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UNIFORM APPROXIMATION ON THE GRAPH OF A SMOOTH MAP IN $\mathbf{C}^n$

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1. Introduction. Let X be a compact set in $\mathbf{C}^n$, and let $f_1, \ldots, f_m$, $m \geqq n$, be continuous, complex-valued functions on X which have C^1 extensions to some neighborhood of X. We wish to describe the algebra A of continuous complex-valued functions on X which can be approximated uniformly by polynomials in the functions $z_1, \ldots, z_n, f_1, \ldots, f_m$. For this purpose we introduce the sets

$$E = \{z \in X : \text{rank } (\partial f_i / \partial \bar{z}_j) < n\}$$

and

$$\tilde{X} = \{(z, f_1(z), \ldots, f_m(z)) \in \mathbf{C}^{n+m} : z \in X\}.$$

Our description of the algebra A is given by the following theorem:

THEOREM. *Assume $\tilde{X}$ is a polynomially convex subset of $\mathbf{C}^{n+m}$. Then A consists of those continuous functions on X which agree with some element of A on E.*

The first result of this type was proved by Wermer [**6**] for the case $n = m = 1$. He obtained the substantially stronger conclusion that A consists of those continuous functions which can be approximated uniformly on E by rational functions with no poles on E.

Proofs of this theorem in the more general setting of functions defined on a manifold were obtained by Freeman [**2**] in the real-analytic case and by Fornaess [**1**] for the case when the functions and the manifold are differentiable of sufficiently high order. The case $n = 1$, m arbitrary, E empty is presented in [**7**]. When E is empty the theorem is a special case of the theorem that every continuous function on a compact subset of a totally real C^1 submanifold of $\mathbf{C}^n$ is the uniform limit of holomorphic functions. This result was proved by Harvey and Wells [**3**]. The methods used in the present paper are more elementary than those of Harvey and Wells, since no use is made of uniform estimates for the Cauchy-Riemann operator.

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lecture at the International Conference on Complex Analysis, Laval University, July 1978. After the present manuscript was completed the author received a preprint "Integral kernels and approximation on totally real submanifolds of C^n" by Bo Berndtsson, Chalmers University of Technology and the University of Göteborg, which contains an interesting new proof of the Harvey-Wells Theorem cited above by methods related to those employed here.

The proof presented here is a generalization of Wermer's original proof in [**6**]. The essential idea is to replace Wermer's use of the Cauchy integral formula by a suitable Cauchy-Fantappiè kernel, in somewhat the same manner as in the author's earlier proof in [**5**] of a local version of the theorem in the case E is empty.

2. Construction of the Cauchy-Fantappiè kernel. It suffices to show that if μ is a complex Borel measure on X such that $\int f d\mu = 0$ for all $f \in A$ then $\mu = 0$ on $X - E$, or equivalently, that each point of $X - E$ has a neighborhood U in $\mathbf{C}^n$ such that

$$\int \phi(z) d\mu(z) = 0$$

for all $\phi \in C_0^\infty(U)$.

Let $G_1, \ldots, G_n \in C^1(U \times M)$ where M is an open neighborhood of X containing U. Define G on $U \times M$ by

$$G(\zeta, z) = \sum (\zeta_j - z_j) G_j(\zeta, z).$$

Suppose that

(1) $G(\zeta, z)$ vanishes only when $\zeta = z$

and

(2) for each j, $G_j(\cdot, z) G(\cdot, z)^{-n}$ belongs to L_{loc}^1, uniformly for z in compact subsets of M.

