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A CLASS OF STRONGLY DEGENERATE ELLIPTIC OPERATORS

DUONG MINH DUC

Using a weighted Poincaré inequality, we study $(\omega_1, \ldots, \omega_n)$-elliptic operators. This method is applied to solve singular elliptic equations with boundary conditions in $W^{1,2}$. We also obtain a result about the regularity of solutions of singular elliptic equations. An application to $(\omega_1, \ldots, \omega_n)$-parabolic equations is given.

0. INTRODUCTION

Let a_{ij} be continuous functions on the closure $\overline{\Omega}$ of an open subset Ω of $\mathbb{R}^n$, such that for any i, j in $\{1, \ldots, n\}$, $a_{ij} = a_{ji}$. Let b_j, for $1 \leqslant j \leqslant n$, and c be measurable functions on $\overline{\Omega}$. Denote $\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial y}$ by D_j, D_t and D_y respectively. We consider the following boundary value Dirichlet problem

$$\text{(P)} \qquad \begin{cases} Au \equiv -\sum\limits_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) + \sum\limits_{j=1}^{n} b_j D_j u + cu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

If Ω and the coefficient functions of A are bounded and sufficiently regular, also if A is strongly elliptic, the problem (P) has been solved. In [**9, 10, 11**] Fichera studied (P), in the case when A is a degenerate elliptic operator and its coefficient functions are smooth and defined on a bounded open subset Ω. He solved problem (P) under certain conditions on the derivatives of the coefficient functions. Under these conditions c is positive and sufficiently large in many cases [**11**, pp.118–119]. The regularity of solutions of the degenerate elliptic equation (P) is studied by Kohn, Nirenberg, Oleinik and Radkevich [**20, 24**]. Stampacchia studied the case in which the coefficient functions of (P) may be discontinuous in [**30, 31**]. In [**32**] Trudinger studied (P) under the following condition

$$\lambda(x)\sum_{j=1}^{n}\xi_j^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda(x)\sum_{j=1}^{n}\xi_j^2 \qquad \forall x \in \Omega \quad (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

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where λ^{-r} and Λ are integrable on Ω, with $r \geqslant \frac{n}{2}$.

In this paper we suppose that there exist nonnegative real functions $\omega_1, \dots, \omega_n$ and a constant M such that

$$\sum_{j=1}^{n} \omega_j(x)\xi_j^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant M \sum_{j=1}^{n} \omega_j(x)\xi_j^2.$$

In this case A is called an $(\omega_1, .., \omega_n)$-elliptic operator. This operator is more general than the one in [**32**] and is applicable to the case studied in [**6**].

If A is an $(\omega_1, \dots, \omega_n)$-elliptic operator, the Dirichlet problem (P) may be not only degenerate, but also singular. We solve the problem (P) for this operator, defined on a general domain. These results generalise to those in [**7**] (see Theorem 2.1).

In [**23**] Mikhailov studied the Dirichlet problem with boundary conditions in L^2 and smooth coefficient functions. Chabrowski and Thompson [**4**, **5**], extended these results to cases in which the coefficient functions may be nonsmooth and unbounded. The part of a C^2-domain which is near its boundary can be described as a flow-domain (see Definition 1.2). Using this property, Mikhailov, Chabrowski and Thompson obtained an a priori estimate and solved the problem.

In this paper we improve a weighted Poincaré inequality for the flow-domains, established in [**8**]. Using this inequality we solve the singular Dirichlet porblem with boundary conditions in $W^{1,2}$, which is more singular than the one in [**4**, **5**, **23**]. We also answer an open problem, posed by Chabrowski (see Theorem 3.1). Actually, we can solve the Dirichlet problem, which is not only singular, but also degenerate in the sense of Collaps (see Theorem 3.4). Our method is also applicable to the cases in which the domain is unbounded or nonsmooth. We get the regularity of solutions of singular elliptic equations. This result is similar to the classical one for nonsingular elliptic equations (see Theorem 3.3). If $c \equiv 0$ and $b_j \equiv 0$ for any j, the regularity of solutions of degenerate $(\omega_1, \dots, \omega_n)$-elliptic equations is studied in [**12**, **13**].

On the other hand let A be a uniformly elliptic partial differential operator. Under certain conditions on the smoothness and boundedness of coefficient functions and domain, it is well-known that $-(A + \lambda I)$ is the infinitesimal generator of an analytic semigroup of contractions, where λ is a sufficiently large real number [**25**, p.210].

In this paper, we get an explicit condition under which $-A$ is the infinitesimal generator of an analytic semigroup of contractions, where A is an $(\omega_1, \dots \omega_n)$-elliptic operator and may be degenerate or singular. We remark that we do not need any positive real number λ here. This property is useful in the study of nonlinear evolution equations (see Section 6.3 in [**25**]).

