Ergod. Th. & Dynam. Sys. (1990), **10**, 411-420
Printed in Great Britain

Minimal convergence on L^p spaces

I. ASSANI
Department of Mathematics, University of Toronto, Ontario, Canada, M5S 1A1

(*Received* 26 *May* 1988, *revised* 14 *August* 1989)

Abstract. Let (X, F, μ) be a probability measure space, p and β real numbers such that $1 \le p < +\infty$ and $0 < \beta < p$. For any linear positive operator T satisfying $T\mathbf{1}$, $T^*\mathbf{1} = \mathbf{1}$ we prove the norm and pointwise convergence of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} |T^i f|^\beta \operatorname{sgn} T^i f \quad \text{for any } f \in L^p(\mu).$$

We get then the pointwise and norm convergence in L^p, $0 < \beta \ge 1 < p < 2$, of the sequence $n^{-1}\sum_{i=0}^{n-1} |S^i f|^\beta \operatorname{sgn} S^i f$ for any positive linear operator on $L^p(\Omega, A, \mu)$ (μ-σ-finite) verifying $\|(1-\alpha)I + \alpha S\|_p \le 1$ for a real number $0 < \alpha \le 1$. In the particular case $\alpha = 1$, (S is a contraction), $\beta = p - 1$, this result gives the pointwise and norm convergence of the sequences $s_n^{(p)}$ introduced by Beauzamy and Enflo in 1985 to the asymptotic center of the sequence $(T^n f)_{n \in \mathbb{N}}$.

0. *Introduction*

Let E be a uniformly convex Banach space and x_n a bounded sequence in E. We are interested in this paper in two minimal procedures.

The first one introduced by Edelstein [**6**] leads to the notion of the asymptotic center of the sequence (x_n). He considered for each integer $m \ge 1$ the unique element c_m which minimizes the function

$$r_m(y) = \sup_{k \ge m} \|x_k - y\|$$

and proved the norm convergence of c_m to an element c in E. If we denote by $r(y) = \lim_m r_m(y)$ then $r(c) < r(y)$ for $y \ne c$. This element c is called the asymptotic center of the sequence x_n. When the sequence x_n is given by the iterates $T^n x$ of a contraction $T: C \to C$ (closed convex subset of E). Then c is a fixed point of T.

The second procedure was introduced by Beauzamy and Enflo [**2**]. For any real number p, $1 < p < +\infty$ and fixed x in C, $s_n^{(p)}$ is the unique element in E which minimizes the function

$$\phi_n^{(p)}: y \to \phi_n^{(p)}(y) = \frac{1}{n}\sum_{i=0}^{n-1} \|T^i x - y\|^p \quad y \text{ in } E.$$

One of the interests of these sequences is that in any Hilbert space and for any p,

$$1 < p < +\infty, \; s_n^{(p)} = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$$

the Cesaro averages of the sequence $(T^i x)_{i\in\mathbb{N}}$. Of course the same procedure can be defined for a general bounded sequence (x_n).

When E is not a Hilbert space there is no explicit expression for $s_n^{(p)}$. Even for linear operators the process of creation of $s_n^{(p)}$ is not linear. So one wonders if these sequences still enjoy the same convergence properties of the Cesaro averages.

The problem of the pointwise convergence of the sequences $s_n^{(p)}$ for a linear positive contraction $T: L^p \to L^p$ has been studied by Guerre and Benozene in [7] and [3]. But the problem of the pointwise and norm convergence when $1<p<2$ remained open. If we denote by

$$d\phi_n^{(p)}(y) = \frac{1}{n}\sum_{i=0}^{n-1} |T^i f - y|^{p-1}\,\mathrm{sgn}\,(T^i f - y)$$

and c the asymptotic center of the sequence $(T^i f)_{i\in\mathbb{N}}$, f in L^p, the pointwise convergence of the sequence $s_n^{(p)}$ appears as a consequence of the pointwise convergence of $d\phi_n^{(p)}(c)$. But c being a fixed point of T we have

$$\begin{aligned} d\phi_n^{(p)}(c) &= \frac{1}{n}\sum_{i=0}^{n-1} |T^i(f-c)|^{p-1}\,\mathrm{sgn}\,(T^i(f-c)) \\ &= \frac{1}{n}\sum_{i=0}^{n-1} |T^i(g)|^{p-1}\,\mathrm{sgn}\,(T^i(g)). \end{aligned}$$

