \nabla \cdot \left\{ \sum_s \frac{n_s m_s}{B} \left[\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right] \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right\} \\ &= \nabla \cdot \left[-\frac{j_{\parallel} \mathbf{b}}{c} \left(1 - 4\pi \frac{p_{\parallel} - p_{\perp}}{B^2} \right) - p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) - \frac{\nabla \times \mathbf{b}}{B} (p_{\parallel} - p_{\perp}) \right] \\ & - \mathbf{B} \cdot \nabla \left(\frac{\nabla^2 q_{\parallel s}}{B^2 \Omega_s} \right) + \nabla \cdot \left[\mathbf{b} \cdot \nabla \left(\frac{u_{\parallel s} \nabla p_{\perp s}}{B \Omega_s} \right) \right] \\ & + \frac{1}{2B \Omega_s} [(\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla] q_{\parallel s} + \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \mathbf{P}_2 \right), \end{aligned} \quad (\text{B } 64)$$

in which we have abbreviated

$$q_{\parallel s} = \int d^3\mathbf{w} \frac{m_s w_{\perp}^2}{2} w_{\parallel} f_s. \quad (\text{B } 65)$$

We postpone the flux-surface averaging of this equation to appendix C.

B.7. Gyroaverages

In the previous section we needed to gyroaverage terms like $\mathbf{W} \mathbf{w} \mathbf{w}$. These all stem from the following identity

$$\langle w_{\perp i} w_{\perp j} w_{\perp k} w_{\perp l} \rangle_{\mathbf{r}} = \frac{w_{\perp}^4}{8} (\delta_{ij}^{\perp} \delta_{kl}^{\perp} + \delta_{ik}^{\perp} \delta_{jl}^{\perp} + \delta_{il}^{\perp} \delta_{jk}^{\perp}), \quad (\text{B } 66)$$

where

$$\delta_{ij}^{\perp} = \delta_{ij} - b_i b_j, \quad (\text{B } 67)$$

with δ the usual Kronecker delta. The tensor structure of the right-hand side of (B 66) follows from the absence of a preferred perpendicular direction. The coefficient in front can be calculated by picking a particular component and using e.g. $(\cos^4 \vartheta) = 3/8$. We can then perform gyroaverages like

$$\langle \mathbf{W} \mathbf{w}_{\perp} \mathbf{w}_{\perp} \rangle_{\mathbf{r}} = \frac{-1}{4} \epsilon_{jmn} b_n \langle w_{\perp i} w_{\perp m} w_{\perp k} w_{\perp l} \rangle_{\mathbf{r}} + (i \leftrightarrow j), \quad (\text{B } 68)$$

which lead immediately to (B 31) and (B 32).

Appendix C. Flux-surface-averaged vorticity

Here we take the flux-surface average of the slow time scale vorticity equation (B 64) to obtain (3.21). The usual expressions for the flux-surface average are needed (see Dhaeseleer *et al.* (1991)), namely

$$\langle g \rangle_\psi = \frac{1}{V'} \oint \frac{d\ell d\alpha}{B}, \tag{C1}$$

where g is an arbitrary function, V' is the area of the flux surface in question and ψ, α, ℓ are a set of Clebsch coordinates for the field \mathbf{B} , and

$$\langle \nabla \cdot \mathbf{Y} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \mathbf{Y} \cdot \nabla \psi \rangle_\psi, \tag{C2}$$

with \mathbf{Y} an arbitrary vector field. With these identities, we see immediately that

$$\left\langle \nabla \cdot \left[\frac{j_\parallel \mathbf{b}}{c} \left(1 - 4\pi \frac{p_\parallel - p_\perp}{B^2} \right) \right] \right\rangle_\psi = 0, \tag{C3}$$

and

$$\begin{aligned} \left\langle \nabla \cdot \left(\frac{p_\parallel - p_\perp}{B} \nabla \times \mathbf{b} \right) \right\rangle_\psi &= \left\langle \nabla \cdot \left[\mathbf{b} \times \nabla \left(\frac{p_\parallel - p_\perp}{B} \right) \right] \right\rangle_\psi \\ &= \frac{1}{V'} \frac{\partial}{\partial \psi} V' \oint d\ell d\alpha \frac{\partial}{\partial \alpha} \left(\frac{p_\parallel - p_\perp}{B} \right), \\ &= 0, \end{aligned} \tag{C4}$$

where in both we have used only the properties of the flux surfaces, including α -periodicity, but not used any ordering assumptions. These are identities that hold to all orders in ϵ .

Finally we tackle the term

$$\left\langle \nabla \cdot \left(p_\perp \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right) \right\rangle_\psi. \tag{C5}$$

To analyse this term we need the following expression for the perpendicular current, derived from the exact momentum equation under the inter-ELM orderings

$$\mathbf{j}_\perp = \frac{c}{B} \mathbf{b} \times \nabla p_\perp + c \frac{p_\parallel - p_\perp}{B} (\nabla \times \mathbf{b} - \mathbf{b} \mathbf{b} \cdot \nabla \times \mathbf{b}) + O\left(\sqrt{\epsilon} \frac{c}{B} \frac{p_\perp}{L_\parallel}\right), \tag{C6}$$

which can be trivially rewritten as

$$\mathbf{j} = j_\parallel \mathbf{b} \left[1 + \frac{4\pi}{B^2} (p_\parallel - p_\perp) \right] + \frac{c}{B} \mathbf{b} \times \nabla p_\perp + c \frac{p_\parallel - p_\perp}{B} (\nabla \times \mathbf{b}) + O\left(\sqrt{\epsilon} \frac{c}{B} \frac{p_\perp}{L_\parallel}\right). \tag{C7}$$

Returning to (C5) we use the identity

$$\mathbf{b} \times \nabla B = B \nabla \times \mathbf{b} - \frac{4\pi}{c} \mathbf{j}, \tag{C8}$$

which follows from Ampère’s law to obtain

$$\begin{aligned} \left\langle \nabla \cdot \left(p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right) \right\rangle_{\psi} &= \left\langle \nabla \cdot \left[p_{\perp} \left(\frac{\nabla \times \mathbf{b}}{B} + \frac{\mathbf{b} \times \nabla B}{B^2} \right) \right] \right\rangle_{\psi} \\ &= \left\langle \nabla \cdot \left(\frac{4\pi}{c} p_{\perp} \mathbf{j} \right) \right\rangle_{\psi}, \end{aligned} \tag{C9}$$

where we have used the same results that lead to (C4) to eliminate terms containing $\langle \nabla \cdot (p_{\perp} \nabla \times \mathbf{b}/B) \rangle_{\psi}$.

Now, substituting (C7) into this result, we immediately obtain

$$\left\langle \nabla \cdot \left(p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right) \right\rangle_{\psi} = \left\langle j_{\parallel} \mathbf{b} \cdot \nabla \left(\frac{4\pi p_{\perp}}{cB^2} \right) \right\rangle_{\psi} + O\left(\frac{\epsilon p_{\perp}}{BL_{\perp}L_{\parallel}} \right), \tag{C10}$$

where we retain the j_{\parallel} term because of our ordering for the shear length as $\sqrt{L_{\perp}L_{\parallel}}$. However, from our solution (2.58) for the large part of j_{\parallel} , we see that this entire expression becomes

$$\left\langle \nabla \cdot \left(p_{\perp} \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right) \right\rangle_{\psi} = \frac{4\pi K(\psi)}{c} \left\langle \mathbf{B} \cdot \nabla \left(\frac{4\pi p_{\perp}}{cB^2} \right) \right\rangle_{\psi} = 0, \tag{C11}$$

and so the entire term is small.

