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ON FIXED POINTS OF DOUBLY SYMMETRIC RIEMANN SURFACES

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Abstract. In this paper, we study ovals of symmetries and the fixed points of their products on Riemann surfaces of genus $g \geq 2$. We show how the number of these points affects the total number of ovals of symmetries. We give a generalisation of Bujalance, Costa and Singerman's theorems in which we show upper bounds for the total number of ovals of two symmetries in terms of g, the order n and the number m of the fixed points of their product, and we show their attainments for n holding some divisibility conditions. Finally, we give an upper bound for m in terms of n and g, and we study conditions under which it has given parity.

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1. Introduction. Let X be a compact Riemann surface of genus $g > 1$. By a *symmetry* of X we mean, in this paper, an antiholomorphic involution a of X, which has fixed points. By the classical result of Harnack, the set of fixed points of a consists of at most $g + 1$ disjoint simple closed curves, which, following classical Hilbert terminology, are called *ovals*. If a has $g + 1 - q$ ovals, then following Natanzon [**5**], we shall call it an $(M - q)$-*symmetry*.

In [**1**] (see also [**2**]), the bounds for the total number of ovals of two symmetries in terms of g and the order n of their product were given. Here, using a theorem of Macbeath from [**4**] and a result from [**6**], we give a generalisation of these results, which takes into account the number m of the fixed points of the product of symmetries. We also show the sharpness of our bounds for infinitely many n.

In the remainder of the work, we focus attention on possible values of m for given n and g. We find an upper bound for it and we study its attainments. Finally, we look for the conditions that guarantee specified parity of m.

2. Preliminaries. We shall prove our results using the theory of non-Euclidean crystallographic groups (*NEC groups* in short) by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$.

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The algebraic structure of such group Λ is determined by the signature:

$$s(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \tag{1}$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called *the period cycles*, the integers n_{ij} are the *link periods*, m_i *proper periods* , and finally, g the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators*:

$x_1, \ldots, x_r, e_i, c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, \ldots, a_g, b_g$ if the sign is + or $d_1, \ldots, d_g$ otherwise,

and relators:

$x_i^{m_i}, i = 1, \ldots, r, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, i = 1, \ldots, k, j = 0, \ldots, s_i$

and

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \text{ or } x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2,$$

according to whether the sign is + or −. The elements x_i are elliptic transformations a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i or else to a power of the product of two consecutive canonical reflections.

Now an abstract group with such presentation can be realized as an NEC group Λ if and only if the value

$$2\pi\left(\varepsilon g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ or 1 according to the sign being + or − is positive. This value turns out to be the hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for such group, and we have the following Hurwitz–Riemann formula:

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda)$$

for a subgroup Λ' of finite index in an NEC group Λ.

Now NEC groups having no orientation reversing elements are classical Fuchsian groups. They have signatures $(g; +; [m_1, \ldots, m_r]; \{-\})$, which shall be abbreviated as $(g; m_1, \ldots, m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* Λ and for a group with signature (1), it has, by **[7]**, signature

$$(\varepsilon g + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{2}$$

A torsion-free Fuchsian group Γ is called a *surface group* and it has signature $(g; -)$. In such case, $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g and conversely, each compact Riemann surface can be represented as such orbit space for some Γ. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ.

Let $C(G, g)$ denote the centralizer of an element g in G. The following result from [**6**] and the next theorem of Macbeath from [**4**] are crucial for the paper.

THEOREM 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with a group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then, the number of ovals of a symmetry a of X equals*

$$\sum [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to a.

For a symmetry a, we shall denote by $\|a\|$ the number of its ovals. The index $w_i = [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|a\|$.

