A martingale central limit theorem without negligibility conditions

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We obtain sufficient conditions for the convergence of martingale
triangular arrays to infinitely divisible laws with finite
variances, without making the usual assumptions of uniform
asymptotic negligibility. Our results generalise known results
for both the martingale case under a negligibility assumption and
the classical (independence) case without such assumptions.

1.   Introduction

The theory of convergence in law of sums of independent random
variables has recently been enlarged to include the situation when the
summands do not satisfy asymptotic negligibility conditions. In [4]
Dvoretzky raised the problem of how these results carry over to the
situation when the summands are no longer independent. In [1] this problem
was solved for the case when the summands form a martingale difference
sequence and the limit law is gaussian, and we now consider the same
problem when the limit law is infinitely divisible with finite variance.
Our result generalises the martingale results of Brown and Eagleson [3] and
Dvoretzky [4] (Theorem 2), as well as the sufficiency part of a result of
Machis [6] for the independence case without negligibility assumptions.

2.   The limit theorem

Following [3], consider a double array of random variables, which we
take without loss of generality to be a triangular array, whose rows are
martingale difference sequences; that is to say, for each $n = 1, 2, \ldots$,

we have random variables $X_{n1}, \ldots, X_{nn}$ on a probability space $(\Omega, \underline{\underline{I}}, P)$, with sub σ-fields $\underline{\underline{I}}_{n0} \subset \underline{\underline{I}}_{n1} \subset \ldots \subset \underline{\underline{I}}_{nn}$ of $\underline{\underline{I}}$ such that X_{nk} is $\underline{\underline{I}}_{nk}$-measurable and $E\left(X_{nk} \mid \underline{\underline{I}}_{n,k-1}\right) = 0$ almost surely for $k = 1, 2, \ldots, n$. Such arrays are called *martingale triangular arrays*. Let

$$S_{nk} = \sum_{j=1}^{k} X_{nj} \ , \quad \sigma_{nk}^2 = E\left(X_{nk}^2\right) \ ,$$

$$\tilde{\sigma}_{nk}^2 = E\left(X_{nk}^2 \mid \underline{\underline{I}}_{n,k-1}\right) \ .$$

Let $K(x)$ be a bounded nondecreasing function, which we take as fixed for the remainder of the paper, and let $\sigma^2 = K(\infty)$. Denote by $G(x)$ the distribution function of the infinitely divisible distribution with variance σ^2 whose characteristic function $\phi(t)$ is given by

$$(1) \qquad \qquad \log \phi(t) = \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right) x^{-2} dK(x) \ .$$

For each n we shall denote by $\left\{G_{nk}, k = 1, 2, \ldots, n\right\}$ a decomposition of G into n components, each infinitely divisible with finite variance σ_{nk}^2, and we shall denote the spectral function of G_{nk} by K_{nk}. (See [5] for further definitions, and the proof of the existence of such a decomposition for any n.) For such a decomposition, put

$$\Delta_{nk}(x) = P\left(X_{nk} \leq x \mid \underline{\underline{I}}_{n,k-1}\right) - G_{nk}(x) \ .$$

THEOREM. *Let* γ_n *be a bounded sequence of positive real numbers. If there exist a decomposition* $\left\{G_{nk}\right\}$ *of* G *of the above form and a sequence of positive numbers* A_n *for which*

$$(2) \qquad \gamma_n^{-1}\left[A_n \alpha_n + \sup_{k \in \bar{U}_n} \left\{1 - \left[G_{nk}(A_n) - G_{nk}(-A_n)\right]\right\}\right] \to 0 \quad as \quad n \to \infty \ ,$$

where

(3) $$\alpha_n = \sup_{k \in \overline{U}_n} \sup_x E|\Delta_{nk}(x)|$$

and U_n *is the set of values of the index* k *for which* $\sigma_{nk}^2 < \gamma_n$ *, then in order that* S_{nn} *converge in law as* $n \to \infty$ *to a random variable with distribution function* G *it is sufficient that the following conditions hold:*

(4) $$\max_{k \in U_n} \tilde{\sigma}_{nj}^2 \xrightarrow{P} 0 \quad as \quad n \to \infty \; ;$$