Using these identities, the flux-surface average of (B64) is

$$\begin{aligned} &\left\langle \nabla \cdot \left\{ \sum_s \frac{n_s m_s}{B} \left[\frac{\partial}{\partial t} + u_{\parallel s} \mathbf{b} \cdot \nabla + \mathbf{u}_E \cdot \nabla \right] \left(\frac{\nabla p_{\perp s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right\} \right\rangle_{\psi} \\ &= \left\langle \nabla \cdot \left[\mathbf{b} \cdot \nabla \left(\frac{u_{\parallel s} \nabla p_{\perp s}}{B \Omega_s} \right) \right] \right\rangle_{\psi} + \left\langle \frac{1}{2B \Omega_s} [(\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla] q_{\parallel s} \right\rangle_{\psi} \\ &+ \left\langle \nabla \cdot \left(\frac{\mathbf{b}}{B} \times \nabla \cdot \mathbf{P}_2 \right) \right\rangle_{\psi}, \end{aligned} \tag{C12}$$

again, with the abbreviation for $q_{\parallel s}$.

Appendix D. Derivation of the ballooning equation

In this appendix we demonstrate that our equations for ELM dynamics reduce in the appropriate limit to the ballooning equation of Tang, Connor & Hastie (1980). The equations that we will linearise are (2.45) and (2.49) for the distribution functions, along with (2.47) and (2.53). For the background magnetic field, we will assume that it has topologically toroidal flux surfaces labelled by ψ – the entire equilibrium field is contained in \mathbf{B}_0 , there is no equilibrium A_{\parallel} . We will linearise around Maxwellian equilibria for both ions and electrons, with the density and temperature of the equilibria being flux functions. Consistently, we will assume that there is no equilibrium φ .

We chose a form for our fluctuating quantities that parallels that in Tang *et al.* (1980), but simplifies our notation. We will assume we are linearising in a local flux tube (Beer *et al.* 1995) and so every fluctuating quantity can be expressed in a Fourier series perpendicular to the field line, with a wave vector \mathbf{k}_{\perp} . The wave

vector is assumed to be large so that $k_{\perp}L_{\perp} \gg 1$, and we really are looking at a local perturbation. This is appropriate for a ballooning mode as we know that the most unstable mode occurs for $n \rightarrow \infty$, where n is the toroidal mode number. Naturally, we assume exponential time dependence with frequency ω . We also define the following auxiliary frequencies to parallel the definitions in Tang *et al.* (1980):

$$\omega_{*s} = \frac{cT_s}{Z_s e B} \mathbf{b} \times \mathbf{k}_{\perp} \cdot \nabla \ln n_s, \tag{D 1}$$

$$\omega_{*s}^T = \omega_{*s} \left[1 + \eta_s \left(\frac{m_s v^2}{2T_s} - \frac{3}{2} \right) \right], \tag{D 2}$$

with η_s , ω_{*p} , ω_{κ} and ω_B defined exactly as in Tang *et al.* (1980).

Let us start by linearising and solving the ion kinetic equation (2.49) to obtain:

$$\delta f_i = -\frac{\omega_{*i}^T Z_i e \delta \varphi}{\omega T_i} f_i. \tag{D 3}$$

Hence, the perturbed ion pressure is isotropic and given by

$$\delta p_i = -p_i \frac{\omega_{*i}}{\omega} (1 + \eta_i) \frac{Z_i e \delta \varphi}{T_i}. \tag{D 4}$$

We now linearise the electron kinetic equation (2.45), resulting in

$$\begin{aligned} -i\omega \delta f_e + w_{\parallel} \mathbf{b} \cdot \frac{\partial \delta f_e}{\partial \mathbf{R}_e} + \frac{ic}{B} \mathbf{b} \times \mathbf{k} \cdot \nabla f_{0e} \delta \varphi - e w_{\parallel} \left(\mathbf{b} \cdot \nabla \delta \varphi + \frac{1}{c} \frac{\partial \delta A_{\parallel}}{\partial t} \right) \frac{f_{0e}}{T_e} \\ - w_{\parallel} \mathbf{b} \times \nabla \delta A_{\parallel} \cdot \nabla f_{0e} = C_L [\delta f_e], \end{aligned} \tag{D 5}$$

where $C_L[\cdot]$ is the collision operator linearised about the Maxwellian equilibria. To solve this equation we need to do two things. Firstly, as is usual in linear calculations we introduce a new field ξ defined by

$$\frac{1}{c} \frac{\partial \delta A_{\parallel}}{\partial t} = \mathbf{b} \cdot \nabla \xi. \tag{D 6}$$

Note, this representation is not in general possible, but because we are dealing with perturbations that have non-zero n , neither δA_{\parallel} nor ξ can have non-zero toroidal averages, and so this representation is both unique and well defined. Secondly, in order to make contact with equation (3.40) of Tang *et al.* (1980), we will need to assume that the electron transit and frequencies are large compared to the frequencies of interest, and to self-consistently neglect trapped-particle effects we take $v_{ee} \sim v_{th_e}/L_{\parallel} \gg \omega$. Using this subsidiary ordering, we can immediately solve (D 5) to find

$$\delta f_e = \left[\frac{e \delta \varphi}{T_e} + \frac{e \xi}{T_e} \left(1 - \frac{\omega_{*e}^T}{\omega} \right) \right] f_e + O \left(\frac{\omega}{k_{\parallel} v_{th_e}} \right). \tag{D 7}$$

Unfortunately, this solution satisfies the linearisation of (2.47) identically. Thus, we need to go to the next order in $\omega/k_{\parallel}v_{th_e}$ to obtain a relationship between $\delta \varphi$ and ξ .

This is most rapidly done by simply integrating (D 5) over all velocities and using the fact that any parallel velocity in δf_e must vanish:

$$-i\omega\delta n_e + \frac{ic}{B}\mathbf{b} \times \mathbf{k} \cdot \nabla n_e \delta\varphi = 0. \tag{D 8}$$

Using the lowest-order solution for δn_e above, we obtain

$$(\delta\varphi + \xi) \left(1 - \frac{\omega_{*e}}{\omega}\right) \frac{e}{T_e} = 0. \tag{D 9}$$

Thus,

$$\delta\varphi = -\xi. \tag{D 10}$$

This is just the usual condition of $\delta E_{\parallel} = 0$, which always obtains when the electron diamagnetic corrections to Ohm’s law can be neglected due to rapid electron motion.