COROLLARY 2.2. *Let $\mathrm{D}_n = \Lambda/\Gamma$ be the group of automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ generated by two non-central symmetries a and b and let $C = (n_1, \ldots, n_s)$ be a period cycle of Λ. If n is odd, then the reflections corresponding to C contribute to $\|a\|$ and $\|b\|$ with at most 2 ovals in total. If n is even, then the reflections corresponding to C contribute to $\|a\|$ and $\|b\|$ with at most t ovals in total, where t is the number of even link periods if $s \geq 1$ and some n_i is even and with at most 2 ovals in total for the remaining cases.*

Proof. Let $\theta : \Lambda \to \mathrm{D}_n$ be the canonical epimorphism. The case of odd n is trivial; here all canonical reflections belonging to C are conjugate, $C(\mathrm{D}_n, \theta(c))$ has order 2 and $c \in C(\Lambda, c)$.

Now for n even, the centralizer of any non-central element of D_n has order 4. Since $c_i \in C(\Lambda, c_i)$, we have that $w_i \leq 2$ and since a and b are not conjugate, we can assume that $s \geq 2$ or $s = 1$ and n_1 is even. If c belongs to two odd link periods, then we can assume that c neither contributes to $\|a\|$ nor to $\|b\|$, while if c belongs to an even link period n_1 and cc' has order n_1, then $(cc')^{n_1/2} \in C(\Lambda, c)$. Now $\theta((cc')^{n_1/2}c) \neq 1$ since $\ker\theta$ is a Fuchsian group, and therefore, we see that $\theta(C(\Lambda, c))$ has order 4. □

We also need the following result of Macbeath from [**4**] concerning the number of fixed points of an automorphism of a Riemann surface (c.f. [**3**] for the case of non-orientable Riemann surfaces). By $N_G(\langle g\rangle)$, we mean the normalizer in G of the group generated by g.

THEOREM 2.3. *Let $G = \Delta/\Gamma$ be the group of orientation preserving automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ and let $x_1, x_2, \ldots, x_r$ be the set of canonical elliptic generators of Δ with periods $m_1, \ldots, m_r$, respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism. Then, the number m of points of X fixed by $g \in G$ is given by the formula*

$$m = |N_G(\langle g\rangle)| \sum 1/m_i,$$

where the sum is taken over those i for which g is conjugate to a power of $\theta(x_i)$. □

We shall study the number of fixed points of the product of two symmetries a and b of a Riemann surface X, which has the order n. Let $X = \mathcal{H}/\Gamma$ and $\langle a, b\rangle = \Lambda/\Gamma$ for some NEC group Λ with signature (1) and Fuchsian surface group Γ. Let r and s denote, respectively, the numbers of proper periods and link periods equal to n in the signature of Λ. The subgroup of Λ/Γ of orientation preserving automorphisms is

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generated by the product ab and is Λ^+/Γ. Hence, using the above theorem and (2), we obtain the following.

COROLLARY 2.4. *The product of two symmetries a and b of a Riemann surface X, whose order is equal to n, has $2r + s$ fixed points.*

3. On ovals of two symmetries with specified number of fixed points of their product. Here we study how the number m of the fixed points of the product of two symmetries of a Riemann surface of genus g affects the total number t of their ovals, and we give upper bounds for t depending on the parity of n. We also show that, with some small exceptions, our bounds are sharp for arbitrary arithmetically admissible m, n and $g \geq 2$, that is for n dividing $2g + m - 2$. Throughout the remainder of the paper, a and b will denote two symmetries whose product has order n and has m fixed points.

THEOREM 3.1. *Two symmetries a and b of Riemann surface X of genus g whose product has order n and has m fixed points have at most*

$$4g/n + m - 2(m-2)(n-1)/n$$

ovals in total.

