16 More supersymmetric dynamics

While motivated in part by the hopes of building phenomenologically successful models of particle physics, we have uncovered in our study of supersymmetric theories a rich treasure trove of field theory phenomena. Supersymmetry provides powerful constraints on the dynamics. In this chapter we will discover more remarkable features of supersymmetric field theories. We will first study classes of (super)conformally invariant field theories. Then we will turn to the dynamics of supersymmetric QCD with $N_\mathrm{f} \geq N_\mathrm{c}$, where we will encounter new, and rather unfamiliar, types of behavior.

16.1 Conformally invariant field theories

In quantum field theory, theories which are classically scale invariant are typically not scale invariant at the quantum level. Quantum chromodynamics is a familiar example. In the absence of quark masses we believe that the theory predicts confinement and has a mass gap. The CP^N models are an example where we were able to show systematically how a mass gap can arise in a scale-invariant theory. In all these cases the breaking of scale invariance is associated with the need to impose a cutoff on the high-energy behavior of the theory. In a more Wilsonian language one needs to specify a scale where the theory is defined, and this requirement breaks the scale invariance.

There is, however, a subset of field theories which are indeed scale invariant. We have seen this in the case of $N = 4$ supersymmetric field theories in four dimensions. In this section we will see that this phenomenon can occur in $N = 1$ theories and will explore some of its consequences. In the next section we will discuss a set of dualities among $N = 1$ supersymmetric field theories, in which conformal invariance plays a crucial role.

In order that a theory exhibit conformal invariance it is necessary that its beta function vanish. At first sight it would seem difficult to use perturbation theory to find such theories. For example, one might try to choose the number of flavors and colors in such a that the one-loop beta function vanishes. But then the two-loop beta function will generally not vanish. One could try to balance the first term against the second, but this would generally require $g^4 \sim g^2$, and there would not be a good perturbation expansion. Banks and Zaks pointed out that one can find such theories by adopting a different strategy. By taking the number of flavors and colors to be large, one can arrange that the coefficient of the one-loop beta function almost vanishes, and can choose the coupling so that it cancels the two-loop beta function. In this situation one can arrange a cancelation perturbatively, order by order. The small parameter is $1/N$, where N is the number of colors.

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We can illustrate this idea in the framework of supersymmetric theories with N colors and N_f flavors. The beta function, through two loops, is given by

$$\beta(g) = -\frac{g^3}{16\pi^2} b_0 - \frac{g^5}{(16\pi^2)^2} b_1, \tag{16.1}$$

where

$$b_0 = 3N - N_f, \quad b_1 = 6N^2 - 2NN_F - 4N_F \frac{(N^2-1)}{2N}. \tag{16.2}$$

In the limit of very large N and N_f, we write $N_f = 3N - \epsilon$, where ϵ is an integer of order one. Then, to leading order in $1/N$, the beta function vanishes for a particular coupling, g_0, given by

$$\frac{g_0^2}{16\pi^2} = \frac{\epsilon}{6N^2}. \tag{16.3}$$

Perturbative diagrams behave as $(g^2N)^n$, and g^2N is small. So, at each order, one can make small adjustments in g^2 so as to make the beta function vanish.

A theory in which the beta function vanishes is genuinely conformally invariant. We will not give a detailed discussion of the conformal group here. The exercises at the end of this chapter guide the reader through some features of the conformal group; good reviews are described in the suggested reading. Here we will just mention a few general features and then perform some computations for our Banks–Zaks fixed point theories to verify these.

Without supersymmetry the generators of the conformal group include the Lorentz generators and the translations,

$$M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad P_\mu = -i\partial_\mu, \tag{16.4}$$

and the generators of "special conformal transformations" and dilatations,

$$K_\mu = -i(x^2\partial_\mu - 2x_\mu x_\alpha\partial^\alpha), \quad D = ix_\alpha\partial^\alpha. \tag{16.5}$$

In the presence of supersymmetry the group is enlarged. In addition to the bosonic generators above and the supersymmetry generators, there is a group of *superconformal* generators

$$S_\alpha = X_\mu\sigma^\mu_{\alpha\dot\alpha}Q^{\dot\alpha}. \tag{16.6}$$

We encountered these in our analysis of the zero modes of the Yang–Mills instanton. The superconformal algebra also includes an R symmetry current.