(5)   *there exists a finite constant* C *for which*

$$\lim_{n \to \infty} P\left\{ \sum_{k \in U_n} \tilde{\sigma}_{nj}^2 > C \right\} = 0 \; ;$$

(6) $$\sum_{k \in U_n} E\left\{ X_{nk}^2 I(a < X_{nk} \le b) \mid \underline{\underline{\mathcal{I}}}_{n,k-1} \right\} + \sum_{k \in \overline{U}_n} \left(K_{nk}(b) - K_{nk}(a) \right)$$

$$\xrightarrow{P} K(b) - K(a) \quad as \quad n \to \infty$$

for all continuity points a, b *of* K *.*

REMARKS 1. (a) When the martingale differences X_{nk} are actually independent, it is easy to see that the above result contains the sufficiency part of Theorem 1 of [6], by choosing

$$\gamma_n^3 = \sup_{1 \le k \le n} L\left(P(X_{nk} \le x), G_{nk}(x) \right) \; ,$$

where L is the Levy metric.

(b) If the martingale triangular array satisfies the conditions of Theorem 1 of [3], then clearly $\sum_{k=1}^{n} \sigma_{nj}^2 < C$ for all n, so that taking $\gamma_n = 2C$ ensures that the array also satisfies the conditions of our theorem.

Proof of the theorem. We commence by introducing a new sequence of random variables, Y_{nk}. For every $k \in \overline{U}_n$ let Y_{nk} have distribution function G_{nk}, where the G_{nk} are given in the statement of the theorem,

and let these be distributed independently of each other and the σ-field

generated by $\bigcup\limits_{k=1}^{n} \underline{\underline{I}}_{nk}$. For $k \in U_n$ simply set $Y_{nk} = X_{nk}$. Now define

the sum R_{nk} by

$$R_{nk} = \sum_{j=1}^{k} Y_{nj} + \sum_{j=k+1}^{n} X_{nj} \ .$$

We shall first show that, as $n \to \infty$,

(7) $|E\big(\exp(itR_{nn})\big) - E\big(\exp(itS_{nn})\big)| \to 0 \ ,$

so that to prove the theorem we can replace the X_{nk}'s by the Y_{nk}'s

without any loss of generality. As in equation (15) of [1] it is easy to

show that the left hand side of (7) is not greater than

(8) $\sum_{k \in \overline{U}_n} E|E\big(\exp(itY_{nk}) - \exp(itX_{nk}) \ | \ \underline{\underline{I}}_{n,k-1}\big)| \ .$

Each term of (8) can be written as

(9) $E\Big|\int \exp(itx)d\Delta_{nk}(x)\Big|$

$\qquad \leq E\Big|\int_{-\infty}^{-A_n} e^{itx}d\Delta_{nk}(x)\Big| + E\Big|\int_{-A_n}^{A_n} e^{itx}d\Delta_{nk}(x)\Big| + E\Big|\int_{A_n}^{\infty} e^{itx}d\Delta_{nk}(x)\Big|$

$\qquad = I_1 + I_2 + I_3 \ ,$ say.

Consider now each of these terms separately:

(10) $I_1 \leq E \int_{-\infty}^{-A_n} |d\Delta n_k(x)| \leq E\big(P\big(X_{nk} \leq -A_n \ | \ \underline{\underline{I}}_{n,k-1}\big) + G_{nk}(-A_n)\big)$

$\qquad\qquad\qquad \leq \alpha_n + 2G_{nk}(-A_n)$

from (3). Furthermore, from an integration by parts we have

(11) $I_2 \leq E\Bigg(\Big|\int_{-A_n}^{A_n} ite^{itx}\Delta_{nk}(x)dx\Big| + \Big|\Big[e^{itx}\Delta_{nk}(x)\Big]_{-A_n}^{A_n}\Big|\Bigg)$

$\qquad\qquad \leq 2tA_n\alpha_n + 2\alpha_n \ .$

But $k \in \overline{U}_n$ entails $\sigma_{nk}^2 > \gamma_n$, and since $\sum\limits_{k \in \overline{U}_n} \sigma_{nk}^2 \leq \sigma^2$, there are at

most $\sigma^2 \gamma_n^{-1}$ indices in $\overline{U}_n$. Combining this fact, (9)-(11), and a similar

bound for I_3 gives that (8) is not greater than

$$\sigma^2 \gamma_n^{-1} [4\alpha_n + 2tA_n\alpha_n + 4 \sup_{k \in \overline{U}_n} \{1 - [G_{nk}(A_n) - G_{nk}(-A_n)]\}] ,$$

which, under the conditions of the theorem, tends to zero as $n \to \infty$. This
establishes (7).