So to get the pointwise convergence in L^p, $1<p<2$ we just need to consider the pointwise convergence of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} |T^i(g)|^{p-1}\,\mathrm{sgn}\,(T^i g) \quad \text{for any } g\in L^p.$$

We are going in fact to prove the pointwise and norm convergence of the sequence $n^{-1}\sum_{i=0}^{n-1} |T^i f|^\beta\,\mathrm{sgn}\,(T^i f)$ when T is not necessarily a contraction on L^p. More precisely we shall prove the pointwise and norm convergence for the class C_α of linear positive operators T such that $\|(1-\alpha)I+\alpha T\|_p \le 1$ for a real number α, $0<\alpha<1$, for such operators we recently obtained [1] a dominated and pointwise ergodic theorem in L^p. Let us remark that for $\alpha=1$ we get the set of linear positive contractions on L^p and that there exists simple examples of operators T satisfying $\|(1-\alpha)I+\alpha T\|_p \le 1$ and which are not contractions. For instance $T = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, $\varepsilon = \frac{3}{2}$, $\alpha = \frac{1}{2}$, $p=2$.

The present article is divided in two parts. In the first we prove the pointwise and norm convergence of the sequence $n^{-1}\sum_{i=0}^{n-1} |T^i f|^\beta\,\mathrm{sgn}\,(T^i f)$ for f in L^p, $1\le p< +\infty$, $0<\beta<p$. The measure space is a probability one and the linear positive operator satisfies $T\mathbf{1}=\mathbf{1}$, $T^*\mathbf{1}=\mathbf{1}$. We distinguish in this first part two cases: (i) $1<\beta<p$, (ii) $0<\beta<1\le p$. The first case appears as a direct consequence of the subadditive ergodic theorem for Markovian operators satisfying $T\mathbf{1}=\mathbf{1}$. In the second case we use an almost subadditive property to get the norm convergence in L^1. This result does not appear as a simple consequence of known results on subadditive theorems. The words 'almost subadditive property' come from [**5**]. We will use also some ideas developed in [**5**].

Then we get the pointwise convergence in L^1. In the second part we use these results to get the pointwise and norm convergence of the sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn} T^if$ when T belongs to the class C_α, $1<p\leq 2$ and $f\in L^p$.

This gives us the pointwise and norm convergence of the sequence s_n^p.

1.

The probability measure space is (Ω, A, m). The sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta$ $\operatorname{sgn}(T^if)$ can be written as

$$\frac{1}{n}\sum_{i=0}^{n-1}((T^if)^+)^\beta-\frac{1}{n}\sum_{i=0}^{n-1}((T^if)^-)^\beta.$$

So it is enough to consider $n^{-1}\sum_{i=0}^{n-1}(T^if)^{+\beta}$. The change $f\to -f$ will give the result for $n^{-1}\sum_{i=0}^{n-1}((T^if)^-)^\beta$. As we said in the introduction we distinguish two cases. $1<\beta<p$, $0<\beta<1\leq p$. The following lemma can be proved as the one well-known for conditional expectations. (See also Lemma I.7.4 in [**8**].)

LEMMA I.1. *For any positive linear operator S on $L^1(\Omega, A, m)$, f, $g\geq 0$, $f\in L^r(m)$, $g\in L^{r^*}(m)$, $1<r<+\infty$, $r^*=r/(r-1)$, we have*

$$S(f\cdot g)\leq (S(f^r))^{1/r}\cdot (S(g^{r^*}))^{1/r^*}.$$

(A) $1<\beta<p$.