Conformal invariance implies the vanishing of T^μ_μ. In the superconformal case the superconformal generators and the divergence of the R current also vanish. One can prove a relation between the dimension and the R charge:

$$d \geq \frac{3}{2}|R|. \tag{16.7}$$

States for which the inequality is satisfied are known as *chiral primaries*. An interesting case is provided by the fixed point theories introduced above. For these, the charge of the chiral fields, Q and $\bar{Q}$, under the *non-anomalous* symmetry is

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$$R_{Q,\bar{Q}} = \frac{N_f - N}{N_f}. \tag{16.8}$$

Assuming that these fields are chiral primaries, it follows that their dimension d satisfies

$$d - 1 = -\frac{3N - N_f}{2N_f} = -\frac{\epsilon}{6N}. \tag{16.9}$$

At weak coupling, however, the anomalous dimensions of these fields are known:

$$\gamma = -\frac{g^2}{16\pi^2} N = -\frac{\epsilon}{6N}. \tag{16.10}$$

In this chapter we will see that supersymmetric QCD, for a range of N_f and N, exhibits conformal fixed points for which the coupling is not small.

16.2 More supersymmetric QCD

We have studied the dynamics of supersymmetric QCD with $N_f < N$ and observed a range of phenomena: non-perturbative effects which lift the degeneracy among different vacua and non-perturbative supersymmetry breaking. In the case $N_f \geq N_c$ there are exact moduli, even non-perturbatively. In the context of phenomenology such theories are probably of no relevance, but Seiberg realized that, from a theoretical point of view, these theories are a bonanza. The existence of moduli implies a great deal of control over the dynamics. One can understand much about the strongly coupled regimes of these theories, allowing insights into non-perturbative dynamics unavailable in theories without supersymmetry. We will be able to answer questions such as: are there unbroken global symmetries in some region of the moduli space? In regions of strong coupling, are there massless composite particles?

16.3 $N_f = N_c$

The case $N_f = N_c$ already raises issues beyond those of $N_f < N_c$. First, we have seen that there is no invariant superpotential that one can write down. As a result, there is an exact moduli space, perturbatively and non-perturbatively. Yet there is still an interesting quantum modification of the theory, first discussed by Seiberg.

Consider, first, the classical moduli space. Now, in addition to the vacua with $Q = \bar{Q}$ (up-to-flavor transformations) which we found previously, we can also have

$$Q = vI, \quad \bar{Q} = 0 \quad \text{or} \quad Q \leftrightarrow \bar{Q}. \tag{16.11}$$

This is referred to as the "baryonic branch", since now the operator

$$B = \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Q^{i_1}_{j_1} \cdots Q^{i_N}_{j_N} \tag{16.12}$$

is non-vanishing (similarly for the corresponding antibaryon branch).

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Classically these two sets of possibilities can be summarized in the condition

$$\det \bar{Q}Q = \bar{B}B. \tag{16.13}$$

Now, this condition is subject to quantum modifications. Both sides are completely neutral under the various flavor symmetries; in principle any function of $B\bar{B}$ or the determinant would be permitted as a modification. But we can use anomalous symmetries (with the anomalies canceled by shifts in S) to constrain any possible corrections. Consider, in particular, possible instanton corrections. These are proportional to

$$v^{2N} e^{-8\pi^2/g^2(v)} \sim \Lambda^{2N} \tag{16.14}$$

and transform just like the left-hand side under the anomalous R symmetry for which

$$Q \to e^{i\alpha Q}. \tag{16.15}$$

So, at the quantum level the moduli space satisfies the condition

$$\det \bar{Q}Q - \bar{B}B = c\Lambda^{2N}. \tag{16.16}$$

This is of just the right form to be generated by a one-instanton correction. We will not do the calculation here; it shows that the right-hand side is indeed generated. We can outline the main features. There are now two superconformal zero modes, two supersymmetry zero modes, $4N-4$ zero modes associated with the gluinos in the $(2, N-2)$ representation of the $SU(2) \times SU(N-2)$ subgroup of $SU(N)$ distinguished by the instanton and $2N$ matter zero modes. We want to compute the expectation value of an operator involving N scalars. To obtain a non-vanishing result it is necessary to replace some fields with their classical values. Others must be contracted with Yukawa terms. The scalar field propagators in the instanton background are known, and the full calculation is reasonably straightforward. Because the classical condition which defines the moduli space is modified, the moduli space of the $N_f = N_c$ theory is referred to as the *quantum moduli space*. This phenomenon appears for other choices of gauge group as well.

