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NUMBERS OF CONJUGACY CLASS SIZES AND DERIVED LENGTHS FOR A-GROUPS

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ABSTRACT. An A-group is a finite solvable group all of whose Sylow subgroups are abelian. In this paper, we are interested in bounding the derived length of an A-group G as a function of the number of distinct sizes of the conjugacy classes of G. Although we do not find a specific bound of this type, we do prove that such a bound exists. We also prove that if G is an A-group with a faithful and completely reducible G-module V, then the derived length of G is bounded by a function of the number of distinct orbit sizes under the action of G on V.

1. **Introduction.** The concern in this paper is with finite solvable groups all of whose Sylow subgroups are abelian. Such groups will be referred to as A-groups. We wish to find, for an A-group G, a bound on the derived length of G as a function of the number of distinct sizes of the conjugacy classes of G. Although we do not find a specific bound of this type, we do prove that such a bound exists, as stated in Theorem B. Here, we use the symbol $d\ell(G)$ to denote the derived length of G. Also, we write $\mathrm{cs}(G)$ to denote the set of all conjugacy class sizes of G, that is,

$$\mathrm{cs}(G) = \{|G : \mathbf{C}_G(x)| \mid x \in G\}.$$

THEOREM B. *There exists a function $g: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ such that*

$$d\ell(G) \leq g\big(|\,\mathrm{cs}(G)|\big)$$

for every A-group G.

The following result is also proved, and it is the key to our proof of Theorem B.

THEOREM A. *There exists a function $f: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ for which the following holds: If G is an A-group and V is a faithful and completely reducible G-module, then $d\ell(G) \leq f(b)$, where b is the number of distinct orbit sizes under the action of G on V.*

2. **Preliminary Lemmas.** The first two lemmas appear as Hilfssatz 14.17 and Satz 14.18a in [2].

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LEMMA 1. *Suppose G is an A-group and a subgroup of* $\mathrm{GL}(n,F)$, *where n is an integer and F is a field whose characteristic does not divide $|G|$. Then $d\ell(G) \leq n$.*

PROOF. Note that if K is a field containing F, then $G \subseteq \mathrm{GL}(n,K)$. Thus, we may replace F by its algebraic closure and assume that F is algebraically closed. We induct on n. If $n = 1$, then $G \subseteq F^{\times}$ and so G is abelian and $d\ell(G) \leq 1$, as needed. Thus, we may assume that $n \geq 2$.

Since 1_G is an F-representation of G with F algebraically closed and since G is an M-group, we know that every element of G is a monomial matrix. For each $g \in G$, define $\mathcal{N}(g) \in \mathrm{GL}(n,F)$ by

$$\mathcal{N}(g)_{ij} = \begin{cases} 1 & \text{if } g_{ij} \neq 0 \\ 0 & \text{if } g_{ij} = 0. \end{cases}$$

That is, we obtain $\mathcal{N}(g)$ by replacing all nonzero entries of the matrix g by 1's. One can check that $\mathcal{N}\colon G \longrightarrow \mathrm{GL}(n,F)$ is an F-representation of G. Note that if $\{v_1, v_2, \ldots, v_n\}$ is an F-basis for F^n, then for any $g \in G$, multiplication by $\mathcal{N}(g)$ fixes the 1-dimensional subspace $\langle v_1 + v_2 + \cdots + v_n\rangle$ of F^n, since each $\mathcal{N}(g)$ is a monomial matrix. Thus, by possibly replacing $\mathcal{N}$ by a similar F-representation of G, we may assume that there exists an F-representation $\mathcal{N}_1$ of G such that

$$\mathcal{N}(g) = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{N}_1(g) \end{bmatrix}$$

for each $g \in G$. Then $\ker \mathcal{N} = \ker \mathcal{N}_1$ and we see that $G/\ker \mathcal{N} = G/\ker \mathcal{N}_1$ is isomorphic to a subgroup of $\mathrm{GL}(n-1,F)$. By the inductive hypothesis, it follows that $d\ell(G/\ker \mathcal{N}) \leq n-1$. But $\ker \mathcal{N}$ is the set of all diagonal matrices in G, and so $\ker \mathcal{N}$ is abelian. Hence, $d\ell(G) \leq n$. ∎