THEOREM A.1. *For any positive linear operator on $L^1(\Omega, A, m)$ such that $T\mathbf{1}=\mathbf{1}$, $T^*\mathbf{1}=\mathbf{1}$ and any f in $L^p(m)$ the sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn}(T^if)$ converges* a.e. *and in norm in L^1.*

Proof. It is enough to prove the result for the sequence $n^{-1}\sum_{i=0}^{n-1}(T^if)^{+\beta}$. If we write $S_n=\sum_{i=0}^{n-1}(T^if)^{+\beta}$ we have

$$\begin{aligned} T^k(S_n) &= \sum_{i=0}^{n-1} T^k((T^if)^{+\beta}) \\ &\geq \sum_{i=0}^{n-1}(T^k(T^if)^+)^\beta \quad \text{by Lemma I.1} \\ &\geq \sum_{i=0}^{n-1}(T^k(T^if))^{+\beta} \\ &= S_{n+k}-S_k \end{aligned}$$

So $S_{n+k}\leq T^k(S_n)+S_k$. We have a subadditive sequence with respect to $\bar{T}$ and the sequence S_n/n converges a.e. and in norm by the subadditive theorem for Markovian operator [**8**]. (In this case $T\mathbf{1}=\mathbf{1}$ and the conclusion can follow from a simpler argument.)

(B) $0<\beta<1\leq p$.

We need just to consider the case $p=1$.

LEMMA B.2. *Let T be a Markovian operator on $L^1(\Omega, A, m)$ satisfying $T\mathbf{1}=\mathbf{1}$. Then for any f in $L^1(m)$ the sequence $n^{-1}\sum_{i=1}^{n}|T^if|$ converge* a.e. *and in norm in L^1.*

Proof. Let us note $S_n = \sum_{i=1}^{n} |T^i f|$. Then for any integer k we have

$$T^k(S_n) = \sum_{i=1}^{n} T^k(|T^i f|)$$

$$\geq \sum_{i=1}^{n} |T^{k+i} f| = S_{n+k} - S_k$$

and the result follows from the subadditive theorem for Markovian operators.

PROPOSITION B.3. *Let T be a Markovian contraction on $L^1(m)$ verifying $T1 = 1$ and f in $L^1(m)$. If we denote by $S_n = -\sum_{i=1}^{n} (T^i f)^{+\beta}$ then for any integers n, k $n \geq 1$ we have*

$$S_{n+k} \leq S_k + T^k(S_n) + \sum_{i=1}^{n} \left(\frac{T^k(|T^i f|) - |T^{k+i} f|}{2} \right)^{\beta}.$$

Proof. If $S'_n = \sum_{i=1}^{n} (T^i f)^{+\beta}$ then

$$T^k(S'_n) = T^k \left(\sum_{i=1}^{n} \left(\frac{T^i f + |T^i f|}{2} \right)^{\beta} \right)$$

$$\leq \sum_{i=1}^{n} \left(T^k \left(\frac{T^i f + |T^i f|}{2} \right) \right)^{\beta} \quad \text{by Lemma I.1}$$

$$= \sum_{i=1}^{n} \left((T^k(T^i f))^{+} + \frac{T^k(|T^i f|) - |T^{k+i} f|}{2} \right)^{\beta}$$

$$\leq \sum_{i=1}^{n} (T^{k+i} f)^{+\beta} + \sum_{i=1}^{n} \left(\frac{T^k(|T^i f|) - |T^{k+i} f|}{2} \right)^{\beta}$$

$$\geq S'_{n+k} - S'_k + \sum_{i=1}^{n} \left(\frac{T^k(|T^i f|) - |T^{k+i} f|}{2} \right)^{\beta}.$$

PROPOSITION B.4. *Under the assumptions of Proposition B.3 the sequence*

$$\gamma_n = \int \frac{S_n}{n} \cdot dm \quad \text{converges to a real number } \gamma.$$

Proof. Let us fix the integer $n > 1$ and note $m = nl + r$, $0 \leq r < n$. Then using Proposition B.3 we have

$$S_m \leq \sum_{j=0}^{l-1} T^{nj}(S_n) + T^{nl}(S_r) + \sum_{i=0}^{r} \left(\frac{T^{nl}(|T^i f|) - |T^{nl+i} f|}{2} \right)^{\beta}$$

$$+ \sum_{j=0}^{l-1} \sum_{i=1}^{n} \left(\frac{T^{nj}(|T^i f|) - |T^{nj+i} f|}{2} \right)^{\beta}.$$