Turning now to the vorticity equation, we can drop all terms involving pressure anisotropy, and thus we have

$$\begin{aligned} \nabla \cdot \left[\frac{n_i m_i}{B} \frac{\partial}{\partial t} \left(\frac{\nabla \delta p_i}{n_i m_i \Omega_i} + \frac{c}{B} \nabla \delta\varphi \right) + \frac{cn_i m_i}{B^2} \mathbf{b} \times \nabla \delta\varphi \cdot \nabla \left(\frac{\nabla p_i}{n_i m_i \Omega_i} \right) \right] \\ = \nabla \cdot \left[-\frac{j_{\parallel} \mathbf{b}}{c} - (\delta p_i + \delta p_e) \nabla \times \left(\frac{\mathbf{b}}{B} \right) \right]. \end{aligned} \tag{D 11}$$

Now, for consistency with the Tang *et al.* (1980) approach, we will furthermore assume $\rho_i^2/L_n L_T \ll \omega/\Omega_i$ which allows us to drop the final term on the left-hand side of (D 11). This is in fact a consequence of our local assumption – if $\omega \sim \omega_*$ then

$$\frac{\omega L_n L_T}{\rho_i^2 \Omega_i} \sim k_{\perp} L_T \gg 1. \tag{D 12}$$

We will also lean on the local approximation on the right-hand side of (D 11) and assume that $k_{\perp} L_s \ll 1$, where L_s is the shear length; thus eliminating the kink drive (i.e. $\delta \mathbf{b} \cdot \nabla j_{\parallel 0}$) from (D 11).

With these approximations we obtain

$$\begin{aligned} \frac{i\omega n_i T_i}{B \Omega_i} k_{\perp}^2 \left[1 - \frac{\omega_{*i}}{\omega} (1 + \eta_i) \right] \frac{Z_i e \xi}{T_i} \\ = -\frac{1}{4\pi} \mathbf{B} \cdot \nabla \left(\frac{k_{\perp}^2 c i}{\omega B} \mathbf{b} \cdot \nabla \xi \right) - i \frac{m_i n_i \Omega_i}{B} \left(2\omega_{\kappa} + \frac{\beta_i}{2} \omega_{*p} \right) \frac{\omega_{*p}}{\omega} \frac{Z_i e \xi}{T_i}, \end{aligned} \tag{D 13}$$

where we have used

$$\mathbf{k}_{\perp} \cdot \nabla \times \left(\frac{\mathbf{b}_0}{B_0} \right) = \frac{m_i \Omega_i}{B T_i} \left(2\omega_{\kappa} + \frac{\beta_i}{2} \omega_{*p} \right). \tag{D 14}$$

Rearranging (D 13) and multiplying by $4\pi T_i \omega / i Z_i e \Omega_i B$, we obtain

$$\frac{1}{B} \mathbf{b} \cdot \nabla (b_i B \mathbf{b} \cdot \nabla \xi) = -\frac{\omega^2}{v_A^2} b_i \left[1 - \frac{\omega_{*i}}{\omega} (1 + \eta_i) \right] \xi - \frac{1}{v_A^2} \left(2\omega_{\kappa} + \frac{\beta_i}{2} \omega_{*p} \right) \omega_{*p} \xi, \tag{D 15}$$

where we have used the fact that $\mathbf{b} \cdot \nabla T_i = 0$ and introduced the notation $b = k_{\perp}^2 \rho_i^2 / 2$.

Inserting explicit geometric expressions for $\mathbf{b} \cdot \nabla$, and dropping the β_i term, (D 15) matches the first line of (3.40) of Tang *et al.* (1980). The second line of the ballooning equation in Tang *et al.* (1980) is small in our ordering, because $\omega_*/\omega_k \sim L_n/R \ll 1$. Taking the JET-ILW pedestals from Hatch *et al.* (2017) as typical, then $R/L_n \gtrsim 50$ in the pedestal region.

Hence we see that both formulations capture the key physics of ballooning modes, with our model retaining more electron and finite- β physics, and the model of Tang *et al.* (1980) retaining plasma compressibility effects (the final term in 3.6 and 3.40 of that paper).

Appendix E. Derivations for the inter-ELM equations

In this appendix, we perform the derivations required for the inter-ELM equations.

E.1. Electron behaviour on the slow time scale

In this section we discuss the possible solutions to the lowest-order electron kinetic equation, under the inter-ELM orderings. Taking (2.25) for the electrons, and applying the inter-ELM orderings, we see that the largest terms are those involving electron velocities:

$$w_{\parallel} \mathbf{b} \cdot \nabla f_e + e w_{\parallel} \mathbf{b} \cdot \nabla \varphi \frac{\partial f_e}{\partial \varepsilon_s} = C[f_e]. \quad (\text{E } 1)$$

We multiply this by $1 + \ln f_e$, and integrate over all velocities to obtain

$$\nabla \cdot (\mathbf{b} \int d^3 \mathbf{v} w_{\parallel} f_e \ln f_e) = \int d^3 \mathbf{v} \ln f_e C[f_e], \quad (\text{E } 2)$$

where we use the fact that $f_e \rightarrow 0$ implies $f_e \ln f_e \rightarrow 0$ at high energies. We will now *assume* that the magnetic field has good flux surfaces, at least to some approximation. Then, averaging over these surfaces, and applying the H -theorem for the nonlinear collision operator, we obtain Maxwell–Boltzmann electrons

$$f_e = \frac{N_e(\psi)}{\pi^{3/2} v_{\text{th}_e}^3} e^{-\varepsilon_e - e\varphi/T_e(\psi)}, \quad (\text{E } 3)$$

where N_e and T_e are flux functions. The parameter N_e is related to the electron density n_e by

$$N_e(\psi) = n_e e^{-e\varphi/T_e}. \quad (\text{E } 4)$$

We can rewrite this equation as

$$\mathbf{b} \cdot \nabla \left(\frac{T_e}{e} \ln n_e - \varphi \right) = 0. \quad (\text{E } 5)$$

Note that because the induced parallel electric field is small compared to the electrostatic field there is a field-line preserving flow (Newcomb 1958; Abel & Cowley 2013). Thus, the assumption of good magnetic surfaces is consistent – if we assume that they exist initially for some time then they will be preserved by this flow.

Going to the next order in $\sqrt{\epsilon}$, we expand $f_e = f_e^{(0)} + f_e^{(1)} + \dots$ where $f_e^{(1)} \sim \sqrt{\epsilon} f_e^{(0)}$ and so forth. Using this expansion, we obtain the kinetic equation

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_e} \right) f_e^{(0)} + w_{\parallel} \mathbf{b} \cdot \nabla f_e^{(1)} + e w_{\parallel} \left(\mathbf{b} \cdot \nabla \varphi^{(1)} + \frac{\partial A_{\parallel}}{\partial t} \right) \frac{\partial f_e^{(0)}}{\partial \epsilon_e} \\ & + e w_{\parallel} \mathbf{b} \cdot \nabla \varphi \frac{\partial f_e^{(1)}}{\partial \epsilon_s} + \left\langle \left\langle \delta \mathbf{a}_e \cdot \frac{\partial \delta f_e}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_{\mathbf{R}} = C_L [f_e^{(1)}] + S_e. \end{aligned} \quad (\text{E } 6)$$

Integrating this over all velocities we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e + \nabla \cdot (\mathbf{b} u_{\parallel e} n_e) + \nabla \cdot \left\langle \int d^3 \mathbf{v} \langle \mathbf{V}_{\chi} \rangle_{\mathbf{R}} \delta f_e \right\rangle_{\text{turb}} = S_e^{(n)}, \quad (\text{E } 7)$$

where we have used (E 23) for the fluctuations, and introduced

$$u_{\parallel e} = \int d^3 \mathbf{v} w_{\parallel} f_e^{(1)}, \quad (\text{E } 8)$$

because there is no electron parallel velocity in $f_e^{(0)}$.