Proof. Let t denote total number of ovals of a and b and let $G = \langle a, b\rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k, (n_1), \ldots, (n_l), (-), \overset{u}{\ldots}, (-)\}), \tag{3}$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$ with $s_i \geq 2$ or $s_i = 1$ and n_{i1} even and $n_1, \ldots, n_l$ are odd. Throughout the paper, let $p = \varepsilon h + k + l + u - 1$, where $\varepsilon = 2$ or 1 according to the sign of Λ being $+$ or $-$; it is known as the algebraic genus of Λ. Let $s = s' + s''$ where s' denotes number of link periods n and similarly let $r = r' + r''$ where r' denotes number of proper periods equal n. Now by Corollary 2.4, we have $m = 2r' + s'$. Also $t \leq 2u + 2l + s' + s''$ by Corollary 2.2, and thus,

$$\begin{aligned}
2\pi(g-1)/n &= \mu(\Lambda)\\
&\geq 2\pi(p - 1 + r'(1 - 1/n) + r''/2 + s'(1 - 1/n)/2 + s''/4)\\
&\geq 2\pi(-1 + (1 - 1/n)m/2 + r''/2 + s''/4 + (u + l)/2)\\
&\geq 2\pi(-1 + (1 - 1/n)m/2 + r''/2 + (t - s')/4)\\
&\geq 2\pi(-1 + (1 - 1/n)m/2 + r''/2 + t/4 - m/4)\\
&\geq (\pi/2)(-4 + 2m(1 - 1/n) + t - m)
\end{aligned}$$

which gives $t \leq 4(g-1)/n + 4 + m - 2m + 2m/n = (4g + 2m - 4)/n + 4 - m = 4g/n + m - 2(m-2)(n-1)/n$. □

THEOREM 3.2. *The bound in the previous theorem is attained for every g, $m \geq 2$ and every even n such that $2g + m \equiv 2 \bmod n$.*

Proof. Let Λ be an NEC group with signature

$$(0; +; [-]; \{(2, \overset{s}{\ldots}, 2, n, \overset{m}{\ldots}, n)\}),$$

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where $s = (4g + 2m - 4)/n + 4 - 2m$ and consider an epimorphism $\theta : \Lambda \to D_n = \langle a, b \mid a^2, b^2, (ab)^n \rangle$ defined by $\theta(e_1) = 1$ and which sends consecutive canonical reflections corresponding to the period cycle into

$$\underbrace{a\ b(ab)^{n/2-1}\ a\ b(ab)^{n/2-1}\ \cdots\ a}_{s+1}\ \underbrace{b\ a\ b\ a\ \cdots\ a}_{m}$$

So by the Hurwitz–Riemann formula for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ has genus g and by Theorem 2.1, symmetries a and b have $4g/n + m - 2(m-2)(n-1)/n$ ovals in common. □

COROLLARY 3.3. *The bound from Theorem* 3.1 *is not attained for* $m = 0$ *and* $m = 1$.

Proof. By [**1**], the total number of ovals of two such symmetries does not exceed $4g/n + 2$. On the other hand, for $m \le 1$, $4g/n + m - 2(m-2)(n-1)/n > 4g/n + 2$. □

If $m = 1, n = 2$, then the total number of ovals is $< 2g + (3/2)$ and the next theorem deals with the case $m = 0, n = 2$.

THEOREM 3.4. *Two commuting symmetries, whose product does not have fixed points have at most* $g + 3$ *ovals in total and this bound is attained for every odd* $g > 2$. *The product of commuting symmetries on a Riemann surface of even genus has fixed points.*

Proof. We know that $G = D_2 = \Lambda/\Gamma$ and as $m = 0$ a group Λ has signature

$$(h; \pm; [-]; \{(-), \overset{l}{\ldots}, (-)\}), \tag{4}$$

by Corollary 2.4, where $\varepsilon h + l \ge 3$ as $\mu(\Lambda) > 0$. By the Corollary 2.2, we have $t \le 2l$ and also we know that $\pi(g-1) = \mu(\Gamma)/4 = \mu(\Lambda) = 2\pi(\varepsilon h + l - 2) \ge \pi(2l - 4) \ge \pi(t-4)$. So the first statement follows and also we see that g is odd as $g = 2\varepsilon h + 2l - 3$.