The operator T being Markovian we have

$$\gamma_m \leq \frac{nl}{m} \cdot \gamma_n + \frac{1}{m} \int S_r \cdot dm + \sum_{i=0}^{r} \frac{1}{m} \int \left(\frac{T^{nl}(|T^i f|) - |T^{nl+i} f|}{2} \right)^{\beta} dm$$

$$+ \frac{nl}{m} \cdot \frac{1}{nl} \int \sum_{j=0}^{l-1} \sum_{i=1}^{n} \left(\frac{T^{nj}(|T^i f|) - |T^{nj+i} f|}{2} \right)^{\beta} \cdot dm.$$

By using the concavity of the function $x \to x^\beta$ and the fact that $(\int |F|^\beta \cdot dm) \le (\int |F|\, dm)^\beta$ for any function F in $L^1(m)$ the last term of the previous inequality is bounded by

$$\frac{nl}{m}\left(\int \left(\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{i=1}^{n}\left(\frac{T^{nj}(|T^i f|) - |T^{nj+i} f|}{2}\right)\right)^\beta \cdot dm\right.$$

$$\le \frac{nl}{m}\left(\int \left(\frac{1}{2n}\sum_{i=1}^{n}|T^i f| - \frac{1}{2nl}\sum_{k=1}^{nl}|T^k f|\right) dm\right)^\beta.$$

If $f^* = \lim_N N^{-1}\sum_{i=0}^N |T^i f|$ which exists by lemma B.2 we have

$$\limsup_m \gamma_m \le \gamma_n + \left(\int \left(\frac{\sum_{i=1}^n |T^i f|}{2n} - \frac{f^*}{2}\right) dm\right)^\beta.$$

By using again Lemma B.2 we have

$$\limsup_m \gamma_m \le \liminf_n \gamma_n,$$

which implies the convergence of the sequence γ_n.

PROPOSITION B.5. *The sequence S_n/n converges in L^1 norm to*

$$\lim_l \frac{1}{l}\left[\lim_j \frac{1}{j}\sum_{i=1}^{j} T^{il}(S_l)\right].$$

Proof. We fix again the integer n and take $m = nl + r$. Then as $T^{nl}(S_r) \le 0$ we have

$$S_m \le \sum_{j=0}^{l-1} T^{nj}(S_n) + \sum_{j=0}^{l-1}\sum_{i=1}^{n}\left(\frac{T^{nj}(|T^i f|) - |T^{nl+i} f|}{2}\right)^\beta$$

$$+ \sum_{i=0}^{r}\left(\frac{T^{nl}(|T^i f|) - |T^{nl+i} f|}{2}\right)^\beta.$$

By the ergodic theorem for T^n (see [7]) $\lim_l l^{-1}\sum_{j=0}^{l-1} T^{nj}(S_n) = h_n$ exists a.e. and in L^1 norm.

As

$$\int \left(\frac{S_m}{m} - \frac{1}{n}\cdot h_n\right)^+ \cdot dm \le \frac{1}{m}\sum_{i=0}^{r}\int \left(\frac{T^{nl}(|T^i f|) - |T^{nl+i} f|}{2}\right)^\beta \cdot dm$$

$$+ \frac{1}{m}\int \left(\sum_{j=0}^{l-1}\sum_{i=1}^{n}\left(\frac{T^{nj}(|T^i f|) - |T^{nj+i} f|}{2}\right)^\beta\right) dm$$

$$+ \int \left(\frac{1}{m}\sum_{j=0}^{l-1} T^{nj}(S_n) - \frac{1}{n}\cdot h_n\right)^+ dm$$

we have

$$\limsup_m \int \left(\frac{S_m}{m} - \frac{1}{n}\cdot h_n\right)^+ \cdot dm \le \left(\int \left(\sum_{i=1}^{n}\frac{|T^i f|}{2n} - \frac{f^*}{2}\right) dm\right)^\beta$$

by the same arguments as those used in the proof of Proposition B.4.