If we multiply (E 6) by $(\epsilon_e/T_e - 3/2)$ and integrate over all velocities we find that

$$\begin{aligned} & \frac{3}{2} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e T_e + \nabla \cdot \left\langle \int d^3 \mathbf{v} \left(\frac{\epsilon_e}{T_e} - \frac{3}{2} \right) \langle \mathbf{V}_{\chi} \rangle_{\mathbf{R}} \delta f_e \right\rangle_{\text{turb}} \\ & = e n_e u_{\parallel e} \mathbf{b} \cdot \nabla \varphi + \left\langle e \left(\mathbf{b} \cdot \nabla \delta \varphi + \frac{1}{c} \frac{\partial \delta A_{\parallel}}{\partial t} \right) \delta u_{\parallel e} \right\rangle_{\text{turb}} - \nabla \cdot (q_{\parallel e}^{(1)} \mathbf{b}) + S_e^{(E)}, \end{aligned} \quad (\text{E } 9)$$

where again we have used (E 23) for the fluctuations, and $q_{\parallel e}^{(1)}$ is a moment of $f_e^{(1)}$ which is not solved for. As T_e is a flux function, we average this equation over the flux surfaces to finally obtain

$$\begin{aligned} & \frac{3}{2} \left\langle \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e T_e \right\rangle_{\psi} + \left\langle \nabla \cdot \left\langle \int d^3 \mathbf{v} \left(\frac{\epsilon_e}{T_e} - \frac{3}{2} \right) \langle \mathbf{V}_{\chi} \rangle_{\mathbf{R}} \delta f_e \right\rangle_{\text{turb}} \right\rangle_{\psi} \\ & = \langle e n_e u_{\parallel e} \mathbf{b} \cdot \nabla \varphi \rangle_{\psi} - \langle \langle e \delta E_{\parallel} \delta u_{\parallel e} \rangle_{\text{turb}} \rangle_{\psi} + \langle S_e^{(E)} \rangle_{\psi}. \end{aligned} \quad (\text{E } 10)$$

E.2. Pedestal gyrokinetic turbulence

In this section, we derive the equations governing the turbulence in the pedestal. To obtain the equations for the inter-ELM turbulence, we take the fluctuating part of the fundamental kinetic equation, written in Catto-transformed variables

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{R}}_s \cdot \frac{\partial}{\partial \mathbf{R}_s} + \dot{\epsilon}_s \frac{\partial}{\partial \epsilon_s} + \dot{\vartheta} \frac{\partial}{\partial \vartheta} \right) \delta f_s + \delta \mathbf{a}_s \cdot \left(\frac{\partial f_s}{\partial \mathbf{v}} + \frac{\partial \delta f_s}{\partial \mathbf{v}} \right) = C_L [\delta f_s]. \quad (\text{E } 11)$$

The leading order of this equation is

$$\Omega_s \frac{\partial \delta f_s}{\partial \vartheta} = -\delta \mathbf{a}_s \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{Z_s e}{m_s} \mathbf{v}_{\perp} \cdot \nabla \delta \varphi \frac{\partial f_s}{\partial \epsilon_s} + \frac{Z_s e}{m_s} \mathbf{v}_{\perp} \cdot \nabla \delta \varphi \frac{\partial f_s}{\partial \mu_s}, \quad (\text{E } 12)$$

whose solution can be written as

$$\delta f_s = h_s(\mathbf{R}_s, \epsilon_s, \mu_s, t) + Z_s e \delta \varphi \frac{\partial f_s}{\partial \epsilon_s} + Z_s e (\delta \varphi - \langle \delta \varphi \rangle_{\mathbf{R}}) \frac{\partial f_s}{\partial \mu_s}, \quad (\text{E } 13)$$

where we have split out part of the gyrophase-independent piece to more closely mirror other derivations.

Substituting this back into (E 11) and gyroaveraging gives

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + (w_{\parallel} \mathbf{b}(\mathbf{R}_s) + \mathbf{u}_E) \cdot \frac{\partial}{\partial \mathbf{R}_s} \right] h_s + \left\langle \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\mathbf{R}} \\ &= Z_s e \frac{\partial f_s}{\partial \varepsilon_s} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \frac{\partial}{\partial \mathbf{R}_s} \right) \langle \delta \chi \rangle_{\mathbf{R}} - \frac{c}{B} \mathbf{b} \times \nabla \langle \delta \chi \rangle_{\mathbf{R}} \cdot \nabla f_s \\ &+ \langle C_L [h_s] \rangle_{\mathbf{R}} + O(\epsilon \Omega_s h_s), \end{aligned} \tag{E 14}$$

where we have used explicit forms for the time derivatives, written $f_s = f_s(\mathbf{R}_s, \varepsilon_s, \mu_s, t)$ and used the chain rule. By exactly the same manipulations as we use in the next section for the nonlinear term in the transport equations, we can show that

$$\left\langle \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\mathbf{R}} = \frac{c}{B} \mathbf{b} \times \nabla \langle \delta \chi \rangle_{\mathbf{R}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s}. \tag{E 15}$$

All that remains to derive the gyrokinetic equations in §3.3 is to specialise the above results to electrons and ions. The field equations follow immediately from the fluctuating parts of quasineutrality and Ampère’s law.

E.3. Turbulent transport in the pedestal

To derive the terms involving the fluctuations in the inter-ELM equations, we need the following results. Firstly, the nonlinear term in the averaged kinetic equation is

$$\left\langle \left\langle \delta \mathbf{a}_s \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_{\mathbf{R}}, \tag{E 16}$$

and so in order to perform the gyroaverage, we need to convert the velocity derivative into a derivative of the gyrokinetic variables. We require this term up only to $O(v_{\text{th}i} f_s / L_{\parallel})$ and will systematically neglect any higher-order terms. We first do this for the non-adiabatic part of the distribution function:

$$\begin{aligned} & \left\langle \left\langle \delta \mathbf{a}_s \cdot \frac{\partial h_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_{\mathbf{R}} \\ &= \left\langle \left\langle \delta \mathbf{a}_s \cdot \frac{\partial \mathbf{R}_s}{\partial \mathbf{v}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} + m_s \delta \mathbf{a}_s \cdot \mathbf{v} \frac{\partial h_s}{\partial \varepsilon_s} + \frac{m_s}{B} \delta \mathbf{a}_s \cdot \mathbf{v}_{\perp} \frac{\partial h_s}{\partial \mu_s} \right\rangle_{\mathbf{R}} \right\rangle_{\text{turb}}. \end{aligned} \tag{E 17}$$

Using the explicit form of the fluctuating acceleration, we see that

$$\langle \delta \mathbf{a}_s \cdot \mathbf{v} \rangle_{\mathbf{R}} = -\frac{Z_s e}{m_s} w_{\parallel} \langle \mathbf{b} \cdot \nabla \delta \varphi \rangle_{\mathbf{R}}, \tag{E 18}$$

and

$$\langle \delta \mathbf{a}_s \cdot \mathbf{v}_{\perp} \rangle_{\mathbf{R}} = 0, \tag{E 19}$$

where we have used $\langle \mathbf{v}_{\perp} \cdot \nabla \delta G \rangle_{\mathbf{R}} = 0$ for any fluctuating field δG that is independent of gyrophase at fixed \mathbf{r} (see (A.4) of (Abel *et al.* 2013)).