LEMMA 2. *Suppose G is an A-group and G has no normal elementary abelian subgroup with rank greater than k. Then $d\ell(G) \leq k+1$.*

PROOF. For a prime q, let $B_q = \mathbf{O}_q(G)$ and let $A_q = \Omega_1(B_q)$. Using Fitting's Theorem and the fact that Sylow subgroups of G are abelian, we have $\mathbf{C}_G(B_q) = \mathbf{C}_G(A_q)$. Then $G/\mathbf{C}_G(B_q)$ is isomorphically embedded in $\mathrm{Aut}(A_q)$, which is isomorphic to $\mathrm{GL}\big(\mathrm{rank}(A_q), q\big)$. Also, since a Sylow q-subgroup centralizes B_q, $G/\mathbf{C}_G(B_q)$ must be a q'-group. Thus, by Lemma 1, $d\ell\big(G/\mathbf{C}_G(B_q)\big) \leq \mathrm{rank}(A_q)$. But $\mathrm{rank}(A_q) \leq k$. Therefore, $d\ell\big(G/\mathbf{C}_G(B_q)\big) \leq k$ for all primes q.

We now have $G^{(k)} \subseteq \cap \mathbf{C}_G(B_q)$. But since $\mathbf{F}(G) = \Pi B_q$, we also have $\mathbf{C}_G\big(\mathbf{F}(G)\big) = \cap \mathbf{C}_G(B_q)$. Hence, $G^{(k)} \subseteq \mathbf{C}_G\big(\mathbf{F}(G)\big) = \mathbf{F}(G)$. Finally, since $\mathbf{F}(G)$ is abelian, it follows that $d\ell(G) \leq k+1$. ∎

The final lemma is rather technical and is designed to simplify the proof of Theorem A.

LEMMA 3. *Suppose A is an elementary abelian q-group with* $\mathrm{rank}(A) > k$. *Suppose $\{K_i \mid 1 \leq i \leq r\}$ is a collection of subgroups of A with A/K_i cyclic for each i. Then no subcollection of k or fewer of the K_i's can intersect trivially.*

PROOF. Assume $\ell \leq k$ and that a set of ℓ of the K_i's intersect trivially. Without loss of generality, assume that $K_1 \cap K_1 \cap \cdots \cap K_\ell = 1$. For each $t \in \{1, 2, \ldots, \ell\}$, let

$N_t = K_1 \cap \cdots \cap K_t$. Note that since A/K_i is both cyclic and elementary abelian, we must have $|A : K_i| \leq q$ for each $i \in \{1, 2, \ldots, \ell\}$. Now, $|A : N_1| = |A : K_1| \leq q$. Also, for $t \in \{1, 2, \ldots, \ell - 1\}$, we have

$$|N_t : N_{t+1}| = |N_t : N_t \cap K_{t+1}| = |N_t K_{t+1} : K_{t+1}|.$$

But since $K_{t+1} \subseteq N_t K_{t+1} \subseteq A$, this implies that $|N_t : N_{t+1}| \leq |A : K_{t+1}| \leq q$ for each $t \in \{1, 2, \ldots, \ell - 1\}$. Therefore, since

$$1 = N_\ell \subseteq N_{\ell-1} \subseteq \cdots \subseteq N_2 \subseteq N_1 \subseteq A,$$

we have

$$q^k < |A| = |A : N_1||N_1 : N_2| \cdots |N_{\ell-1} : N_\ell| \leq q^\ell.$$

Hence, $\ell > k$. This is a contradiction, as needed, and the lemma is proved. ∎

3. **Proofs of Theorems.** As a tool to be used in the proof of Theorem A, we first define a sequence of positive integers,

$$1 = f(1) < f(2) < \cdots,$$

as follows. Put $f(1) = 1$. Then, whenever $f(i) \geq 1$ is given, note that the exponential function

$$2^{x - f(i)}$$

grows faster than the polynomial function

$$\binom{x}{f(i)}.$$

Thus, given $f(i) \geq 1$, we can define $f(i+1)$ to be the smallest integer such that $f(i+1) > f(i)$ and

$$2^{f(i+1) - f(i)} > \binom{f(i+1)}{f(i)}.$$

This defines a strictly increasing function $f \colon \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ to be used in the proof of Theorem A.