For $n \ge n_0(\varepsilon)$ we have then

$$\limsup_m \int \left(\frac{S_m}{m} - \frac{1}{n} h_n\right)^+ dm \le \varepsilon \quad \text{and for } m$$

https://doi.org/10.1017/S0143385700005666 Published online by Cambridge University Press

large enough greater than n

$$\int\left(\frac{S_m}{m}-\frac{1}{n}h_n\right)^+ dm \le \varepsilon.$$

We can choose $n_0(\varepsilon)$ such that

$$\left|\frac{1}{n}\cdot\gamma_n-\gamma\right|<\varepsilon.$$

Then as $\int|g|\,dm=2\int g^+\cdot dm-\int g\cdot dm$ we have

$$\begin{aligned}\int\left|\frac{S_m}{m}-\frac{1}{n}\cdot h_n\right|\cdot dm &\le 2\int\left(\frac{S_m}{m}-\frac{1}{n}\cdot h_n\right)^+ dm+\left|\int\left(\frac{S_m}{m}-\frac{1}{n}h_n\right)dm\right|\\ &\le 2\varepsilon+|\gamma_m-\gamma_n| \quad \text{because} \int h_n\,dm=\int S_n\,dm\\ &\le 2\varepsilon+\varepsilon=3\varepsilon.\end{aligned}$$

The sequences S_m/m and h_n/n are Cauchy sequences in L^1. They converge to the same limit in L^1 norm:

$$\lim_l \frac{1}{l}\left[\lim_j \frac{1}{j}\sum_{i=1}^{j} T^{il}(S_l)\right].$$

THEOREM B.6. *For any positive linear contraction T on $L^1(\Omega, A, m)$ such that $T\mathbf{1}=\mathbf{1}$, $T^*\mathbf{1}=\mathbf{1}$ and any f in $L^1(m)$ the sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta$ sgn (T^if) converges* a.e. (*for* $0<\beta\le 1$) *in* L^1.

Proof. It is enough to prove the pointwise convergence of

$$\frac{1}{n}\sum_{i=0}^{n-1}(T^if)^{+\beta}.$$

We remark first that for any positive contraction U on $L^1(m)$ verifying $U\mathbf{1}=\mathbf{1}$ the sequence $n^{-1}\sum_{i=0}^{n=1}(U^i(g))^\beta$ converges a.e. In fact if we note $V_n=\sum_{i=1}^n(U^ig)^\beta$ then

$$\begin{aligned}U^kV_n&=\sum_{i=1}^n U^k(U^ig)^\beta\\ &\le\sum_{i=1}^n(U^k(U^ig))^\beta\\ &=V_{n+k}-V_k \quad (g\ge 0),\end{aligned}$$

for any integers k and n. The result follows again by the subadditive (or superadditive) ergodic theorem.

Now we fix the integer $n\ge 1$, we have for any $l\ge 1$

$$\begin{aligned}\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=0}^{n}(T^{nj}(T^kf))^{+\beta}&\le\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}(T^{nj}(T^kf)^+)^\beta\\ &\le\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}\left((T^{nj}(T^kf))^{+\beta}+\left(\frac{T^{nj}(|T^kf|)-|T^{nj+k}f|}{2}\right)^\beta\right)\end{aligned}$$

https://doi.org/10.1017/S0143385700005666 Published online by Cambridge University Press

by using the inequality

$$(T^{nj}(g^+))^\beta \le (T^{nj}g)^{+\beta} + \left(\frac{T^{nj}(|g|) - |T^{nj}g|}{2}\right)^\beta.$$

So if we note $S_m = \sum_{k=1}^m (T^k f)^{+\beta}$ we have

$$(*) \qquad \frac{S_{nl}}{nl} \le \frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}(T^{nj}(T^k f)^+)^\beta \le \frac{S_{nl}}{nl} + \left(\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}\frac{T^{nj}(|T^k f|) - |T^{nj+k}f|}{2}\right)^\beta$$

(by the concavity of $x \to x^\beta$).