To show the attainment of this bound for odd $g > 2$, consider an NEC group Λ with signature $(0; +; [-]; \{(-), \overset{l}{\ldots}, (-)\})$ where $l = (g+3)/2$ and an epimorphism $\theta : \Lambda \to D_2$ that sends all e_i into 1 and canonical reflections alternatively to a and b. As each period cycle produces two ovals in a or b, by Theorem 2.1, θ defines desired configuration of symmetries. □

Now we will show that, like in [**1**], the bound in Theorem 3.1 can be significantly improved for odd n.

THEOREM 3.5. *Two symmetries* a *and* b *of a Riemann surface of genus* g, *whose product has odd order* n *and has* m *fixed points have at most*

$$2(g-1)/n + 4 - m(n-1)/n$$

ovals in total.

Proof. As in the proof of Theorem 3.1, we have $G = \langle a, b \rangle = D_n = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (3), where $s_i \ge 2$. Now by Corollary 2.2, we have $t \le 2k + 2l + 2u$ and so

$$\begin{aligned} 2\pi(g-1)/n &= \mu(\Lambda) \\ &\ge 2\pi(\varepsilon h + l + k + u - 2 + r'(1 - 1/n) + s'(1 - 1/n)/2) \\ &\ge 2\pi(\varepsilon h + t/2 - 2 + (1 - 1/n)m/2) \\ &\ge \pi(-4 + t + m(1 - 1/n)) \end{aligned}$$

which gives $t \le 2(g-1)/n + m/n + 4 - m = 2(g-1)/n + 4 - m(n-1)/n$. □

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THEOREM 3.6. *The bound in the previous theorem is attained for every m, n and $g \geq 2$ for which $2g + m \equiv 2 \bmod n$.*

Proof. Let Λ be an NEC group with signature

$$(0; +; [-]; \{(n, \overset{m}{\ldots}, n), (-), \overset{l-1}{\ldots}, (-)\})$$

where $l = (g + m/2 - 1)/n + 2 - m/2$. Consider an epimorphism $\theta : \Lambda \to G$ defined by $\theta(c_{i0}) = a$ for all $i > 1$, $\theta(e_i) = 1$ for all i and sending canonical reflections corresponding to the non-empty period cycle alternatively to a and b starting with a for even m and if m is odd defined on all canonical generators in the same way as before except $\theta(c_{1\,m-1}) = aba$ with $\theta(c_{1\,m}) = a$. This gives rise to the desired configuration of symmetries. □

COROLLARY 3.7. *If a and b are two non-commuting symmetries of a Riemann surface of genus g whose product has m fixed points, then the total number of their ovals does not exceed $g + 3 - m/2$.*

Proof. It follows directly from Theorems 3.1 and 3.5. □

By the degree of hyperellipticity of a conformal involution ρ of a Riemann surface X, we shall understand the genus of the orbit space X/ρ.

COROLLARY 3.8. *Two $(M - q)$- and $(M - q')$-symmetries of a Riemann surface of genus g, whose product has m fixed points, commute for $g \geq q + q' + 2 - m/2$. Furthermore, in such case, $m = 2g + 2 - 4p$, where p denotes the degree of hyperellipticity of the involution ab.*

Proof. Assume to a contrary that these symmetries do not commute. Then, we have $2g + 2 - q - q' \leq g + 3 - m/2$ and so $g \leq q + q' + 1 - m/2$. Now if a and b are two commuting symmetries of a Riemann surface of genus g, then $G = \langle a, b\rangle = D_2 = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [2, \overset{r}{\ldots}, 2]; \{(2, \overset{s_1}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2), (-) \overset{l}{\ldots}, (-)\}). \tag{5}$$

Since the algebraic genus p of Λ is just the degree of hyperellipticity of ab, then for $p = \varepsilon h + k + l - 1$, we have $\pi(g - 1) = \mu(\Gamma)/4 = \mu(\Lambda) = 2\pi(p - 1 + m/4) = \pi(2p - 2 + m/2)$ and so $m = 2g + 2 - 4p$. □

4. On the number of fixed points of the product of two symmetries.

THEOREM 4.1. *Let a and b be two symmetries of a Riemann surface X of genus g whose product has order n. Then, ab has at most $2(g + n - 1)/(n - 1)$ fixed points.*