If we define $\Omega(\zeta, z)$ by

$$\Omega(\zeta, z) = (n-1)!(2\pi i)^{-n} G(\zeta, z)^{-n} \sum (-1)^j G_j(\zeta, z) \bigwedge_{k \neq j} \bar{\partial}_\zeta G_k \wedge \alpha$$

where $\alpha = d\zeta_1 \wedge \ldots \wedge d\zeta_n$ then it is well-known (cf. [**5**]) that every $\phi \in C_0^\infty(U)$ admits the representation

$$\phi(z) = \int \Omega(\zeta, z) \wedge \bar{\partial} \phi(\zeta)$$

with equality for all $z \in M$ (that is, the right side vanishes also for $z \in M - U$). If we rewrite $\Omega(\zeta, z)$ in the form

$$\Omega(\zeta, z) = \sum K_j(\zeta, z) \bigwedge_{k \neq j} d\bar{\zeta}_k \wedge \alpha$$

then we conclude from Fubini's theorem that

$$\int \phi(z)d\mu(z) = \int \left| \sum \int K_j(\zeta, z)d\mu(z) \right| \bar{\partial}\phi(\zeta) \wedge_{k\neq j} d\bar{z}_k \wedge \alpha$$

where the z-integration is over U.

Thus we will have proved the theorem if we can construct $G_1, \ldots, G_n$ satisfying (1) and (2) and such that for almost all $z \in U$,

$$\text{(3)} \quad \int K_j(\zeta, z)d\mu(z) = 0.$$

Fix $p \in X - E$. Without loss of generality we may assume that the principal $n \times n$ submatrix of $(\partial f_i/\partial\bar{z}_j(p))$ is non-singular. Let $T(p)$ denote this submatrix, and let $S(p)$ denote the corresponding submatrix of $(\partial f_i/\partial z_j(p))$. If $w \in \mathbf{C}^m$ we let $w' = (w_1, \ldots, w_n)$. Similarly $f' = (f_1, \ldots, f_n)$.

Define $g(\zeta, z, w)$ by

$$g(\zeta, z, w) = T(p)^{-1}(f'(\zeta) - w' - S(p)(\zeta - z)).$$

LEMMA 1. *There is a neighborhood U_1 of p such that if $\zeta, z \in U_1$ then*

$$|g(\zeta, z, f(z)) - (\bar{\zeta} - \bar{z})| < \tfrac{3}{4}|\zeta - z|.$$

Proof. Define $R(\zeta, z)$ by

$$f'(\zeta) = f'(z) + S(z)(\zeta - z) + T(z)(\bar{\zeta} - \bar{z}) + R(\zeta, z).$$

Let $C = \|T(p)^{-1}\|$. Choose a neighborhood V of p such that

$$\|S(z) - S(p)\| < (4C)^{-1} \quad \text{and} \quad \|T(z) - T(p)\| < (4C)^{-1} \quad \text{if} \quad z \in V.$$

Choose $\epsilon > 0$ such that $|R(\zeta, z)| < (4C)^{-1}|\zeta - z|$ if $\zeta, z \in V$ and $|\zeta - z| < \epsilon$. Let

$$U_1 = V \cap \{|\zeta - p| < \epsilon/2\}.$$

Then

$$\begin{aligned} &|g(\zeta, z, f(z)) - (\bar{\zeta} - \bar{z})| = \\ &|T(p)^{-1}\{R(\zeta, z) + (S(z) - S(p))(\zeta - z) + (T(z) - T(p))(\bar{\zeta} - \bar{z})\}| \\ &\leqq 3/4|\zeta - z|. \end{aligned}$$

COROLLARY. *Let $\Gamma(\zeta, z, w) = (\zeta - z)\cdot g(\zeta, z, w)$ where $\alpha\cdot\beta$ denotes the standard bilinear form on $\mathbf{C}^n$;*

(i) *Γ is holomorphic in z and w for fixed ζ, and Γ is of class C^1*

(ii) $|\Gamma(\zeta, z, f(z))| \geqq 1/4|\zeta - z|^2 \quad \zeta, z \in U_1$

(iii) Re $\Gamma(\zeta, z, f(z)) > 0$ *if* $\zeta \neq z \quad \zeta, z \in U_1$

(iv) $|\Gamma(\zeta, z, f(z))| \leqq 7/4|\zeta - z|^2 \quad \zeta, z \in U_1.$

Since $\tilde{X}$ is polynomially convex we can find a neighborhood $\tilde{M}$ of $\tilde{X}$ which is a domain of holomorphy and open subsets V, W of $\tilde{M}$ with the following properties:

(a) $\{V, W\}$ is an open convering of $\tilde{M}$
(b) if $z \in U_1$ then $(z, f(z)) \in V$
(c) there is an open neighborhood U_2 of p, $U_2 \subset U_1$, such that $z \in U_1$ and $(z, f(z)) \in V \cap W$ imply $z \notin U_2$
(d) Re $\Gamma(\zeta, z, w) > 0$ on $U_2 \times (V \cap W)$.