By the ergodic theorem applied to T^n and Lemma B.2

$$\lim_l \left(\frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n} T^k(|T^{nj}f|) - \frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}|T^{nj+k}f|\right)$$

$$= h_n^* - f^* \quad \text{a.e. where} \int h_n^*\, dm = \int \frac{1}{n}\sum_{k=1}^{n}|T^k f|\, dm.$$

So from (*) and the fact that $\lim_{m\to\infty} S_{m+1}/(m+1) - (S_m/m) = 0$ a.e. we have

$$\overline{\lim}\, \frac{S_m}{m} \le \overline{\lim_l}\, \frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}(T^{nj}(T^k f)^+)^\beta$$

$$= \lim_l \frac{1}{nl}\sum_{j=0}^{l-1}\sum_{k=1}^{n}(T^{nj}(T^k f)^+)^\beta$$

(by the remark made at the beginning of the proof)

$$\le \underline{\lim}\, \frac{S_m}{m} + \left(\frac{h_n^* - f^*}{2}\right)^\beta.$$

So

$$\int \left(\overline{\lim}\, \frac{S_m}{m} - \underline{\lim}\, \frac{S_m}{m}\right) dm \le \int \left(\frac{h_n^* - f^*}{2}\right)^\beta dm$$

$$\le \left(\int \left(\frac{h_n^* - f^*}{2}\right) dm\right)^\beta.$$

If we let n go to the infinity we get

$$\int \left(\overline{\lim}\, \frac{S_m}{m} - \underline{\lim}\, \frac{S_m}{m}\right) dm = 0$$

which proves the pointwise convergence of S_n/n. The convergence holds in L^1 because of the norm convergence in L^1 proved in Proposition B.4.

COROLLARY B.7. *Under the assumptions of Theorem* B.6 *but $f \in L^p$ for $1 < p < +\infty$, then the pointwise and norm convergence holds in $L^{p/\beta}$.*

Proof. It suffices to prove that

$$\sup_n \frac{1}{n}|T^i f|^\beta \in L^{p/\beta}.$$

We have

$$\sup_{n<N}\left(\frac{1}{n}|T^i f|^\beta\right) \le \sup_{n<N}\left(\frac{1}{n}T^i|f|\right)^\beta$$

https://doi.org/10.1017/S0143385700005666 Published online by Cambridge University Press

so

$$\int \left(\sup_{n\le N} \frac{1}{n} |T^i f|^\beta\right)^{p/\beta} dm \le \int \left(\sup_{n\le N} \frac{1}{n} T^i|f|\right)^p dm \le K \int |f|^p\, dm$$

(as a particular case of the estimate in [1]) and $\|\sup_n n^{-1}|T^i f|^\beta\|_{p/\beta} \le K^{\beta/p}\|f\|_p^\beta$.

Remarks

(1) It is clear that when f is in L^p, $1<p<+\infty$ by using only the proof of Theorem B.6 and the fact that $\sup_n n^{-1}\sum_{i=0}^{n-1}|T^i f|^\beta \in L^{p/\beta}$ we can get the pointwise and norm convergence in $L^{p/\beta}$.

(2) A consequence of Theorem B.6 is the pointwise and norm convergence in L^1 of the sequence $n^{-1}\sum_{i=0}^{n-1}|T^i f|^\beta$. We can also get the same conclusion when T is not necessarily positive but its linear modulus $\boldsymbol{T}$ satisfies $\boldsymbol{T}\mathbf{1}=\mathbf{1}$ and $\boldsymbol{T}^*\mathbf{1}=\mathbf{1}$.

We apply now the results of this first part to operators in any class C_α, $0<\alpha\le 1$.

2.

PROPOSITION 2.1. *Let T be a positive linear operator on $L^p(X, F, \mu)$ $1<p<+\infty$ such that $\|(1-\alpha)I+\alpha T\|_p \le 1$ for a real number α, $0<\alpha\le 1$. Then there exists a decomposition of the space X in two disjoint parts E and E^c invariant by T. (i.e. $T(L^p(E))\subset L^p(E)$ and $T(L^p(E^c))\subset L^p(E^c)$). Furthermore there exists h in $L^p(\mu)$ such that* supp $h=E$, *$Th=h$ and $T^*(h^{p-1})=h^{p-1}$.*

Proof. Let h be a function in L^p invariant by T with maximal support E. This function h is also invariant with maximal support for $S=(1-\alpha)I+\alpha T$. As

$$\int S(h)\cdot h^{p-1}\, d\mu = \int h^p\, d\mu = \int hS^*(h^{p-1})\, d\mu,$$

we have also $S^*(h^{p-1})=h^{p-1}$ and E is the maximal support of the invariant functions of S^* and then of T^*. (Note: h^{p-1} is the only element in L^q (such that $\int h\cdot h^{p-1}\, d\mu = \|h\|_p^p = \|h^{p-1}\|_q^q$.) We have to show that E and E^c are invariant by T (and also T^*). We have

$$\int T(\mathbf{1}_{E^c}\cdot f)\cdot h^{p-1}\, d\mu = \int \mathbf{1}_{E^c} f\cdot T^*(h^{p-1})\, d\mu$$
$$= \int \mathbf{1}_{E^c} f\cdot h^{p-1}\, d\mu = 0.$$