Proof. As before, let $G = \langle a, b\rangle = D_n = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (3). Then,

$$\begin{aligned} 2\pi(g - 1)/n &= \mu(\Lambda) \\ &\geq 2\pi(p - 1 + r'(1 - 1/n) + r''/2 + s'(1 - 1/n)/2 + s''/4) \\ &\geq 2\pi(-1 + (1 - 1/n)m/2) \\ &= \pi(-2 + m(1 - 1/n)). \end{aligned}$$

So the statement follows since $m \leq (2(g - 1)/n + 2)(n/(n - 1)) = 2(g + n - 1)/(n - 1)$ □

From the next theorem, it follows in particular that the above bound is attained for arbitrary arithmetically admissible g and n with n even.

THEOREM 4.2. *Given $g \geq 2, m$, an even n such that $g \equiv 0 \bmod (n-1)$ and $2(g+n-1)/(n-1) \equiv m \bmod 4$, there exists a Riemann surface X of genus g having two symmetries whose product has order n and has m fixed points.*

Proof. Let $2g/(n-1)+2$ be denoted as M and so $m = M - 4k$ for some integer k. Consider an NEC group Λ with signature

$$(0; +; [-]; \{(-)^k, (n, \overset{m}{\ldots}, n, n/2, \overset{2k}{\ldots}, n/2)\})$$

and an epimorphism $\theta : \Lambda \to D_n = \langle a, b\rangle$ defined by $\theta(e_i) = 1$ for all i, $\theta(c_{i0}) = a$ for reflections corresponding to empty period cycles and which sends canonical reflections corresponding to the non-empty period cycle onto

$$\underbrace{a\ \ b\ a\ b\ \cdots\ a}_{m+1}\ \underbrace{bab\ \ a\ bab\ a\ \cdots\ a}_{2k}$$

Here again by the Hurwitz–Riemann formula, we have get a Riemann surface of genus g that admits two symmetries a and b whose product has m fixed points. □

Now we shall give some conditions under which m have specified parity. Particularly interesting is the case when the product of symmetries has an odd number of fixed points.

THEOREM 4.3. *If n is a power of 2, then m is even. If n is even but is not a power of 2, then for infinitely many g, there exists a Riemann surface of genus g having two symmetries, whose product has order n and m is odd.*

Proof. Let as always $G = \langle a, b\rangle = \Lambda/\Gamma$ and let $\theta : \Lambda \to G$ be the corresponding epimorphism. If there are no link periods equal to n, then by Corollary 2.4, $m = 2r$ for r being the number of proper periods equal to n in the signature of Λ. So we can assume that there are link periods n. As both symmetries have ovals, the order of symmetry conjugate to a and symmetry conjugate to b is n and $\theta(c_{i0})$ is conjugate to $\theta(c_{is_i})$, the number of link periods n is even in each of the non-empty period cycles. Hence, the number of fixed points of ab is even by 2.4.

For the second part, let $k \neq 1$ be the smallest odd divisor of n. Given an integer u consider an NEC group Λ with signature

$$(0; +; [n, \overset{u}{\ldots}, n, \mu]; \{(n, k, k, n/k)\}),$$

where $\mu = n/\gcd(n, u + (k-1)/2)$ and an epimorphism given by $\theta(x_i) = ab$ for $i = 1, \ldots, u$, $\theta(x_{u+1}) = (ba)^{u+(k-1)/2}$, $\theta(e) = (ab)^{(k-1)/2}$ and which sends reflections corresponding to non-empty period cycle to

$$a\ \ b\ \ a(ba)^{n/k-1}\ \ b\ \ a(ba)^{k-1}.$$

Then, θ gives rise to a configuration of two symmetries whose product has order n and has $2u+1$ or $2u+3$ fixed points on a Riemann surface of genus $g = (u+2)n - u - n/\mu - n/k - (k-1)/2$. □

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