These results are then substituted back into (E 17) to arrive at

$$\left\langle \left\langle \delta \mathbf{a}_s \cdot \frac{\partial h_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_R = \left\langle \frac{1}{\Omega_s} \langle \delta \mathbf{a}_s \rangle_R \cdot \mathbf{b} \times \frac{\partial h_s}{\partial \mathbf{R}_s} \right\rangle_{\text{turb}} - \frac{Z_s e}{m_s} w_{\parallel} \left\langle \langle \mathbf{b} \cdot \nabla \delta \varphi \rangle_R \frac{\partial h_s}{\partial \varepsilon_s} \right\rangle_{\text{turb}}. \quad (\text{E } 20)$$

If we also use

$$\langle \delta \mathbf{a}_s \rangle_R = -\frac{Z_s e}{m_s} \nabla \left\langle \delta \varphi - \frac{1}{c} w_{\parallel} \delta A_{\parallel} \right\rangle, \quad (\text{E } 21)$$

then we obtain

$$\left\langle \left\langle \delta \mathbf{a}_s \cdot \frac{\partial h_s}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_R = \left\langle \langle \mathbf{V}_x \rangle_R \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} \right\rangle_{\text{turb}} - \frac{Z_s}{m_s} w_{\parallel} \langle \langle \mathbf{b} \cdot \nabla \delta \varphi \rangle_R \rangle_{\text{turb}}. \quad (\text{E } 22)$$

To find a full expression for the original nonlinear term, we need to also handle the Boltzmann-like pieces of δf_s .

For the electrons this is simple, and we obtain

$$\left\langle \left\langle \delta \mathbf{a}_e \cdot \frac{\partial \delta f_e}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_R = \left\langle \mathbf{V}_x \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} \right\rangle_{\text{turb}} + \frac{e}{m_e} w_{\parallel} \left\langle \langle \mathbf{b} \cdot \nabla \delta \varphi \rangle_R \frac{\partial h_s}{\partial \varepsilon_s} \right\rangle_{\text{turb}}, \quad (\text{E } 23)$$

where gyroaverages have been removed as the electron gyroradius is smaller than all scales of interest. The first term will vanish when summing over the signs of w_{\parallel} and so does not appear in any resulting equations.

For the ions, we can neglect terms involving w_{\parallel} as they are small in the mass ratio compared to the same terms for electrons. Thus we have

$$\left\langle \left\langle \delta \mathbf{a}_i \cdot \frac{\partial \delta f_i}{\partial \mathbf{v}} \right\rangle_{\text{turb}} \right\rangle_R = \left\langle \frac{c}{B} \mathbf{b} \times \nabla \langle \delta \varphi \rangle_R \cdot \frac{\partial h_i}{\partial \mathbf{R}_i} \right\rangle_{\text{turb}}, \quad (\text{E } 24)$$

because all the non-adiabatic pieces only contain terms proportional to $\delta \varphi$, which vanish under the turbulence average.

Appendix F. Matching to multiscale gyrokinetics

In this appendix we detail how our inter-ELM transport equations can be matched at the top of the pedestal to multiscale gyrokinetics. We will demonstrate that there exists a subsidiary expansion of multiscale gyrokinetics and a different subsidiary expansion of our inter-ELM equations that both result in the same system of equations. In the main text we have explained the concepts behind these subsidiary orderings, and provided the orderings themselves. Thus, all that remains is to actually perform the subsidiary expansions.

F.1. Subsidiary expansion of multiscale gyrokinetics

In this section we now apply the ordering from § 4.2 to the low-Mach-number equations of Abel *et al.* (2013) (henceforth in this appendix we will refer to this as paper I, and refer to equations in that reference with a prefix I-). Consistently with the ordering discussed in § 4.2, we drop all flow-related terms, either because they are small, or because their shear is small and so they can be removed by a

Galilean transformation. Taking (I-248) and retaining the leading order in ξ , the gyrokinetic equation for ions is,

$$\begin{aligned} \frac{\partial h_i}{\partial t} + \frac{c}{B} \mathbf{b} \times \nabla \langle \delta\varphi \rangle_{\mathbf{R}} \cdot \frac{\partial h_i}{\partial \mathbf{R}_s} - \langle C_L [h_i] \rangle_{\mathbf{R}} \\ = \frac{Z_i e F_{0i}}{T_i} \frac{\partial \langle \delta\varphi \rangle_{\mathbf{R}}}{\partial t} - \frac{c}{B} \mathbf{b} \times \nabla \langle \delta\varphi \rangle_{\mathbf{R}} \cdot \nabla \psi \frac{\partial F_{0s}}{\partial \psi}, \end{aligned} \tag{F1}$$

where we have used the definition (I-250) of χ and the orderings for the magnetic field to replace χ with $\delta\varphi$ in this equation. For electrons, we have

$$\begin{aligned} \frac{\partial h_e}{\partial t} + \left[v_{\parallel} (\mathbf{b} + \delta\mathbf{b}) + \frac{c}{B} \mathbf{b} \times \nabla \delta\varphi \right] \cdot \frac{\partial h_e}{\partial \mathbf{R}_s} - C_L [h_e] \\ = - \frac{e F_{0e}}{T_e} \frac{\partial \chi}{\partial t} - \frac{\partial F_{0s}}{\partial \psi} \frac{c}{B} \mathbf{b} \times \nabla \chi \cdot \nabla \psi, \end{aligned} \tag{F2}$$

where here we have been able to drop all gyroaverages because $k_{\perp} \rho_e \sim \xi \ll 1$. Because of the ordering of δB_{\parallel} , χ is now given by the simpler form

$$\chi = \delta\varphi - \frac{v_{\parallel}}{c} \delta A_{\parallel}. \tag{F3}$$

As the flow is subsonic, we have replaced $w_{\parallel} \approx v_{\parallel}$ in these expressions. In addition, we have the field equations (I-251) and (I-149)

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi}{T_s} = \sum_s Z_s e \int d^3 \mathbf{w} \langle h_s \rangle_{\mathbf{r}} \tag{F4}$$

and

$$-\nabla_{\perp}^2 \delta A_{\parallel} = \frac{4\pi}{c} \sum_s Z_s e \int d^3 \mathbf{w} v_{\parallel} \langle h_s \rangle_{\mathbf{r}}, \tag{F5}$$

which determine $\delta\varphi$ and δA_{\parallel} respectively.

Turning now to the transport equations, we first have (I-252), for particle transport

$$\frac{1}{V'} \frac{\partial}{\partial t} \Big|_{\psi} V' n_s + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \Gamma_s \rangle_{\psi} = \langle S_s^{(n)} \rangle_{\psi}, \tag{F6}$$

in which we have not dropped any terms, because the time scale of the sources and the evolution of the equilibrium is defined by the turbulent transport time. So all terms are of the same order in ξ by definition. However, evaluating the terms in the particle flux (I-170) gives

$$\langle \Gamma_s \rangle_{\psi} = \left\langle \left\langle \int d^3 \mathbf{w} \langle h_s \mathbf{V}_{\chi} \rangle_{\mathbf{r}} \cdot \nabla \psi \right\rangle_{\text{turb}} \right\rangle_{\psi}, \tag{F7}$$

where we have dropped both collisional transport terms as they are $O(v_{ss} \rho_s^2 |\nabla \psi| / L_{\perp})$, and we have dropped the electric field term as $\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\psi}$ is small in the mass ratio