By analogy we also have

$$\int T^*(\mathbf{1}_{E^c} g)\cdot h\, d\mu = \int \mathbf{1}_{E^c} g\cdot h\, d\mu = 0,$$

which proves that E^c is invariant by T and T^*. Let us denote by P and P^*, the projections obtained by the mean ergodic theorem (a consequence of the result obtained in [1]). If $\tilde g$ (resp. $\tilde f$) is a strictly positive function such that

$$E^c = \text{supp}\, h(\tilde g - P^*\tilde g) \quad (\text{resp. } E^c = \text{supp}\, h(\tilde f - P(\tilde f))$$

then we have

$$\int T(\mathbf{1}_E f)\cdot(\tilde g - P^*(\tilde g))\,d\mu = \int (\mathbf{1}_E f)\cdot(T^*(\tilde g) - T^*P^*(\tilde g))\,d\mu$$

$$= 0 \quad \text{as } E^c \text{ is invariant by } T^*.$$

By analogy we have

$$\int T^*(\mathbf{1}_E g)\cdot(\tilde f - P(\tilde f))\,d\mu = 0 \quad \text{because } E^c \text{ is invariant.}$$

THEOREM 2.3. *Let $0<\beta<1<p\le 2$, for any positive operator T on $L^p(\mu)$ such that $\|(1-\alpha)I+\alpha T\|_p\le 1$ and for any function f in $L^p(\mu)$ the sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn}(T^if)$ converges* a.e. *and in norm in $L^{p/\beta}$.*

Proof. We have

$$\frac{1}{n}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn}(T^if)| \le \left(\frac{1}{n}\sum_{i=0}^{n-1} T^i|f|\right)^\beta.$$

By Proposition 2.2 the space Ω can be divided in two disjoint parts E and E^c both invariant by T. There exists also h in $L^p(\mu)$ such that $\operatorname{supp} h = E$, $Th = h$ and $T^*(h^{p-1}) = h^{p-1}$.

Because of the pointwise ergodic theorem [1] we have

$$\mathbf{1}_{E^c}\cdot\frac{1}{n}\left(\sum_{i=0}^{n-1} T^i(|f|)\right)\to 0 \quad \text{a.e. and so}$$

$$\mathbf{1}_{E^c}\cdot\left|\frac{1}{n}\sum_{i=0}^{n-1}|T^if|^p \operatorname{sgn}(T^if)\right|\to 0 \quad \text{a.e.}$$

The operator $S: S(g) = T(g\cdot h)/h$ on the space $L^p(E, m)$ (where $m(A) = \int_A h^p\,d\mu$) is a Markovian operator contraction on $L^1(m)$ verifying also $S\mathbf{1} = \mathbf{1}$ as

$$S^*(s) = \frac{T^*(s\cdot h^{p-1})}{h^{p-1}}.$$

As $S^i(g) = T^i(g\cdot h)/h$ for any integer $i\ge 0$ the pointwise convergence of the sequence $n^{-1}\sum_{i=0}^{n-1}|S^i(g)|^\beta \operatorname{sgn} S^ig$ valid by Theorem B.6 implies the same con-sequence for the sequence $n^{-1}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn} T^if$ on E. The norm convergence follows from the following inequality consequence of the dominated estimate in [1] and the concavity of $x\to x^\beta$

$$\left\|\sup_n \frac{1}{n}\sum_{i=0}^{n-1}|T^if|^\beta \operatorname{sgn} T^if\right\|_{p/\beta} \le (\gamma(\alpha))^{\beta/p}\left(\frac{p}{p-1}\right)^\beta \|f\|_p^\beta. \qquad \square$$