For fixed $\zeta \in U_2$, $\log \Gamma$ is holomorphic on $V \cap W$. By [**4**, Proposition 2] there exist C^1 functions P on $U_2 \times V$ and Q on $U_2 \times W$ which are holomorphic in V and W respectively for fixed $\zeta \in U_2$ and which satisfy

$$\log \Gamma = Q - P \text{ on } U_2 \times (V \cap W).$$

If we now define $\tilde{G}(\zeta, z, w)$ on $U_2 \times \tilde{M}$ to be e^Q on $U_2 \times W$ and Γe^P on $U_2 \times V$ then $\tilde{G}$ is (well-defined and) holomorphic in M for fixed $\zeta \in U_2$ and $\tilde{G}$ is of class C^1 in $U_2 \times \tilde{M}$.

Furthermore, we may assume with no loss of generality that $P(p, p, f(p)) = 0$ and that therefore

$$|e^{P(\zeta, z, w)} - 1| < 1/\sqrt{2}$$

on some neighborhood of $(p, p, f(p))$ of the form $U_3 \times U_3 \times Z$ where $U_3 \subset U_2$. Thus, if $(\zeta, z) \in U_3 \times U_3$ and $\zeta \neq z$,

$$|\tilde{G}(\zeta, z, f(z)) - \Gamma(\zeta, z, f(z))| < 2^{-1/2}|\Gamma(\zeta, z, f(z)|.$$

Since on $U_2 \times V$ the function $\tilde{G}$ vanishes only where Γ does, and since $\tilde{G}$ is nowhere zero on $U_2 \times W$ we have the following result which we record as Lemma 2 for easy reference.

LEMMA 2. *There exists* $\epsilon > 0$ *such that if* $(\zeta, z) \in U_3 \times U_3$ *then* $\tilde{G}(\zeta, z, f(z))$ *lies in the circular sector* $3/4\pi \leqq \theta \leqq 5/4$, $0 \leqq r \leqq \epsilon$ *only if* $\tilde{G}(\zeta, z, f(z)) = 0$.

Let M be a neighborhood of X such that $z \in M$ implies $(z, f(z)) \in \tilde{M}$. Define $G(\zeta, z)$ on $U_3 \times M$ by $G(\zeta, z) = \tilde{G}(\zeta, z, f(z))$.

LEMMA 3. *There exist* $G_1, \ldots, G_n \in C^1(U_3 \times M)$ *such that*
(i) $G(\zeta, z) = \sum(\zeta_j - z_j)G_j(\zeta, z)$
(ii) *for fixed* $\zeta \in U_3$, $G_j(\zeta, \cdot) \in A$, $1 \leqq j \leqq n$
(iii) $|G_j(\zeta, z)| \leqq C|\zeta - z|$ *if* $\zeta, z \in U_3$.
Furthermore,
(iv) $\lambda > 0$ *such that* $|G(\zeta, z)| \geqq \lambda|\zeta - z|^2$ *for* $(\zeta, z) \in U_3 \times M$.

Proof. By [**4**, Proposition 4] we can find functions $R_1, \ldots, R_n, S_1, \ldots, S_n$ of class C^1 on $U_3 \times (\tilde{M} \times \tilde{M})$, holomorphic in $\tilde{M} \times \tilde{M}$ for fixed $\zeta \in U_3$,