(see appendix C of Abel & Cowley (2013)). Secondly, we have the pressure evolution equation (I-254):

$$\frac{3}{2} \frac{1}{V'} \frac{\partial}{\partial t} \Big|_{\psi} V' p_s + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle q_s \rangle_{\psi} = P_s^{\text{comp}} + P_s^{\text{turb}} + \langle S_s^{(E)} \rangle_{\psi}, \quad (\text{F8})$$

where again we have dropped the flow related terms, including the viscous heating, because the shearing rate is small, and the ohmic heating is dropped because $\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\psi}$ is small in the mass ratio. Similarly, the collisional energy exchange term is small. The heat flux is now

$$\langle q_s \rangle_{\psi} = \left\langle \left\langle \int d^3 \mathbf{w} \varepsilon_s \langle h_s \mathbf{V}_x \rangle_r \cdot \nabla \psi \right\rangle_{\text{turb}} \right\rangle_{\psi}, \quad (\text{F9})$$

and the turbulent heating is given by (I-259):

$$P_s^{\text{turb}} = Z_s e \left\langle \left\langle \int d^3 \mathbf{w} \left\langle h_s \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \chi \right\rangle_r \right\rangle_{\text{turb}} \right\rangle_{\psi} = P_s^{\text{diss}} - P_s^{\text{drive}}. \quad (\text{F10})$$

Examining the size of the terms in this formula, we immediately see that the turbulent heating occurs on a rate

$$P_s^{\text{turb}} \sim \frac{\epsilon^3}{\xi^3} \Omega_i, \quad (\text{F11})$$

one order in ξ too small to appear in our transport equation. This is due to the fact that the rate of perpendicular transport is enhanced by a power of ξ due to the decreasing distance L_{\perp} over which transport has to occur, whilst both terms benefit from the increased amplitude and frequency of the turbulence.

At this point we have to make an assumption on the background magnetic field. We will assume that ψ does not change by $O(1)$ on the transport time scale – this is for consistency with our β ordering. Thus, we order

$$\frac{\partial \psi}{\partial t} \sim \xi \frac{\psi}{\tau_E}. \quad (\text{F12})$$

Applying this ordering means that we drop the compressional heating term P_s^{comp} from (F8). Thus, there are no heat sources in our transport equation save for the explicit source.

F.2. Subsidiary expansion of inter-ELM equations

Now we turn to obtaining the same equations from the inter-ELM system. We apply our subsidiary ordering first to (3.20) to find

$$\mathbf{b} \cdot \nabla \varphi = -\frac{T_e}{e} \mathbf{b} \cdot \nabla \ln n_e = O\left(\zeta \frac{\varphi}{L_{\parallel}}\right), \quad (\text{F13})$$

so, to lowest order in ζ , $\varphi = \bar{\varphi}(\psi)$. Then the ion kinetic equation (3.17), to lowest order in ζ , becomes

$$w_{\parallel} \mathbf{b} \cdot \nabla f_i = C [f_i]. \quad (\text{F14})$$

The general solution of this equation is, by the usual arguments, a stationary Maxwellian, with no parallel flow (i.e. $u_{\parallel i} \ll v_{\text{th}i}$). In addition, the ion densities and temperatures are functions only of ψ .

Examining parallel Ampère's law, we see that

$$e \left(n_e u_{\parallel e} - \sum_{s=i} Z_s n_s u_{\parallel s} \right) = O \left(e n_e v_{\text{th}i} \frac{\beta \rho_i}{L_s} \right), \quad (\text{F } 15)$$

and so, as the shear length L_s gets longer as ζ^{-1} , we have that the electron parallel flow is also small in ζ relative to the ion sound speed. Again, the electron density is now a flux function to lowest order in ζ . This in fact holds to two orders in ζ as we can iterate (F 13) to show φ is a flux function to $O(\zeta^2)$, assume axisymmetry and repeat the above argument.

As the ions are Maxwellian, we can integrate (3.17) over all velocities to find a continuity equation equivalent to (3.14) for the electrons. Flux-surface averaging any one of these continuity equations, we obtain

$$\frac{1}{V'} \frac{\partial}{\partial t} \Big|_{\psi} V' n_s + \frac{1}{V'} \frac{\partial}{\partial \psi} \left\langle \left\langle \int d^3 \mathbf{w} \langle h_s \mathbf{V}_{\varphi} \rangle_{\mathbf{r}} \cdot \nabla \psi \right\rangle_{\text{turb}} \right\rangle_{\psi} = \langle S_s^{(n)} \rangle_{\psi}, \quad (\text{F } 16)$$

where we have used identities from section 3.4 of I for the flux-surface averages. This is the same transport equation as we obtained from gyrokinetics.

Multiplying (3.17) by $\varepsilon_s - 3T_s/2$, and integrating gives us the following heat transport equation

$$\frac{3}{2} \left\langle \left(\frac{\partial}{\partial t} \right) n_i T_i \right\rangle_{\psi} + \left\langle \nabla \cdot \left\langle \int d^3 \mathbf{v} \left(\frac{\varepsilon_i}{T_i} - \frac{3}{2} \right) \langle \mathbf{V}_{\chi} \rangle_{\mathbf{R}} \delta f_i \right\rangle_{\text{turb}} \right\rangle_{\psi} = \langle S_i^{(E)} \rangle_{\psi}, \quad (\text{F } 17)$$

which, upon using the usual identities for the flux-surface average becomes the gyrokinetic heat transport equation. Similarly, taking (3.16) and using our solution for the mean potential and the electron parallel velocity, we arrive at the same equation. Thus, all heat transport equations match.

However, we still need to show that the equations governing $\delta\varphi$ and h_s are also the same. Using our Maxwellian solution for f_i in (3.23), we have that

$$\frac{\partial h_i}{\partial t} + \frac{c}{B} \{ \langle \delta\varphi \rangle_{\mathbf{R}}, h_i \} = \frac{Z_i e}{T_i} \frac{\partial \langle \delta\varphi \rangle_{\mathbf{R}}}{\partial t} - \frac{c}{B} \mathbf{b} \times \nabla \langle \delta\varphi \rangle_{\mathbf{R}} \cdot \nabla \psi \frac{\partial f_i}{\partial \psi}, \quad (\text{F } 18)$$

where we have used the fact that the differences between the energy variable used here and the one defined by (I-241) are all negligible in ζ . Similarly the distinction between the exact μ_s that we use here and the lowest-order one used in paper I can also be ignored. We have also used the fact that we can transform the mean flow out of this equation due to its low shear. For the electrons there is no such difficulty, there are no flow terms in (3.25) and it is already in exactly the same form as the equation derived from gyrokinetics. The field equations also trivially match in this limit. Thus, we have proved what we set out to. The two subsidiary expansions given in § 4.2 result in identical sets of equations for the pedestal top region.

