# LOCAL CONNECTEDNESS OF EXTENSION SPACES

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**1. Introduction.** An extension $E^*$ of a topological space $E$ (that is, a space containing $E$ as a dense subspace)[1] determines a family of filters $\mathfrak{S}(u)$ on $E$, given by the traces $U \cap E$ of the neighbourhoods $U \subseteq E^*$ of each $u \in E^* - E$. Many topological properties of an extension $E^*$ of a given space $E$ can be related to properties of these trace filters (as we shall call them) belonging to $E^*$. In this respect, the following condition for filters $\mathfrak{A}$ has proved to be of some interest:

(C) If $O \cup P \in \mathfrak{A}$, $O$ and $P$ disjoint open sets, then either $O \in \mathfrak{A}$ or $P \in \mathfrak{A}$.

If, for instance, the trace filters of a locally connected extension $E^*$ of a simply connected space $E$ fulfil (C), then $E^*$ is also simply connected **(2)**. This statement involves previous knowledge of the local connectedness of $E^*$. In the present note, a simple characterisation in terms of trace filters will be given for the local connectedness of extension spaces whose trace filters satisfy condition (C). This will then enable us to show that certain types of extensions, amongst them the Čech compactification of locally compact spaces which are denumerable at infinity, can never be locally connected.

**2. The principal result.** A filter $\mathfrak{A}$ on a topological space $E$ is called *open*, if it has a basis consisting of open sets. Open filters for which condition (C) holds we shall call *connected*. As one can readily see, condition (C) for open filters is an extension of the concept of connectedness from open sets to open filters. If $E^*$ be an extension of $E$, to any open set $O \subseteq E$ let $\tilde{O}$ be the set of all points $u \in E^* - E$ which satisfy $O \in \mathfrak{S}(u)$, $\mathfrak{S}(u)$ being the trace filter belonging to $u$. *$O \cup \tilde{O}$ is open in $E^*$.* There is at least one open $O^* \subseteq E^*$ for which $O = O^* \cap E$. Obviously, one has $O^* \subseteq O \cup \tilde{O}$, hence it follows that each $x \in O$ is an interior point of $O \cup \tilde{O}$ (in $E^*$). Furthermore, to each $u \in \tilde{O}$ there exists, by definition, an open neighbourhood $V$ in $E^*$ for which $V \cap E \subseteq O$, and again one has $V \subseteq O \cup \tilde{O}$.

The passage from $O$ to $O \cup \tilde{O}$ will be used as the main device in proving the following proposition:

*Let $E^*$ be an extension of $E$ each of whose trace filters is connected. Then $E^*$ is locally connected if and only if $E$ is locally connected and each trace filter has a basis consisting of connected open sets.*

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[1]All topological concepts are used in the sense of Bourbaki **(3)**. All spaces considered here are assumed to be separated (=Hausdorff) spaces.

If $E$ is locally connected and the trace filters $\mathfrak{S}(u)$ have the stated property, it is quite obvious that $E^*$ is locally connected: To each neighbourhood $W$ of $u \in E^*$, there is a connected open set (in $E$) $V \subseteq W \cap E$ in $\mathfrak{S}(u)$ or in the neighbourhood filter $\mathfrak{V}(u)$ of $u$ in $E$ if $u \in E$, and therefore an open neighbourhood $U$ of $u$ in $E^*$ for which $U \cap E = V$ and $U \subseteq W$. As $V$ is dense in $U$, $U$ is also connected.

Now, let $E^*$ be locally connected and $U$ any open set (in $E$) from $\mathfrak{S}(u)$. Then there exists a connected open neighbourhood (in $E^*$) $V \subseteq U \cup \tilde{U}$ of $u$ (as $U \cup \tilde{U}$ is open, hence a neighbourhood of $u$) and $W = V \cap E \subseteq U$ belongs to $\mathfrak{S}(u)$. The set $W \cup \tilde{W}$ contains $V$ and, apart from that, only adherence points of $V$; therefore, the connectedness of $V$ implies that of $W \cup \tilde{W}$. Supposing there were a decomposition $W = O \cup P$ of $W$ into disjoint open sets $O$ and $P$ in $E$. Then it would follow that $\tilde{W} = \tilde{O} \cup \tilde{P}$, because, by hypothesis, $O \cup P \in \mathfrak{S}(v)$ implies $O \in \mathfrak{S}(v)$ or $P \in \mathfrak{S}(v)$ for any $v \in E^* - E$. This would, however, mean that $(O \cup \tilde{O}) \cup (P \cup \tilde{P})$ is a decomposition of $W \cup \tilde{W}$ into open disjoint sets, in contradiction to the connectedness of $W \cup \tilde{W}$. Conseqently, $W$ is a connected set, and as $W \subseteq U$ and $U$ was arbitrary, this shows $\mathfrak{S}(u)$ has a basis consisting of connected open sets. The same argument applied to the neighbourhood filter $\mathfrak{V}(x)$ of each $x \in E$ (instead of the $\mathfrak{S}(u)$) proves that each $\mathfrak{V}(x)$ also has a basis consisting of connected open sets, or, in other words, that $E$ is locally connected.