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such that

$$\tilde{G}(\zeta, z, w) - \tilde{G}(\zeta, z', w') = \sum (z_j - z_j')R_j(\zeta, z, w, z', w')$$
$$+ \sum (w_j - w_j')S_j(\zeta, z, w, z', w')$$

for all $\zeta \in U_3$ and all (z, w), $(z', w') \in \tilde{M}$. Let $w = f(z)$, $z' = \zeta$, and $w' = f(\zeta)$, and define

$$G_j(\zeta, z) = -R_j(\zeta, z, f(z), \zeta, f(\zeta)).$$

For fixed ζ, $R_j(\zeta, z, w, \zeta, f(\zeta))$ is holomorphic on $\tilde{M}$, hence is the uniform limit on $\tilde{X}$ of a sequence of polynomials by the Oka-Weil theorem. Consequently, for fixed $\zeta \in U_3$, $G_j(\zeta, z)$ is the uniform limit on X of a sequence of polynomials in z and $f(z)$. Since $\tilde{G}(\zeta, \zeta, f(\zeta)) = 0$ if $\zeta \in U_3$ we have established (i) and (ii).

To prove (iii), observe that in $U_3 \times U_3$ we have

$$|G(\zeta, z)| \leqq C|\zeta - z|^2$$

by the Corollary to Lemma 1. It follows from Taylor's theorem that

$$|G_j(\zeta, z)| \leqq C|\zeta - z|$$

for some constant $c > 0$.

Finally, if V is a small neighborhood of $\bar{U}_3$ then $G(\zeta, z)|\zeta - z|^{-2}$ is bounded below on $U_3 \times (M - V)$, while on $U_3 \times U_3$,

$$G(\zeta, z) = e^P \cdot \Gamma(\zeta, z, f(z))$$

which is bounded below by a multiple of $|\zeta - z|^2$ by the Corollary to Lemma 1.

3. Proof of the theorem. The function $G(\zeta, z) = \sum (\zeta_j - z_j)G_j(\zeta, z)$ defined on $U_3 \times M$ vanishes only when $\zeta = z$ and, by the Corollary to Lemma 1,

$$|G_j(\zeta, z)G(\zeta, z)^{-n}| \leqq C|\zeta - z|^{1-2n}$$

hence, if $\Omega(\zeta, z)$ denotes the Cauchy-Fantappiè form constructed above using the functions G_j, and if K_j is defined as above then, if $E \subset U_3$ and $F \subset M$ are compact,

$$\sup_{z \in F} \int_E |K_j(\zeta, z)| dm(\zeta) < \infty$$

where dm denotes Lebesgue measure on $\mathbf{C}^n$. Hence

$$\int_F \int_E |K_j(\zeta, z)| dm(\zeta) d|\mu|(z)$$

is finite, so by Fubini's theorem,

$$(*) \quad \int |K_j(\zeta, z)| d|\mu|(z) < \infty$$

for almost all ζ in U_3.

LEMMA 4. *Fix $\zeta \in U_3$ such that (*) holds. There exist functions $H_j(\lambda)$, holomorphic on a neighborhood of $\{G(\zeta, z) : z \in X\}$ such that*
(i) $|H_\nu(\lambda)| \leqq 3/|\lambda|$
(ii) $H_\nu(\lambda) \to 1/\lambda \quad \lambda \neq 0$.

Proof. This follows as in [**6**, Lemma 3] in view of Lemma 2 above.

Each of the functions K_j is the product of $G_j \cdot G^{-n}$ with some ζ-derivatives of the functions G_k. Since the ζ-derivatives of the functions $R_j(\zeta, z, w, \zeta, f(\zeta))$ of the previous section are also holomorphic in z and w, it follows from the Oka-Weil theorem once again that the ζ-derivatives of each function G_k belong to A. Moreover, on some neighborhood of $\tilde{X}$ the functions $H_\nu(G(\zeta, \cdot, \cdot))$ are holomorphic, hence $H_\nu(G(\zeta, \cdot, \cdot))$ is the uniform limit on $\tilde{X}$ of polynomials in z and w, so that $H_\nu(G(\zeta, z, f(z))$ is in A. By (i) and (ii) and the remarks preceding Lemma 4,

$$H_\nu(G(\zeta, z, f(z)) \in L^1(d|\mu|(z))$$

and consequently, for each j,

$$K_j G^n H^n \in L^1(d|\mu|).$$

Since $K_j G^n H_\nu^n \to K_j$, and since $|K_j G^n H_\nu^n| \leqq 3|K_j|$,

$$\int K_j(\zeta, z) d\mu(z) = \lim_\nu \int K_j G^n H_\nu^n d\mu(z).$$

But $K_j G^n H_\nu^n \in A$. Thus each integral on the right is zero. This completes the proof of the theorem.