REFERENCES

- ABEL, I. G. & COWLEY, S. C. 2013 Multiscale gyrokinetics for rotating tokamak plasmas II: reduced models for electron dynamics. *New J. Phys.* **15**, 023041.
- ABEL, I. G., PLUNK, G. G., WANG, E., BARNES, M. A., COWLEY, S. C., DORLAND, W. & SCHEKOCHIHIN, A. A. 2013 Multiscale gyrokinetics for rotating tokamak plasmas: fluctuations and transport. *Rep. Prog. Phys.* **76**, 116201.
- BARNES, M. A., ABEL, I. G., DORLAND, W., GÖRLER, T., HAMMETT, G. W. & JENKO, F. 2010 Direct multiscale coupling of a transport code to gyrokinetic turbulence codes. *Phys. Plasmas* **17**, 056109.
- BARNES, M., PARRA, F. I. & SCHEKOCHIHIN, A. A. 2011 Critically balanced ion temperature gradient turbulence in fusion plasmas. *Phys. Rev. Lett.* **107** (11), 115003.
- BEER, M. A., COWLEY, S. C. & HAMMETT, G. W. 1995 Field-aligned coordinates for nonlinear simulations of tokamak turbulence. *Phys. Plasmas* **2**, 2687.
- BERKOWITZ, J. & GARDNER, C. S. 1959 On the asymptotic series expansion of the motion of a charged particle in slowly varying fields. *Commun. Pure Appl. Maths* **12**, 501.
- BEURSKENS, M. N. A., OSBORNE, T. H., SCHNEIDER, P. A., WOLFRUM, E., FRASSINETTI, L., GROEBNER, R., LOMAS, P., NUNES, I., SAARELMA, S., SCANNELL, R. *et al.* 2011 H-mode pedestal scaling in DIII-D, ASDEX upgrade, and JET. *Phys. Plasmas* **18**, 056120.
- BURRELL, K. H., AUSTIN, M. E., BRENNAN, D. P., DEBOO, J. C., DOYLE, E. J., GOHIL, P., GREENFIELD, C. M., GROEBNER, R. J., LAO, L. L., LUCE, T. C. *et al.* 2002 Quiescent H-mode plasmas in the DIII-D tokamak. *Plasma Phys. Control. Fusion* **44**, A253.
- CHANG, C. S., KU, S., ADAMS, M., DAZEVEDO, G., CHEN, Y., CUMMINGS, J., ETHIER, S., GREENGARD, L., HAHM, T. S., HINTON, F. *et al.* & THE CPES TEAM 2006 Integrated particle simulation of neoclassical and turbulence physics in the tokamak pedestal/edge region using XGC. In *Proceedings of the 21st IAEA Fusion Energy Conference*, p. TH/P6–14. IAEA.
- CHANG, C. S., KU, S., LOARTE, A., PARAIL, V., KCHL, F., ROMANELLI, M., MAINGI, R., AHN, J.-W., GRAY, T., HUGHES, J. *et al.* 2017 Gyrokinetic projection of the divertor heat-flux width from present tokamaks to iter. *Nucl. Fusion* **57** (11), 116023.
- CHURCHILL, R. M., CHANG, C. S., KU, S. & DOMINSKI, J. 2017 Pedestal and edge electrostatic turbulence characteristics from an XGC1 gyrokinetic simulation. *Plasma Phys. Control. Fusion* **59**, 105014.
- CONNOR, J. W., HASTIE, R. J., WILSON, H. R. & MILLER, R. L. 1998 Magnetohydrodynamic stability of tokamak edge plasmas. *Phys. Plasmas* **5**, 2687.
- COWLEY, S. C., COWLEY, B., HENNEBERG, S. A. & WILSON, H. R. 2015 Explosive instability and erupting flux tubes in a magnetized plasma. *Proc. R. Soc. Lond. A* **471**, 20140913.
- DICKINSON, D., ROACH, C. M., SAARELMA, S., SCANNELL, R., KIRK, A. & WILSON, H. R. 2012 Kinetic instabilities that limit β in the edge of a tokamak plasma: a picture of an H-mode pedestal. *Phys. Rev. Lett.* **108**, 135002.
- DICKINSON, D., ROACH, C. M., SAARELMA, S., SCANNELL, R., KIRK, A. & WILSON, H. R. 2013 Microtearing modes at the top of the pedestal. *Plasma Phys. Control. Fusion* **55**, 074006.
- DHAESELEER, W. D., HITCHON, W. N. G., CALLEN, J. D. & SHOHET, J. L. 1991 *Flux Coordinates and Magnetic Field Structure*. Springer.
- FREIDBERG, J. P. 2014 *Ideal MHD*. Cambridge University Press.
- FRIEMAN, E. A. & CHEN, L. 1982 Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria. *Phys. Fluids* **25**, 502.
- GOLDSTON, R. J. 2012 Heuristic drift-based model of the power scrape-off width in low-gas-puff H-mode tokamaks. *Nucl. Fusion* **52** (1), 013009.
- HAM, C. J., COWLEY, S. C., BROCHARD, G. & WILSON, H. R. 2016 Nonlinear stability and saturation of ballooning modes in tokamaks. *Phys. Rev. Lett.* **116**, 235001.
- HAMMETT, G. W., BEER, M. A., DORLAND, W., COWLEY, S. C. & SMITH, S. A. 1993 Developments in the gyrofluid approach to tokamak turbulence simulations. *Plasma Phys. Control. Fusion* **35**, 973.
- HASSAM, A. B., ANTONSEN, T. M., DRAKE, J. F. & LIU, C. S. 1991 Spontaneous poloidal spin-up of tokamaks and the transition to the H mode. *Phys. Rev. Lett.* **66**, 309.

