J. Aust. Math. Soc. **81** (2006), 15–20

STRICT FEASIBILITY OF GENERALIZED COMPLEMENTARITY PROBLEMS

Y. R. HE✉ **and K. F. NG**

(Received 17 November 2004; revised 15 February 2005)

Communicated by A. Rubinov

Abstract

The existence of strictly feasible points is shown to be equivalent to the boundedness of solution sets of generalized complementarity problems with stably pseudomonotone mappings. This generalizes some known results in the literature established for complementarity problems with monotone mappings.

2000 *Mathematics subject classification*: primary 90C33, 49J40.
Keywords and phrases: generalized complementarity problem, stably pseudomonotone mapping, strict feasibility, upper hemicontinuous set-valued mapping.

1. Introduction and preliminaries

Throughout this note (unless explicitly stated otherwise), let K denote a nonempty closed convex subset of $\mathbb{R}^n$ and F denote a set-valued mapping from K into $\mathbb{R}^n$. We make the blanket assumption that the mapping F is upper hemicontinuous on K with nonempty compact convex values. The *generalized variational inequality problem*, denoted by GVIP(F, K), is to find vectors $x \in K$ and $x^* \in F(x)$ such that

$$(1) \qquad \langle x^*, y - x\rangle \geq 0 \quad \text{for all} \quad y \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. If K is a nonempty closed convex cone in $\mathbb{R}^n$, then the GVIP(F, K) is the same as the *generalized complementarity problem* GCP(F, K) (see [8]): to find $x \in K$ and $x^* \in F(x)$ such that $x^* \in K^+$ and

The research was supported by an earmarked grant from the Research Grant Council of Hong Kong. Research of the first author was supported by Sichuan Youth Science and Technology Foundation (06ZQ026-13) and grant SZD0406 from Sichuan Province.

https://doi.org/10.1017/S1446788700014609 Published online by Cambridge University Press

$\langle x^*, x\rangle = 0$, where K^+ is the positive polar cone of K. Assume that the interior of K^+ is nonempty, that is, $\operatorname{int}(K^+) \neq \emptyset$. We say that the problem GCP(F, K) is *strictly feasible* if $F(K) \cap \operatorname{int}(K^+) \neq \emptyset$.

A mapping F is said to be *upper semicontinuous* on K if for any open set V, $\{y \in K \mid F(y) \subset V\}$ is open relative to K; F is said to be *upper hemicontinuous* on K if the restriction of F to each line segment of K is upper semicontinuous. Recall also that F is said to be *pseudomonotone* on K if for all $x_1, x_2 \in K$, $x_1^* \in F(x_1)$ and $x_2^* \in F(x_2)$, $\langle x_1^*, x_2 - x_1\rangle \geq 0$ implies $\langle x_2^*, x_2 - x_1\rangle \geq 0$. A pseudomonotone mapping F is said to be *stably pseudomonotone* on K with respect to a set $U \subset \mathbb{R}^n$ if the mapping $F(\cdot) - u$ is pseudomonotone on K for every $u \in U$. For a nonempty convex set K, $\operatorname{barr}(K)$ denotes its *barrier cone* [6] and K^+ denotes the set $\{x^* \in \mathbb{R}^n \mid \langle x^*, x\rangle \geq 0 \text{ for all } x \in K\}$. A function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if $\operatorname{dom}(h)$ defined as the set $\{x \in \mathbb{R}^n \mid h(x) < +\infty\}$ is nonempty; and it is said to be *level-bounded* on K if the set $\{x \in K \mid h(x) \leq r\}$ is bounded for every $r \in \mathbb{R}$.

In Section 2, we prove the following theorem.

THEOREM 2.3. *Let K be a nonempty closed convex cone with* $\operatorname{int}(K^+)$ *nonempty. Assume that there exists a sequence* $\{u_m\} \subset \operatorname{int}(K^+)$ *such that* (a) $\lim_{m\to\infty} u_m = 0$ *and* $\{u_m\}$ *is decreasing with respect to* K^+; (b) *the mapping* $F(\cdot) - u_m$ *is pseudomonotone on K for each* $m \in \mathbb{N}$. *Then* GCP(F, K) *is strictly feasible if and only if its solution set is nonempty and bounded.*

Theorem 2.3 generalizes [5, Theorem 4] and [1, Corollary 1]. The former assumed F is maximal monotone while the latter assumed F is monotone and single-valued mapping, and $K = \mathbb{R}^n_+$. Example 2 in [4] shows that Theorem 2.3 genuinely generalizes the aforementioned results.