16.3.1 Supersymmetry breaking in quantum moduli spaces

We have mentioned that, in the $(3, 2)$ model, in the limit where the $SU(2)$ gauge group is the strong group, supersymmetry breaking can be understood as resulting from an expectation value for QL. The QL vev is non-zero since $N = N_f = 2$. The introduction of a larger class of models, in which a quantum moduli space is responsible for dynamical supersymmetry, is due to Intriligator and Thomas.

Consider a model with gauge group $SU(2)$ and four doublets $Q_I, I = 1-4$ (two "flavors"). Classically, this model has a moduli space labeled by the expectation values of the fields, $M_{IJ} = Q_I Q_J$. These satisfy $\text{Pf}\langle M_{IJ}\rangle = 0$[1] but, as have have just seen, the quantum moduli space is different and satisfies

$$\text{Pf}\langle M_{IJ}\rangle = \Lambda^4. \tag{16.17}$$

[1] In this expression, Pf denotes the Pfaffian. The Pfaffian is defined for $2N \times 2N$ antisymmetric matrices; it is essentially the square root of the determinant of the matrix.

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Now add a set of singlets S_{IJ} to the model, with superpotential couplings

$$W = \lambda_{IJ} S_{IJ} Q_I Q_J. \tag{16.18}$$

Unbroken supersymmetry now requires

$$\frac{\partial W}{\partial S_{IJ}} = Q_I Q_J = 0. \tag{16.19}$$

However, this is incompatible with the quantum constraint. So on the one head the supersymmetry is broken.

On the other hand the model, classically, has flat directions in which $S_{IJ} = s_{IJ}$ and all the other fields vanish. So one might worry that there is runaway behavior in these directions, similar to that we saw in supersymmetric QCD. However, for large s it turns out that the energy is growing at infinity. This can be established as follows. Suppose all the components of S are large, $S \sim s \gg \Lambda_2$. In this limit the low-energy theory is a pure $SU(2)$ gauge theory. In this theory gluinos condense,

$$\langle \lambda\lambda \rangle = \Lambda_{\rm LE}^3 = \lambda s \Lambda_2^2. \tag{16.20}$$

Here, $\Lambda_{\rm LE}$ is the Λ parameter of the low-energy theory.

At this level, then, the superpotential of the model behaves as

$$W_{\rm eff} \sim \lambda S \Lambda_2^2, \tag{16.21}$$

and the potential is a constant,

$$V = |\Lambda_2|^4 |\lambda|^2. \tag{16.22}$$

The natural scale for the coupling, λ, which appears here is $\lambda(s)$. This is the correct answer in this case and implies that for large s the potential is growing, since λ is not asymptotically free. So the potential has a minimum in a region of small s.

16.3.2 $N_{\rm f} = N_{\rm c} + 1$

For $N_{\rm f} > N_{\rm c}$ the classical moduli space is exact. But again Seiberg has, pointed out a rich set of phenomena and given a classification of the different theories. As in the case $N_{\rm f} < N_{\rm c}$, different phenomena occur for different values of $N_{\rm f}$.

First, we need to introduce a new tool: the 't Hooft anomaly-matching conditions. 't Hooft was motivated by the following question. When one looks at the repetitive structure of the quark and lepton generations, it is natural to wonder whether the quarks and leptons themselves are bound states of some simpler constituents. 't Hooft pointed out that if this idea were correct then the masses of the quarks and leptons would be far smaller than the scale of the underlying interactions; even at that time it was known that if these particles have any structure then it is on scales shorter than 100 GeV^{-1}. 't Hooft argued that this could only be understood if the underlying interactions left an unbroken chiral symmetry.