PROOF OF THEOREM A. Let f be the function defined in the preceding paragraph. Note that if $b = 1$, then G acts both faithfully and trivially on V, which means that $G = 1$ and so $d\ell(G) \leq f(1)$. Thus, we may assume $b \geq 2$.

We claim that it is no loss to assume that V is an irreducible G-module. To see this, write $V = W_1 \dot{\times} \cdots \dot{\times} W_k$, where each W_i is an irreducible G-module. Then $G/\mathbf{C}_G(W_i)$ is an A-group and W_i is a faithful irreducible $G/\mathbf{C}_G(W_i)$-module. Also, each orbit size under the action of $G/\mathbf{C}_G(W_i)$ on W_i is also an orbit size under the action of G on V. Hence, assuming the theorem is true in the case that the G-module is irreducible instead of only completely reducible, we have $d\ell\big(G/\mathbf{C}_G(W_i)\big) \leq f(b)$ for all $i \in \{1, 2, \ldots, k\}$.

This means that $G^{(f(b))} \subseteq \bigcap \mathbb{C}_G(W_i) = \mathbb{C}_G(V) = 1$. Hence, $d\ell(G) \leq f(b)$, as needed. Therefore, we may assume that V is irreducible as a G-module.

Note that $\mathbb{C}_V(G)$ is a G-submodule of V. Thus, since V is irreducible, either $\mathbb{C}_V(G) = 1$ or $\mathbb{C}_V(G) = V$. But if $\mathbb{C}_V(G) = V$, then $G = \mathbb{C}_G(V) = 1$ since the action of G on V is faithful. Hence, we have $\mathbb{C}_V(G) = 1$.

Note that if G has no normal elementary abelian subgroup of rank greater than $f(b)-1$, then $d\ell(G) \leq f(b)$ by Lemma 2, and we're done. Thus, we may assume that there exists a normal elementary abelian q-subgroup A of G with $\mathrm{rank}(A) > f(b) - 1$ for some prime q. We will see that this leads to a contradiction.

By Clifford's theorem and since $A \lhd G$, we have

$$V = V_1 \dot{\times} \cdots \dot{\times} V_r,$$

where $\{V_1, \ldots, V_r\}$ are the A-isotypic components of V, and where G transitively permutes $\{V_1, \ldots, V_r\}$. For $i \in \{1, 2, \ldots, r\}$, let $K_i = \mathbb{C}_A(V_i)$, and note that since A/K_i is abelian and since any A-simple submodule of V_i is a faithful irreducible A/K_i-module, we know that each A/K_i is cyclic.

Since $\mathrm{rank}(A) > f(b) - 1$, we know by Lemma 3 that no collection of $f(b) - 1$ or fewer of the K_i's can intersect trivially. Since $\bigcap_{i=1}^r K_i = 1$, we must have $f(b) - 1 \leq r - 1$. Also, since $b \geq 2$, we have $f(b) - 1 \geq f(2) - 1 \geq 2 - 1 = 1$, which means that $K_1 \neq 1$. Hence, since $\bigcap_{i=1}^r K_i = 1$, there exists some K_i which does not contain K_1. Without loss of generality we may assume that $K_1 \not\subseteq K_2$. Now, if $2 \leq f(b) - 1$, then $K_1 \cap K_2 \neq 1$ and without loss of generality we may assume that $K_1 \cap K_2 \not\subseteq K_3$. In general, if $i \leq f(b) - 1$, then we may assume that $K_1 \cap K_2 \cap \cdots \cap K_i \not\subseteq K_{i+1}$.

For $i \in \{1, 2, \ldots, f(b)\}$, let $N_i = K_1 \cap \cdots \cap K_i$. Let $x_1 \in V_1 - \{1\}$. For $i \in \{2, 3, \ldots, f(b)\}$, note that $N_{i-1} \subseteq A$ but $N_{i-1} \not\subseteq K_i = \mathbb{C}_A(V_i)$. Thus, for $i \in \{2, \ldots, f(b)\}$, we can choose $x_i \in V_i$ with $N_{i-1} \not\subseteq \mathbb{C}_G(x_i)$.