LEMMA 1.1. *Let K be a nonempty closed convex subset of* $\mathbb{R}^n$ *and let*

$$\operatorname{int}(\operatorname{barr}(K)) \neq \emptyset.$$

Then $e \in \operatorname{int}(-\operatorname{barr}(K))$ *if and only if the set* $E(r) := \{x \in K \mid \langle e, x\rangle \leq r\}$ *is bounded for every* $r \in \mathbb{R}$.

PROOF. Define a function $f(x) := \langle e, x\rangle + I_K(x)$, where $I_K(x)$ denotes the indicator function of K. Then f is proper convex lower semicontinuous and $E(r)$ is the level set of f. The conclusion now follows from [6, Corollaries 8.7.1 and 14.2.2]. $\square$

We need to introduce an additional function. For $x \in K$ and $u \in \mathbb{R}^n$,

$$\hat{f}(x, u) = \sup\left\{\frac{\langle y^* - u, x - y\rangle}{\max\{\|y^*\|, 1\}\max\{\|y\|, 1\}} \;\middle|\; y^* \in F(y),\, y \in K\right\},$$

The function $\hat{f}(\cdot, 0)$ is considered in [2].

2. Main results

PROPOSITION 2.1. *Let K be a nonempty closed convex set in $\mathbb{R}^n$ and $u \in \mathbb{R}^n$ be such that $F(\cdot) - u$ is pseudomonotone on K. Then $x \in K$ solves* GVIP$(F - u, K)$ *if and only if $\hat{f}(x, u) = 0$.*

PROOF. Assume that $x \in K$ solves GVIP$(F - u, K)$. Then there exists $x^* \in F(x)$ such that $\langle x^* - u, x - y\rangle \le 0$ for all $y \in K$. Since $F(\cdot) - u$ is pseudomonotone, it follows that $\langle y^* - u, x - y\rangle \le 0$ for all $y \in K$ and $y^* \in F(y)$. This shows that $\hat{f}(x, u) \le 0$. On the other hand, the definition of $\hat{f}$ implies that $\hat{f}(x, u) \ge 0$ for every $x \in K$. Hence $\hat{f}(x, u) = 0$.

Conversely, assume that $\hat{f}(x, u) = 0$. Let $y \in K$ and define $y_t = x + t(y - x)$ for $t \in (0, 1]$. Pick $y_t^* \in F(y_t)$. Then the definition of $\hat{f}$ implies $\langle y_t^* - u, x - y\rangle \le 0$. Since F is upper hemicontinuous with compact values, we assume without loss of generality that $(y_t^*) \to z^* \in F(x)$ as $t \to 0+$. Thus $\langle z^* - u, x - y\rangle \le 0$. Since $y \in K$ is arbitrary,

$$\sup_{y \in K} \inf_{x^* \in F(x)} \langle x^* - u, x - y\rangle \le 0.$$

Since $F(x)$ is a nonempty compact convex set and K is a closed convex set, we can apply Sion's minimax theorem [9] to obtain that

$$\inf_{x^* \in F(x)} \sup_{y \in K} \langle x^* - u, x - y\rangle = \sup_{y \in K} \inf_{x^* \in F(x)} \langle x^* - u, x - y\rangle \le 0.$$

Since $x^* \mapsto \sup_{y \in K} \langle x^* - u, x - y\rangle$ is clearly lower semicontinuous and $F(x)$ is compact, it follows that there exists $x^* \in F(x)$ such that $\sup_{y \in K} \langle x^* - u, x - y\rangle \le 0$, so x solves GVIP$(F - u, K)$. □

LEMMA 2.2. *Let $u \in \mathbb{R}^n$ be such that $F(\cdot) - u$ is pseudomonotone on K and that $\hat{f}(\cdot, u)$ is level-bounded on K. Then there exists $x \in K$ such that x solves* GVIP$(F - u, K)$ *and hence $\hat{f}(x, u) = 0$.*

PROOF. Define $K_i = \{x \in K \mid \|x\| \le i\}$, $i \in \mathbb{N}$, and

$$\hat{f}_i(x, u) = \sup\left\{ \frac{\langle y^* - u, x - y\rangle}{\max\{\|y^*\|, 1\} \max\{\|y\|, 1\}} \;\middle|\; y^* \in F(y),\, y \in K_i \right\}.$$