To see how these results can be applied to the sequences $s_n^{(p)}$ we need the following propositions. The first one can be obtained following the proof of Bruck and Reich [4] (taking the function $\theta(y) = \lim\|T^nx-y\|^p$ instead of $\frac{1}{2}\lim\|T^nx-y\|^2$) and the fact that $r(c)<r(y)$ for $y\ne c$ (c is the asymptotic center). The second proposition uses ideas of Beauzamy and Enflo [2] in their proof of the weak convergence in l^p of the sequence $s_n^{(p)}$. (Just use pointwise the scalar inequalities established in Lemma

6 for $2 \leq p < \infty$ and Lemma 6 for $1 < p \leq 2$ in [2], then the mean value theorem.) Both propositions can also be found in [3].

PROPOSITION 2.4. *Let E be a uniformly smooth Banach space and T a contraction (not necessarily linear) $T: E \to E$. Then if for any x in E we denote by c the asymptotic center of the sequence $(T^n x)_{n\in\mathbb{N}}$, and J_ψ the duality map associated with the function $\psi(r) = r^{p-1}$ $(1 < p < +\infty)$ then $d\phi_n^{(p)}(c) = n^{-1} \sum_{j=0}^{n-1} J_\psi(c - T^j x)$ converges weakly to* 0.

PROPOSITION 2.5. *When $E = L^p(\mu)$, $1 < p < +\infty$, (the same p as the one used for s_n^p) and c is the asymptotic center of the sequence $(T^n f)_n$ then if $d\phi_n(p)(c) \to 0$* a.e. *then*

$$s_n^p \to c \quad \text{a.e.}$$

THEOREM 2.6. *Let (Ω, A, μ) be a σ-finite measure space and T a positive linear contraction on $L^p(\mu)$, $1 < p \leq 2$. If we denote by s_n^p the element which minimises the function $\phi_n^p(y) = n^{-1} \sum_{i=0}^{n-1} \|T^n f - y\|^p$ and c the asymptotic center of the sequence $(T^n f)_{n\in\mathbb{N}}$ then s_n^p converges almost everywhere to c.*

Proof. For $\psi(r) = r^{p-1}$ we have

$$J_\psi(c - T^j x) = p|c - T^j x| \operatorname{sgn}(c - T^j x)$$

and $d\phi_n^{(p)}(c) = (p/n) \sum_{j=0}^{n-1} |c - T^j x|^{p-1} \operatorname{sgn}(c - T^j x)$. As c is a fixed point of T

$$d\phi_n^{(p)}(c) = \frac{p}{n} \sum_{j=0}^{n-1} |T^j(c-x)|^{p-1} \operatorname{sgn}(T^j(c-x)).$$

By Theorem 2.3 (for $\beta = p - 1$) the sequence $d\phi_n^{(p)}(c)$ converges a.e. to a function which must be equal to zero a.e. by Proposition 2.4. We conclude then by using Proposition 2.5.

REFERENCES

[1] I. Assani. Estimates of positive linear operators in L^p. *Proc. Amer. Math. Soc.* accepted.

[2] B. Beauzamy & P. Enflo. Théorèmes de points fixes et d'approximation. *Arkiv Math.* **23** (1985), 19–34.

[3] R. Benozene. Sommes minimales des itérés d'une contraction dans les espaces de Hilbert et de Banach. Thèse de 3^{eme} cycle, Université Pierre et Marie Curie, 29 Octobre 1981.

[4] R. E. Bruck & S. Reich. Accretive operators, Banach limits, and dual ergodic theorems. *Bull. de l'Acad. Polon. des Sciences* **19** Nos. 11–12 (1981), 585–589.

[5] Y. Derriennic. Un théorème ergodique presque sous additif. *Ann. of Prob.* **11**, 669–677.

[6] M. Edelstein. The construction of an asymptotic center with a fixed point property. *Bull. Amer. Math. Soc.* **78** (1972), 206–208.

[7] S. Guerre. Procédé de convergence minimale dans les espaces de Banach. Une loi des grands nombres et un théorème ergodique. *Sem. Geom. Espaces de Banach, Ec. polytech., Cent. Math* (1977–1978), Exposé, 1–14.

[8] U. Krengel. Ergodic theorems. *Studies in Mathematics*, de Gruyter, 1985.