For $t \in \{1, 2, \ldots, b\}$, let $y_t = x_1 x_2 \cdots x_{f(t)}$, the product of the first $f(t)$ x_i's. We claim that

$$|\mathbb{C}_G(y_1)| > |\mathbb{C}_G(y_2)| > \cdots > |\mathbb{C}_G(y_b)|.$$

Since $\mathbb{C}_V(G) = 1$, this would provide a list,

$$|G : \mathbb{C}_G(y_1)| < |G : \mathbb{C}_G(y_2)| < \cdots < |G : \mathbb{C}_G(y_b)|,$$

of b nontrivial orbit sizes under the action of G on V, which is the contradiction we need to finish the proof.

Fix $t \in \{1, 2, \ldots, b - 1\}$. Let $T = \mathbb{C}_G(y_t) \cap \mathbb{C}_G(y_{t+1})$. Note that $\mathbb{C}_G(y_{t+1})$ acts on $X = \{x_1, x_2, \ldots, x_{f(t+1)}\}$. Thus, $\mathbb{C}_G(y_{t+1})$ also acts on

$$\Omega = \{X_0 \subseteq X : |X_0| = f(t)\},$$

and $T = \mathbb{C}_G(y_{t+1}) \cap \mathbb{C}_G(y_t)$ is the stabilizer of $X_0 = \{x_1, x_2, \ldots, x_{f(t)}\}$ under this action. Hence, $|\mathbb{C}_G(y_{t+1}) : T|$ is an orbit size under the action of $\mathbb{C}_G(y_{t+1})$ on Ω, which means that $|\mathbb{C}_G(y_{t+1}) : T| \leq |\Omega| = \binom{f(t+1)}{f(t)}$.

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Note that

$$N_{f(t)} = K_1 \cap \cdots \cap K_{f(t)} \subseteq \mathbb{C}_G(x_1 x_2 \cdots x_{f(t)}) = \mathbb{C}_G(y_t).$$

Thus, $T \subseteq N_{f(t)}T \subseteq \mathbb{C}_G(y_t)$, and so

$$|\mathbb{C}_G(y_t) : T| \geq |N_{f(t)}T : T| = |N_{f(t)} : N_{f(t)} \cap T|.$$

Also, since $N_{f(t)} \subseteq A \subseteq N_G(V_i)$ for all $i \in \{1, 2, \ldots, r\}$ and since $N_{f(t)} \subseteq \mathbb{C}_G(y_t)$ we have

$$\begin{aligned} N_{f(t)} \cap T &= N_{f(t)} \cap \mathbb{C}_G(y_t) \cap \mathbb{C}_G(y_{t+1}) \\ &= N_{f(t)} \cap \mathbb{C}_G(y_{t+1}) \\ &= N_{f(t)} \cap \mathbb{C}_G(x_1 x_2 \cdots x_{f(t+1)}) \\ &= N_{f(t)} \cap \mathbb{C}_G(x_1) \cap \mathbb{C}_G(x_2) \cap \cdots \cap \mathbb{C}_G(x_{f(t+1)}) \\ &= N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \mathbb{C}_G(x_{f(t)+2}) \cap \cdots \cap \mathbb{C}_G(x_{f(t+1)}). \end{aligned}$$

Now, note that

$$N_{f(t)} > N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}),$$

since otherwise,

$$N_{f(t)} \subseteq \mathbb{C}_G(x_{f(t)+1}),$$

which we know is not the case. Also, we have

$$N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) > N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \mathbb{C}_G(x_{f(t)+2}),$$

since otherwise,

$$N_{f(t)+1} \subseteq N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \subseteq \mathbb{C}_G(x_{f(t)+2}),$$

which we know is false. In general, for $i \in \{2, 3, \ldots, f(t+1) - f(t)\}$, we have

$$N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \cdots \cap \mathbb{C}_G(x_{f(t)+i-1}) > N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \cdots \cap \mathbb{C}_G(x_{f(t)+i}),$$