The boundedness of K_i implies that (see [3, Corollary 1]) there exists some $x_i \in K_i$ that solves GVIP$(F - u, K_i)$. For some $x_i^* \in F(x_i)$,

$$\langle x_i^* - u, x_i - y\rangle \le 0, \quad \text{for all } y \in K_i. \tag{2}$$

https://doi.org/10.1017/S1446788700014609 Published online by Cambridge University Press

Then $\hat{f}_i(x_i, u) = 0$ by Proposition 2.1. We claim that $\|x_i\| < i$ for some $i \in \mathbb{N}$. Otherwise, $\|x_i\| = i$ for each i. Note that $\{\hat{f}_i(x, u)\}$ is increasing and pointwise converging on K to $\hat{f}(x, u)$. Since $\hat{f}(x, u)$ is level-bounded on K, [7, Corollary 4.12] shows that $\hat{f}_{i_0}(x, u)$ is level-bounded on K for some $i_0 \in \mathbb{N}$. Set

$$C = \left\{x \in K \mid \hat{f}_{i_0}(x, u) \le 1\right\} \quad \text{and} \quad c = \sup_{x \in C} \|x\|.$$

Then C is compact and nonempty. If $i > \max\{i_0, c\}$, then $x_i \notin C$. Hence $\hat{f}_i(x_i, u) > 1$ which contradicts $\hat{f}_i(x_i, u) = 0$. Thus there must exist j such that $\|x_j\| < j$. We claim that x_j solves GVIP$(F - u, K)$. Indeed for each $y \in K$ there exists sufficiently small $t > 0$ such that $x_j + t\,(y - x_j) \in K_j$. It follows from (2) that $t\,\langle x_j^* - u, x_j - y\rangle \le 0$, and hence $\langle x_j^* - u, x_j - y\rangle \le 0$. Thus x_j solves GVIP$(F - u, K)$, and hence $\hat{f}(x_j, u) = 0$ by Proposition 2.1. □

THEOREM 2.3. *Let K be a nonempty closed convex set with* $\operatorname{int}(K^+)$ *nonempty, and suppose that there exists a sequence* $\{u_m\} \subset \operatorname{int}(K^+)$ *decreasing with respect to* K^+ *and converging to zero such that F is stably pseudomonotone on K with respect to* $\{u_m\}$. *Then* $F(K) \cap \operatorname{int}(-\operatorname{barr}(K)) \neq \emptyset$ *if and only if* GVIP(F, K) *has a nonempty and bounded solution set.*

PROOF. Suppose that GVIP(F, K) has a nonempty and bounded solution set. By Proposition 2.1, the set $\{x \in K \mid \hat{f}(x, 0) = 0\}$ is nonempty and bounded. Hence, [6, Corollary 8.7.1] implies that $\hat{f}(\cdot, 0)$ is level-bounded on K. We claim that there exists $m \in \mathbb{N}$ such that $\hat{f}(\cdot, u_m)$ is also level-bounded on K. To see this, we define $h_m : K \to \mathbb{R}$ for each $m \in \mathbb{N}$ by $h_m(x) := \hat{f}(x, 0) - \langle u_m, x\rangle$. Clearly $\{h_m\}$ is increasing and pointwise converges to $\hat{f}(x, 0)$. By [7, Corollary 4.12], h_m is level-bounded on K if m is large enough. Thus our claim will stand if one can show that

$$h_m(x) \le \hat{f}(x, u_m), \quad \text{for all } x \in K,\ m \in \mathbb{N}. \tag{3}$$

To verify (3), let $x \in K$ and $m \in \mathbb{N}$. For any $\varepsilon > 0$, the definition of $\hat{f}(x, 0)$ implies that there exist $y \in K$ and $y^* \in F(y)$ such that

$$\hat{f}(x) < \frac{\langle y^*, x - y\rangle}{\max\{\|y^*\|, 1\} \max\{\|y\|, 1\}} + \varepsilon.$$