4. Further remarks. The algebra A is naturally isomorphic to the algebra $P(\tilde{X})$ of those continuous functions on $\tilde{X}$ which can be uniformly approximated by polynomials in z and w. Let $\tilde{E}$ be the set $\{(z, f(z)) : z \in E\}$. In this setting we can rephrase our theorem as follows:

$$P(\tilde{X}) = \{g \in C(X) : g|\tilde{E} \in P(\tilde{E})\}.$$

Thus the problem of approximation on $\tilde{X}$ by polynomials is reduced to the problem of approximation on $\tilde{E}$ by polynomials.

LEMMA 5. *The set $\tilde{E}$ is polynomially convex.*

Proof. This is probably well-known, but for lack of a convenient reference we give the short argument here. Let h be a complex homo-

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morphism of $P(\tilde{E})$. Then h extends to a complex homomorphism of $P(\tilde{X})$ so there exists $x \in \tilde{X}$ with $h(P) = P(x)$ for all polynomials P. If $x \notin \tilde{E}$ there exists a continuous function g on $\tilde{X}$ which vanishes on $\tilde{E}$ but not at x. Then $0 = h(g) = g(x) \neq 0$. Thus $x \in \tilde{E}$, so $\tilde{E}$ is polynomially convex.

COROLLARY. *Suppose $\tilde{E}$ has the property that every continuous function on $\tilde{E}$ is the uniform limit of a sequence of functions holomorphic in a neighborhood of $\tilde{E}$. Then $P(\tilde{X}) = C(\tilde{X})$, i.e., $A = C(X)$.*

Proof. This follows by Lemma 5 and the Oka-Weil theorem.

A natural question to ask now is: Are there reasonable conditions on $f_1, \ldots, f_n$ which imply that $\tilde{E}$ satisfies the hypotheses of the Corollary? The author hopes to treat this question in future work.

Finally, we observe that the hypothesis of polynomial convexity for $\tilde{X}$ is certainly necessary when E is empty, but this is not the case otherwise. The author is indebted to the referee for the following example:

Let $X = \{(z, w) \in \mathbf{C}^2 : |z| = 1, \ \operatorname{Im} w = 0, \ 0 \leqq \operatorname{Re} w \leqq 1\}$. Let $f_1 = (\operatorname{Re} w)^2$ and $f_2 = \bar{z} f_1$. Then $E = \{|z| = 1, \ w = 0\}$ and $\tilde{X}$ is not polynomially convex. Nevertheless $A = \{g \in C(X) : g|E \in A|E\}$. Indeed, since f_1, $f_1 \operatorname{Re} z$, and $f_1 \operatorname{Im} z \in A$, any pair of points of X at least one of which is not in E can be separated by a real-valued function in A. The desired conclusion now follows by a well-known argument. (Cf. A. Browder, Introduction to Function Algebras, remarks following Theorem 2.7.5.)

REFERENCES

1. J. E. Fornaess, *Uniform approximation on manifolds*, Math. Scand. *31* (1972), 166–170.
2. M. Freeman, *Uniform approximation on a real-analytic manifold*, Trans. Amer. Math. Soc. *143* (1969), 545–553.
3. R. Harvey and R. O. Wells, Jr., *Holomorphic approximation and hyperfunction theory on a C^1 totally real submanifold of a complex manifold*, Math. Ann. *197* (1972), 287–318.
4. B. Weinstock, *Inhomogeneous Cauchy-Riemann systems with smooth dependence on parameters*, Duke Math. J. *40* (1973), 307–312.
5. ——— *A new proof of a theorem of Hörmander and Wermer*, Math. Ann. *200* (1976), 59–64.
6. J. Wermer, *Polynomially convex discs*, Math. Ann. *158* (1965), 6–10.
7. ——— *Banach algebras and several complex variables* (Springer-Verlag, New York, 1976).

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