As we have proved recently **(2)** the maximal open, the maximal regular and the maximal completely regular filters of a space $E$ are connected filters. It is well known that the non-convergent filters in each of these categories form the set of trace filters of certain extensions of $E$: the maximal open filters correspond **(4)** to Katětov's maximal Hausdorff extension $\kappa E$ of $E$; the maximal regular filters, in the case of a regular $E$, to Alexandroff's **(1)** extension $\alpha' E$; the maximal completely regular filters, in the case of a completely regular $E$, to Čech's **(1; 2**, ch. IX, 1, ex. 7**)** compactification $\beta E$ of $E$. As a corollary to the above proposition, we therefore have:

*If $E$ is not locally connected, then $\kappa E$, $\alpha' E$, and $\beta E$ are not locally connected either.*

**3. Application to particular types of spaces.** We now want to prove a similar statement for $\beta E$, $\alpha' E$ and $\kappa E$ in the case of certain types of spaces $E$ which include locally connected spaces as well as others.

Let $E$ be completely regular and suppose there exist denumerably many open sets $O_i \subseteq E$ whose closures are mutually disjoint and have a closed union. *Then $\beta E$ is not locally connected.* In each $O_i$ one can find a descending sequence $O_{i,k} (k = 0, 1, 2, \ldots ; O_{i,0} = O_i)$ of open sets such that for each pair $O_{i,k}$, $O_{i,k+1}$ there exists a continuous function $h_{i,k}$ on $E$, for which

$$0 \leqslant h_{i,k}(x) \leqslant 1, \; x \in E;$$

$h_{i,k}(x) = 1$ on $O_{i,k+1}$ and $h_{i,k}(x) = 0$ outside $O_{i,k}$. Now the sets

$$M_i = \bigcup_{s \geqslant i} O_{s,i}$$

constitute the basis of a completely regular filter $\mathfrak{N}$: one has $M_{i+1} \subseteq M_i$. Furthermore, the function

$$h_i(x) = \begin{cases} h_{s,i}(x), & x \in O_{s,i},\ s \geqslant i \\ 0 & \text{otherwise} \end{cases}$$

is continuous, since

$$\bar{M}_i = \bigcup_{s \geqslant i} \bar{O}_{s,i}$$

(by hypothesis concerning the $O_i$) vanishes outside $M_i$, is equal to 1 on $M_{i+1}$, and assumes only values between 0 and 1. This shows the filter $\mathfrak{N}$ is completely regular.

Now, if $\beta E$ were locally connected, each of the corresponding trace filters (that is, each maximal completely regular filter) would have a basis of connected open sets as proved above. Let $\mathfrak{M} \supseteq \mathfrak{N}$ be maximal completely regular. By Zorn's lemma, or similarly, owing to the compactness of $\beta E$, there exist such $\mathfrak{M}$. Then, as $M_0 \in \mathfrak{M}$, there would exist a connected open set $G \subseteq M_0$ in $\mathfrak{M}$, and as $M_0$ is the union of the disjoint open sets $O_i$, one would have $G \subseteq O_r$ for a certain $r$. This, however, would entail $G \cap M_{r+1} = \phi$, in contradiction to $\mathfrak{N} \subseteq \mathfrak{M}$, which proves $\beta E$ is not locally connected.

In exactly the same way, one obtains the following similar proposition: *If a regular space $E$ contains denumerably many open sets $O_i$ whose closures are mutually disjoint and have a closed union, then the extension $\alpha' E$ of $E$ is not locally connected.* Here one has to construct a regular filter $\mathfrak{N}$ from sequences $O_{i,k}$, $O_{i,0} = O_i$, for which $\overline{O_{i,k+1}} \subseteq O_{i,k}$ holds, and then the proof proceeds as above.

Finally, the same method gives this result: *If a space $E$ contains denumerably many disjoint open sets $O_i$, then the extension $\kappa E$ is not locally connected.* In this case, one need only consider the open filter generated by the sets

$$\bigcup_{i \geqslant n} O_i, \quad (n = 0, 1, 2, \ldots)$$

instead of the filters $\mathfrak{N}$ above.

A class of spaces satisfying the hypothesis required for $E$ in the preceding arguments are the locally compact spaces which are denumerable at infinity. A space $E$ of this type is the union of an ascending sequence $M_i$ $(i = 0, 1, 2, \ldots)$ of open, relatively compact sets for which $\overline{M_i} \subseteq M_{i+1}$. Then $M_{i+1} - \overline{M_i}$ are disjoint open sets and any collection of open sets $O_i$ satisfying $\overline{O_i} \subseteq M_{i+1} - \overline{M_i}$ will have the desired property: for any $x \in \overline{\cup O_i}$ one has $x \in M_s$ for some $s$ and hence $x \in \overline{O_s}$. This means $\cup \overline{O_i}$ is closed.

We have, therefore, the following corollary: *For locally compact spaces $E$ denumerable at infinity, none of the extensions $\beta E$, $\alpha' E$ and $\kappa E$ is locally connected.*

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