This implies that

$$\begin{aligned}
&\hat{f}(x, 0) - \hat{f}(x, u_m) \\
&\qquad \le \frac{\langle y^*, x - y\rangle}{\max\{\|y^*\|, 1\} \max\{\|y\|, 1\}} - \frac{\langle y^* - u_m, x - y\rangle}{\max\{\|y^*\|, 1\} \max\{\|y\|, 1\}} + \varepsilon
\end{aligned}$$

https://doi.org/10.1017/S1446788700014609 Published online by Cambridge University Press

$$
\begin{aligned}
&= \frac{\langle u_m, x - y\rangle}{\max\{\|y^*\|, 1\}\max\{\|y\|, 1\}} + \varepsilon \\
&\leq \frac{\langle u_m, x\rangle}{\max\{\|y^*\|, 1\}\max\{\|y\|, 1\}} + \varepsilon \\
&\leq \langle u_m, x\rangle + \varepsilon,
\end{aligned}
$$

where the last two inequalities hold because $u_m \in K^+$ and $x, y \in K$. Letting $\varepsilon \to 0+$, it follows that $h_m(x) = \hat{f}(x, 0) - \langle u_m, x\rangle \leq \hat{f}(x, u_m)$. Thus (3) holds and $\hat{f}(\cdot, u_m)$ is level-bounded on K. Consequently, by virtue of Lemma 2.2, there exists some $x_m \in K$ such that x_m solves $\mathrm{GVIP}(F - u_m, K)$. This implies that $x_m^* - u_m \in -\operatorname{barr}(K)$ and so $x_m^* \in \operatorname{int}(-\operatorname{barr}(K))$. Therefore $x_m^* \in F(x_m) \cap \operatorname{int}(-\operatorname{barr}(K))$. Conversely, suppose that there exist $x_0 \in K$ and $x_0^* \in F(x_0)$ such that $x_0^* \in \operatorname{int}(-\operatorname{barr}(K))$. Define

$$D := \{x \in K \mid \langle x_0^*, x - x_0\rangle \leq 0\}.$$

By Lemma 1.1, D is bounded. Let $x \in K \setminus D$. Since F is pseudomonotone, $\langle x^*, x - x_0\rangle > 0$ for all $x^* \in F(x)$. Thus one can apply [3, Corollary 1] to conclude that the solution set of $\mathrm{GVIP}(F, K)$ is nonempty and bounded. ☐

We end this note with two simple examples to show that the condition on stable pseudomonotonicity in Theorem 2.3 cannot be dropped. Let

$$K = \left\{(x, y) \in \mathbb{R}^2 \mid x + iy = r\exp(i\theta) \text{ with } \theta \in [\pi/6, \pi/3] \text{ and } r \geq 0\right\}.$$

EXAMPLE 1. For each $(x, y) \in K$ with $x + iy = r\exp(i\theta)$, define

$$F(x, y) = r^8\big(\cos(8\theta - 5\pi/6), \sin(8\theta - 5\pi/6)\big).$$

Then $\mathrm{GCP}(F, K)$ is strictly feasible with unbounded solution set.

EXAMPLE 2. For each $(x, y) \in K$ with $x + iy = r\exp(i\theta)$, define

$$F(x, y) = r^6\sqrt{r}\big(\cos(13\theta/2 - 3\pi/8), \sin(13\theta/2 - 3\pi/8)\big).$$

Then $\mathrm{GCP}(F, K)$ has a bounded solution set, but is not strictly feasible.

References

[1] B. Chen, X. Chen and C. Kanzow, 'A penalized Fischer-Burmeister NCP-function', *Math. Program. Ser. A* **88** (2000), 211–216.

[2] J.-P. Crouzeix, 'Pseudomonotone variational inequality problems: Existence of solutions', *Math. Program. Ser. A* **78** (1997), 305–314.

https://doi.org/10.1017/S1446788700014609 Published online by Cambridge University Press

[3] A. Daniilidis and N. Hadjisavvas, 'Coercivity conditions and variational inequalities', *Math. Program. Ser. A* **86** (1999), 433–438.
[4] Y. He, 'A relationship between pseudomonotone and monotone mappings', *Appl. Math. Lett.* **17** (2004), 459–461.
[5] L. McLinden, 'Stable monotone variational inequalities', *Math. Program. Ser. B* **48** (1990), 303–338.
[6] R. T. Rockafellar, *Convex analysis* (Princeton University Press, Princeton, N.J., 1970).
[7] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis* (Springer, Berlin, 1998).
[8] R. Saigal, 'Extension of the generalized complementarity problem', *Math. Oper. Res.* **1** (1976), 260–266.
[9] M. Sion, 'On general minimax theorems', *Pacific J. Math.* **8** (1958), 171–176.

Department of Mathematics
Sichuan Normal University
Chengdu, Sichuan
P. R. China
e-mail: yiranhe@hotmail.com

Department of Mathematics
The Chinese University of Hong Kong
Shatin, New Territories
Hong Kong
e-mail: kfng@math.cuhk.edu.hk

https://doi.org/10.1017/S1446788700014609 Published online by Cambridge University Press