- HATCH, D. R., KOTSCHENREUTHER, M., MAHAJAN, S., VALANJU, P., JENKO, F., TOLD, D., G??RLER, T. & SAARELMA, S. 2016 Microtearing turbulence limiting the JET-ILW pedestal. *Nucl. Fusion* **56**, 104003.
- HATCH, D. R., KOTSCHENREUTHER, M., MAHAJAN, S., VALANJU, P. & LIU, X. 2017 A gyrokinetic perspective on the JET-ILW pedestal. *Nucl. Fusion* **57**, 036020.
- HAZELTINE, R. D. & MEISS, J. D. 2003 *Plasma Confinement*. Dover Publications.
- HILLESHEIM, J. C., DICKINSON, D., ROACH, C. M., SAARELMA, S., SCANNELL, R., KIRK, A., CROCKER, N. A., PEEBLES, W. A., MEYER, H. & THE MAST TEAM 2016 Intermediate- k density and magnetic field fluctuations during inter-ELM pedestal evolution in MAST. *Plasma Phys. Control. Fusion* **58**, 014020.
- JACKSON, E. A. 1960 Drift instabilities in a Maxwellian plasma. *Phys. Fluids* **3**, 786.
- JORGE, R., RICCI, P. & LOUREIRO, N. F. 2017 A drift-kinetic analytical model for scrape-off layer plasma dynamics at arbitrary collisionality. *J. Plasma Phys.* **83** (6), 905830606.
- KAGAN, G. & CATTO, P. J. 2008 Arbitrary poloidal gyroradius effects in tokamak pedestals and transport barriers. *Plasma Phys. Control. Fusion* **50**, 085010.
- KAGAN, G. & CATTO, P. J. 2010 Neoclassical ion heat flux and poloidal flow in a tokamak pedestal. *Plasma Phys. Control. Fusion* **52**, 055004.
- KOTSCHENREUTHER, M., HATCH, D. R., MAHAJAN, S., VALANJU, P., ZHENG, L. & LIU, X. 2017 Pedestal transport in H-mode plasmas for fusion gain. *Nucl. Fusion* **57**, 064001.
- KRUSKAL, M. 1958 The gyration of a charged particle. *Project Matterhorn Report* PM-S-33.
- KU, S., CHANG, C.-S., ADAMS, M., CUMMINGS, J., HINTON, F., KEYES, D., KLASKY, S., LEE, W., LIN, Z., PARKER, S. *et al.* & THE CPES TEAM 2006 Gyrokinetic particle simulation of neoclassical transport in the pedestal/scrape-off region of a tokamak plasma. *J. Phys.: Conf. Ser.* **46**, 87.
- MILITELLO, F. & FUNDAMENSKI, W. 2011 Multi-machine comparison of drift fluid dimensionless parameters. *Plasma Phys. Control. Fusion* **53**, 095002.
- NEWCOMB, W. A. 1958 Motion of magnetic lines of force. *Ann. Phys.* **3**, 347.
- PARRA, F. I., BARNES, M. & PEETERS, A. G. 2011 Up-down symmetry of the turbulent transport of toroidal angular momentum in tokamaks. *Phys. Plasmas* **18**, 062501.
- PARRA, F. I. & CATTO, P. J. 2008 Limitations of gyrokinetics on transport time scales. *Plasma Phys. Control. Fusion* **50**, 065014.
- RASMUSSEN, J. J., NIELSEN, A. H., MADSEN, J., NAULIN, V. & XU, G. S. 2016 Numerical modeling of the transition from low to high confinement in magnetically confined plasma. *Plasma Phys. Control. Fusion* **58**, 014031.
- RICCI, P., HALPERN, F. D., JOLLIET, S., LOIZU, J., MOSETTO, A., FASOLI, A., FURNO, I. & THEILER, C. 2012 Simulation of plasma turbulence in scrape-off layer conditions: the GBS code, simulation results and code validation. *Plasma Phys. Control. Fusion* **54**, 124047.
- ROGERS, B. N. & DRAKE, J. F. 1999 Diamagnetic stabilization of ideal ballooning modes in the edge pedestal. *Phys. Plasmas* **6**, 2797.
- ROSENBLUTH, M. N. & TAYLOR, J. B. 1969 Plasma diffusion and stability in toroidal systems. *Phys. Rev. Lett.* **23**, 367.
- SAARELMA, S., BEURSKENS, M. N. A., DICKINSON, D., FRASSINETTI, L., LEYLAND, M. J., ROACH, C. M. & CONTRIBUTORS EFDA-JET 2013 MHD and gyro-kinetic stability of JET pedestals. *Nucl. Fusion* **53**, 123012.
- SCHNEIDER, R., BONNIN, X., BORRASS, K., COSTER, D. P., KASTELEWICZ, H., REITER, D., ROZHANSKY, V. A. & BRAAMS, B. J. 2006 Plasma edge physics with B2-Eirene. *Contrib. Plasma Phys.* **46**, 3.
- SCOTT, B. D. 1997 Three-dimensional computation of drift alfvén turbulence. *Plasma Phys. Control. Fusion* **39**, 1635.
- SCOTT, B. D. 2007 Tokamak edge turbulence: background theory and computation. *Plasma Phys. Control. Fusion* **49**, S25.
- SHI, E. L., HAMMETT, G. W., STOLTZFUS-DUECK, T. & HAKIM, A. 2017 Gyrokinetic continuum simulation of turbulence in a straight open-field-line plasma. *J. Plasma Phys.* **83**, 905830304.

- SIMAKOV, A. N. & CATTO, P. J. 2003 Drift-ordered fluid equations for field-aligned modes in low- β collisional plasma with equilibrium pressure pedestals. *Phys. Plasmas* **10**, 4744.
- SNIPES, J. A., LABOMBARD, B., GREENWALD, M., HUTCHINSON, I. H., IRBY, J., LIN, Y., MAZURENKO, A. & PORKOLAB, M. 2001 The quasi-coherent signature of enhanced $D\alpha$ H-mode in Alcator C-Mod. *Plasma Phys. Control. Fusion* **43**, L23.
- SNYDER, P. B., BURRELL, K. H., WILSON, H. R., CHU, M. S., FENSTERMACHER, M. E., LEONARD, A. W., MOYER, R. A., OSBORNE, T. H., UMANSKY, M., WEST, W. P. *et al.* 2007 Stability and dynamics of the edge pedestal in the low collisionality regime: physics mechanisms for steady-state elm-free operation. *Nucl. Fusion* **47**, 961.
- SNYDER, P. B., GROEBNER, R. J., HUGHES, J. W., OSBORNE, T. H., BEURSKENS, M., LEONARD, A. W., WILSON, H. R. & XU, X. Q. 2011 A first-principles predictive model of the pedestal height and width: development, testing and iter optimization with the eped model. *Nucl. Fusion* **51**, 103016.
- SNYDER, P. B., WILSON, H. R., FERRON, J. R., LAO, L. L., LEONARD, A. W., OSBORNE, T. H., TURNBULL, A. D., MOSSESIAN, D., MURAKAMI, M. & XU, X. Q. 2002 Edge localized modes and the pedestal: a model based on coupled peeling–ballooning modes. *Phys. Plasmas* **9**, 2037.
- STRINGER, T. E. 1969 Diffusion in toroidal plasmas with radial electric field. *Phys. Rev. Lett.* **22**, 770.
- SUGAMA, H., OKAMOTO, M., HORTON, W. & WAKATANI, M. 1996 Transport processes and entropy production in toroidal plasmas with gyrokinetic electromagnetic turbulence. *Phys. Plasmas* **3**, 2379.
- TANG, W. M., CONNOR, J. W. & HASTIE, R. J. 1980 Kinetic-ballooning-mode theory in general geometry. *Nucl. Fusion* **20**, 1439.
- THE ASDEX TEAM 1989 The H-Mode of ASDEX. *Nucl. Fusion* **29**, 1959.
- WAGNER, F., BECKER, G., BEHRINGER, K., CAMPBELL, D., EBERHAGEN, A., ENGELHARDT, W., FUSSMANN, G., GEHRE, O., GERNHARDT, J., GIERKE, G. v. *et al.* 1982 Regime of improved confinement and high beta in neutral-beam-heated divertor discharges of the asdex tokamak. *Phys. Rev. Lett.* **49**, 1408.
- WILSON, H. R., SNYDER, P. B., HUYSMANS, G. T. A. & MILLER, R. L. 2002 Numerical studies of edge localized instabilities in tokamaks. *Phys. Plasmas* **9**, 1277.
- XU, X. Q., DUDSON, B. D., SNYDER, P. B., UMANSKY, M. V., WILSON, H. R. & CASPER, T. 2011 Nonlinear elm simulations based on a nonideal peelingballooning model using the BOUT++ code. *Nucl. Fusion* **51** (10), 103040.
- ZEILER, A., DRAKE, J. F. & ROGERS, B. 1997 Nonlinear reduced Braginskii equations with ion thermal dynamics in toroidal plasma. *Phys. Plasmas* **4**, 2134.
- ZOCCO, A., LOUREIRO, N. F., DICKINSON, D., NUMATA, R. & ROACH, C. M. 2015 Kinetic microtearing modes and reconnecting modes in strongly magnetised slab plasmas. *Plasma Phys. Control. Fusion* **57**, 065008.
- ZOCCO, A. & SCHEKOCHIHIN, A. A. 2011 Reduced fluid-kinetic equations for low-frequency dynamics, magnetic reconnection and electron heating in low-beta plasmas. *Phys. Plasmas* **18**, 102309.
- ZOHN, H. 1996 Edge localized modes (ELMs). *Plasma Phys. Control. Fusion* **38**, 105.