since otherwise,

$$N_{f(t)+i-1} \subseteq N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \cdots \cap \mathbb{C}_G(x_{f(t)+i-1}) \subseteq \mathbb{C}_G(x_{f(t)+i}),$$

which is false. In summary, we have

$$\begin{aligned} N_{f(t)} &> N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \\ &> N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \mathbb{C}_G(x_{f(t)+2}) \\ &> \cdots \\ &> N_{f(t)} \cap \mathbb{C}_G(x_{f(t)+1}) \cap \cdots \cap \mathbb{C}_G(x_{f(t+1)}) \\ &= N_{f(t)} \cap T, \end{aligned}$$

where there are a total of $f(t+1) - f(t)$ strict inequalities. Therefore,

$$|N_{f(t)} : N_{f(t)} \cap T| \geq 2^{f(t+1)-f(t)}.$$

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Then since $|\mathbf{C}_G(y_t) : T| \geq |N_{f(t)} : N_{f(t)} \cap T|$, we have

$$|\mathbf{C}_G(y_t) : T| \geq 2^{f(t+1)-f(t)}.$$

However, by the definition of $f(t+1)$, we know that $2^{f(t+1)-f(t)} > \binom{f(t+1)}{f(t)}$, and we have already shown that

$$|\mathbf{C}_G(y_{t+1}) : T| \leq \binom{f(t+1)}{f(t)}.$$

Hence, we have $|\mathbf{C}_G(y_t) : T| > |\mathbf{C}_G(y_{t+1}) : T|$, which implies that $|\mathbf{C}_G(y_t)| > |\mathbf{C}_G(y_{t+1})|$, as needed. As mentioned earlier, this provides a contradiction and the theorem is proved. ∎

We are now ready to prove Theorem B, which is a corollary of Theorem A.

PROOF OF THEOREM B. Let f be the function given by Theorem A, and define g by $g(b) = f(b) + 1$ for all $b \in \mathbb{Z}^+$. Let $b = |\operatorname{cs}(G)|$.

For every prime q, let $B_q = \mathbf{O}_q(G)$ and let $A_q = \Omega_1(B_q)$. Then using the fact that a Sylow q-subgroup of G is abelian, we have $\mathbf{C}_G(B_q) = \mathbf{C}_G(A_q)$ for all primes q, by Fitting's Theorem. Write $C_q = \mathbf{C}_G(B_q) = \mathbf{C}_G(A_q)$.

Fix a prime q. Now, A_q is a faithful $F(G/C_q)$-module, where F is the field with q elements. Also, if $Q \in \operatorname{Syl}_q(G)$, then since $B_q \subseteq Q$ and Q is abelian, we have $Q \subseteq \mathbf{C}_G(B_q) = C_q$. Thus, $\operatorname{char}(F) = q$ does not divide $|G/C_q|$, and so Maschke's theorem implies that A_q is completely reducible as a B/C_q-module. Clearly, every orbit size under the action of G/C_q on A_q is also a conjugacy class size of G. Thus, there are no more than b orbit sizes under the action of G/C_q on the faithful and completely G/C_q-module A_q. Therefore, by Theorem A, we have $d\ell(G/C_q) \leq f(b)$.

By the above paragraph, we have $d\ell(G/C_q) \leq f(b)$ for all primes q. That is, $G^{(f(b))} \subseteq C_q$ for every prime q. But since $\mathbf{F}(G) = \Pi B_q$, we have $\mathbf{C}_G\big(\mathbf{F}(G)\big) = \cap \mathbf{C}_G(B_q) = \cap C_q$. Hence,

$$G^{(f(b))} \subseteq \bigcap C_q = \mathbf{C}_G\big(\mathbf{F}(G)\big) = \mathbf{F}(G).$$

Finally, since $\mathbf{F}(G)$ is abelian, we conclude that $G^{(f(b)+1)} = 1$. That is, $d\ell(G) \leq f(b)+1 = g(b)$, as required. ∎

REFERENCES

1. L. Dornhoff, *Group representation theory*, Dekker, New York, 1967.
2. B. Huppert, *Endliche Gruppen I*, Springer Verlag, Berlin, 1967.
3. W. R. Scott, *Group Theory*, Prentice Hall, Englewood Cliffs, New Jersey, 1964.

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