One could go on and simply postulate that the symmetry is unbroken, but 't Hooft realized that there are strong – and simple – constraints on such a possibility. Assuming that the mechanism is some strongly interacting non-Abelian gauge theory, 't Hooft imagined

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gauging the global symmetries of the theory. In general the resulting theory would be anomalous, but one could always cancel the anomalies by adding some "spectator" fields, fields transforming under the gauged flavor symmetries but not the underlying strong interactions. Below the confinement scale of the strong interactions the flavor symmetries might be spontaneously broken, giving rise to Goldstone bosons, or there might be massless fermions. In either case the low-energy theory must be anomaly-free, so the anomalies of either the Goldstone bosons or the massless fermions must be the same as in the original theory. 't Hooft added another condition, which he called the "decoupling" condition: he asked what happened if one added mass terms for some of the constituent fermions. He went on to show that these conditions are quite powerful and that it is difficult to obtain a theory with unbroken chiral symmetries.

As we will see, Seiberg conjectured various patterns of unbroken symmetries for susy QCD. For these the 't Hooft anomaly conditions provide a strong self-consistency check. In the case $N_f = N_c$ there is no point in the moduli space at which the chiral symmetries are all unbroken. So we will move on to the case $N_f = N_c + 1$. The global symmetry of the model is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \tag{16.23}$$

where, under $U(1)_R$, the quarks and antiquarks transform as

$$Q_f, \bar{Q}_f \to e^{i\alpha/(N+1)} Q_f, \bar{Q}_f. \tag{16.24}$$

In this theory there two sorts of gauge-invariant objects, the mesons, $M_{f\bar{f}} = \bar{Q}_{\bar{f}} Q$, and the baryons, $B_f = \epsilon_f^{\alpha_1 \cdots \alpha_N} \epsilon_{i_1 \cdots i_n} Q^{i_1}_{\alpha_1} Q^{i_2}_{\alpha_2} \cdots Q^{i_N}_{\alpha_N}$. From these we can build a superpotential that is invariant under all the symmetries:

$$W = (\det M - B_{\bar{f}} M_{\bar{f}f} B_f) \frac{1}{\Lambda^{b_0}}. \tag{16.25}$$

As in all our earlier cases, the power of Λ is determined by dimensional arguments but can also be verified by demanding holomorphy in the gauge coupling.

This superpotential has several interesting features. First, it has flat directions, as we would expect, corresponding to the flat directions of the underlying theory. But also, for the first time, there is a vacuum at the point where all the fields vanish, $B = \bar{B} = M = 0$. At this point all the symmetries are unbroken. The 't Hooft anomaly conditions provide an important consistency check on this whole picture. There are several anomalies to check: ($SU(N_f)^3_L, SU(N_f)^3_R$, $SU(N_f)^2_L U(1)_R$, $\mathrm{Tr}\, U(1)_R$, $U(1)^2_B U(1)_R$, $U(1)^3_R$ etc.). The cancelations are quite non-trivial. In the exercises, the reader will have the opportunity to check these.

Another test comes from considering decoupling. If we add a mass for one set of fields, the theory should reduce to the $N_f = N$ case. As in examples with smaller numbers of fields we take advantage of holomorphy, writing down expressions for small values of the mass and continuing to large values. So we add to the superpotential a term

$$m\bar{Q}_{N+1} Q_{N+1} = m M_{N+1,N+1}. \tag{16.26}$$

We want to integrate out the massive fields. Because of the global symmetry, it is consistent to set $M_{f,N+1}$ to zero, where $f \leq N$. Similarly, it is consistent to set $B_f = 0, f \leq N$. So we take the M and B fields to have the form

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$$M = \begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}, \quad B_f = \begin{pmatrix} 0 \\ \vdots \\ B \end{pmatrix}, \quad \bar{B}_f = \begin{pmatrix} 0 \\ \vdots \\ \bar{B} \end{pmatrix}. \tag{16.27}$$

Consider the equation $\partial W/\partial m = 0$. This yields

$$(\det M - \bar{B}B) = m\Lambda^{b_0} \tag{16.28}$$

or

$$(\det M - \bar{B}B) = m\Lambda^{b_0} = \Lambda_{N_\mathrm{f}}^{2N}. \tag{16.29}$$

In the last line we have used the relation between the Λ parameter of the theory with N_f quarks and that with $N_\mathrm{f} + 1$ flavors. This is precisely the expression for the quantum-modified moduli space of the N-flavor theory. Decoupling works perfectly here.