Thus we now need only consider the convergence of R_{mn}. By Theorem
2.3 of [2] we need only show that for any subsequence $\{n'\}$ there exists a
further subsequence $\{n''\}$ along which convergence occurs. Thus, without
any loss of generality, we may assume that

$$(12) \qquad \sum_{k \in \overline{U}_n} K_{nk}(b) - K_{nk}(a) \to K^{(1)}(b) - K^{(1)}(a) ,$$

where $K^{(1)}$ is a nondecreasing function dominated by K. (The existence
of such a function follows from Helly's first theorem.) This condition is
of course equivalent to the condition that $\sum\limits_{k \in \overline{U}_n} Y_{nk}$ converge in a law to

a random variable with characteristic function $\phi^{(1)}(t)$, given by

$$(13) \qquad \log \phi^{(1)}(t) = \int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2} dK^{(1)}(x) .$$

We now define functions $K^{(2)}$ and $\phi^{(2)}$ by

$$(14) \qquad K^{(1)}(x) + K^{(2)}(x) = K(x) , \quad \phi^{(1)}(t)\phi^{(2)}(t) = \phi(t) .$$

Then it follows from (6) and (12) that, as $n \to \infty$,

$$\sum_{k \in \overline{U}_n} E\left[X_{nk}^2 I(a < X_{nk} \leq b) \mid \underset{=}{I}_{n,k-1}\right] \xrightarrow{P} K^{(2)}(b) - K^{(2)}(a) .$$

If we now apply the same type of argument as on pages 52-53 of [1] (that
is, we form a new array from the Y_{nk}, $k \in U_n$) it follows from Theorem 1

of [3] that $\sum_{k \in U_n} Y_{nk} = \sum_{k \in U_n} X_{nk}$ converges in law to the random variable

with characteristic function $\phi^{(2)}(t)$. Combining this with (12)-(14) completes the proof of the theorem.

REMARKS 2. (a) We note that as in [3] the assumption that the rows of the triangular array form a martingale difference sequence is not essential. It is a trivial matter to relax this condition as in Theorem 2 of [1] so that in effect our result also generalises Theorem 2 of [4].

(b) It is of interest to obtain sharper forms of (2) when the limit distribution has a simple form. For example, if the limit law is that of a zero mean, unit variance, gaussian random variable, then (2) becomes

$$\gamma_n^{-1} \alpha_n \left(\log \alpha_n^{-1} \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty \ .$$

This follows simply from (2) by choosing $A_n = \left(2 \log \alpha_n^{-1} \right)^{\frac{1}{2}}$, and then noting that since each G_{nk} can be taken as the distribution function of a gaussian random variable with zero mean and variance $\sigma_{nk}^2 \leq 1$, we can use the well known inequality (for large x)

$$\Phi(-x) = 1 - \Phi(x) \leq \left(x e^{-\frac{1}{2}x^2} \right)^{-1} \ ,$$

where Φ is the standard gaussian distribution function, to bound the terms in (2).

Similarly, when the limit law is Poisson, each G_{nk} can be taken as Poisson with parameter $\sigma_{nk}^2 \leq \sigma^2$. Thus, using the upper tail bound for this distribution given by

$$1 - G_{nk}(A_n) \leq \left(\sigma_{nk}^2 \right)^{A_n} / (A_n) ! \ ,$$

where $\{A_n\}$ is now taken to be a sequence of positive numbers increasing to infinity, we can apply Stirling's approximation to see that

$$1 - G_{nk}\left(A_n\right) \leq \text{constant}\left(\sigma^2 e^1 A_n^{-1}\right)^{A_n + \frac{1}{2}} ,$$

which will tend to zero as $n \to \infty$. Hence, in the Poisson case, (2) can be very simply written as

$$\gamma_n^{-1} \alpha_n A_n \to 0 \quad \text{as} \quad n \to \infty ,$$

for *any* integer sequence $\{A_n\}$ for which $A_n \to \infty$ as $n \to \infty$.

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