16.4 $N_\mathrm{f} > N + 1$

The case $N_\mathrm{f} > N + 1$ poses new challenges. We might try to generalize our analysis of the previous section. Take, for example, $N_\mathrm{f} = N + 2$. Then the baryons are in the second-rank antisymmetric tensor representations of the $SU(N_\mathrm{f})$ gauge groups, B_{fg} and $\bar{B}_{\bar{f}\bar{g}}$. For a term in the superpotential

$$W \sim B_{fg}\bar{B}_{\bar{k}\bar{l}}M^{f\bar{k}}M^{g\bar{l}}, \tag{16.30}$$

this does not respect the non-anomalous R symmetry.

Seiberg suggested a different equivalence. The baryons, in general, have $\tilde{N} = N_\mathrm{f} - N$ indices. So baryons in the same representation of the flavor group can be constructed in a theory with gauge group $SU(\tilde{N})$ and quarks $q_f, \bar{q}_f$ in the fundamental representation. Seiberg postulated that, in the infrared, this theory is dual to the original theory. This is not quite enough. One needs to add a set of gauge-singlet meson fields $M_{\bar{f}f}$, with superpotential

$$W = q^{\bar{f}}M_{\bar{f}f}q^f. \tag{16.31}$$

To check this picture we can first check that the symmetries match. There is an obvious $SU(N_\mathrm{f})_\mathrm{L} \times SU(N_\mathrm{f})_\mathrm{R} \times U(1)_B$. There is also a non-anomalous $U(1)_R$ symmetry. It is important that the dual theory is not asymptotically free, i.e. that it is weakly coupled in the infrared. This is the case for $N > 3N_\mathrm{f}/2$. Again, this duality can only apply for a range of N_f and N.

There are a number of checks on the consistency of this picture. Holomorphic decoupling is again one of the most persuasive. Take the case $N_\mathrm{f} = N + 2$, so that the dual gauge group is $SU(2)$. In this case, working in the flat directions of the $SU(2)$ theory, one can do an instanton computation. One finds a contribution to the superpotential

$$W_{\mathrm{inst}} = \det M. \tag{16.32}$$

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This is consistent with all the symmetries; it is not difficult to see that one can close up all the fermion zero modes with elements of M and q. So one has a superpotential

$$\int d^2\theta(qM\bar{q} - \det M). \tag{16.33}$$

16.5 $N_f \geq 3N/2$

We have noted that Seiberg's duality cannot persist beyond $N_f = 3N/2$. Seiberg also made a proposal for the behavior of the theory in this regime: for $3/2N \leq N_f \leq 3N$ the theories are conformally invariant. Our Banks–Zaks fixed point lies in one corner of this range. As a further piece of evidence, consider the dimension of the operator $\bar{Q}Q$. Under the non-anomalous R symmetry, we have

$$Q \to \exp\left(i\alpha\frac{N_f - N}{2N_f}\right)Q. \tag{16.34}$$

If the theory is superconformal, the dimension of this chiral operator satisfies $d = 3R/2$. As explained in Appendix D, the exact beta function of the theory is given by

$$\beta = -\frac{g^3}{16\pi^2}\frac{3N - N_F + N_f\gamma(g^2)}{1 - N(g^2/8\pi^2)}. \tag{16.35}$$

By assumption this is zero, so

$$\gamma = -\frac{3N - N_f}{N_f}. \tag{16.36}$$

The dimension of $\bar{Q}Q$ is $2 + \gamma$, which is precisely $3R/2$.

We will not pursue this subject further, but there is further evidence that one can provide for all these dualities. They can also be extended to other gauge groups.

Suggested reading

The original papers of Seiberg (1994a,b, 1995a,b; see also Seiberg and Witten 1994) are quite accessible and constitute essential reading on these topics, as the review by Intriligator and Seiberg (1996). Good introductions are provided by the lecture notes of Peskin (1997) and Terning (2003). The use of quantum moduli spaces to break supersymmetry was introduced in Intriligator and Thomas (1996).

Exercises

(1) Discuss the renormalization of the composite operator $\bar{Q}Q$, and verify that the relation $d = 3R/2$ is again satisfied.

(2) Check the anomaly cancelation for the case $N_{\rm f} = N + 1$. You may want to use an algebraic manipulation program, such as MAPLE or Mathematica, to expedite the algebra.