The paper consists of four sections. We establish Poincaré inequalities in the first section. The $(\omega_1, \dots, \omega_n)$-elliptic equations are studied in the second section. The third section is devoted to the study of the singular Dirichlet problem. We consider $(\omega_1, \dots, \omega_n)$-parabolic equations in the last section.

1. Weighted Poincaré inequalities

To study elliptic partial differential equations which are degenerate or singular on a subset V of the boundary $\partial\Omega$ of Ω, where Ω is a bounded domain in $\mathbb{R}^n$, we need a priori estimates of solutions and their derivatives near V. On the other hand, the difficulty is due sometimes to the unboundedness of the domain. In this section we improve a weighted Poincaré inequality for flow-domains proved in [8]. Using this inequality we get a priori estimates in the cases considered.

First we define flow-domain. Let m be a nonnegative integer and $\mathbb{R}^m$ be m-dimensional euclidean space. If $m = 0$, consider $\mathbb{R}^m$ as $\{0\}$. Let n be a positive integer and G be a nonempty open subset of $\mathbb{R}^{n-1}$. Let $x \mapsto b_x$ be a map from G into $(0, \infty]$. We denote by D_1 the set $\{(s,x) \in \mathbb{R} \times G : 0 \leqslant s < b_x\}$. Suppose that the interior D of D_1 is the set $\{(s,x) \in D_1 : 0 < s < b_x\}$. For $n = 1$, D_1 is identified with the interval $[0, b_0)$.

Definition 1.1: Let $h = (h_1, \dots, h_n)$ be a one-to-one continuous mapping from D_1 into $\mathbb{R}^n$, such that h is continuously differentiable on D, with its Jacobian determinant $J_h(s,x) \neq 0$, at every (s,x) in D. Put $\Omega = h(D)$; we say that Ω is a *flow-domain parametrised by* (h, D).

Let Ω be flow-domain parametrised by (h, D); we put:

$$I(h) = \{j \in \{1, \dots, n\} \mid (\exists (s,x) \in D)\Big(\frac{\partial h_j}{\partial s}(s,x) \neq 0\Big)\}; \tag{1}$$

for every (y,x) in D, write

$$s(h,x,y) = \int_0^y \left|\frac{\partial h}{\partial t}(t,x)\right|^2 \mathrm{d}t \tag{2}$$

Let ω_j, for $j \in \{1, \dots, n\}$, be measurable nonnegative functions defined on Ω. For every nonnegative measurable function ω defined on Ω and every $j \in \{i, \dots, n\}$ we put:

$$F(h,j,x,y,b) = \int_y^b \frac{s(h,x,\xi)\omega(h(\xi,x))\,|J_h(\xi,x)|}{\omega_j(h(y,x))\,|J_h(y,x)|}\mathrm{d}\xi, \tag{3}$$

$$d(h,\omega;\omega_j) = \sup\{F(h,j,x,y,b_x) : (x,y) \in D\}, \tag{4}$$

$$d(h,\omega;\omega_1,\dots,\omega_n) = \sup_{j \in I(h)} d(h,\omega;\omega_j), \tag{5}$$

where $d(h, \omega; \omega 1, \ldots, \omega_n)$ may be infinite.

DEFINITION 1.2: Let $\{\Omega_j\}_{j\in J}$ be a family of disjoint flow-domains, parametrised by $\{h_j, D_j\}_{j\in J}$. Let Ω be open in $\mathbb{R}^n$, containing the union of $\{\Omega_j\}_{j\in J}$, and let $m\left(\Omega \setminus \bigcup_{j\in J} \Omega_j\right) = 0$, where m is the Lebesgue measure on $\mathbb{R}^n$. We say Ω is a *flow-domain parametrised by* $\{(h_j, D_j)\}_{j\in J}$.

It is clear that the unit ball $\{x\colon \|x\| < 1\}$, the exterior domain $\{x\colon \|x\| > 1\}$ and the star-shaped domain, with C^2 boundary, are flow-domains. There are other examples of flow-domains in [**8**].

Let Ω be a flow-domain parametrised by $\{(h_j, D_j)\}_{j\in J}$. We write

$$I(\{h_j\}) = \bigcup_{j\in J} I(h_j),$$

$$d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n) = \sup_{j\in J} d\left(h_j, \omega^j; \omega_1^j, \ldots, \omega_n^j\right),$$

where ω^j, $\omega_1^j, \ldots, \omega_n^j$ are, respectively, the restrictions of ω, $\omega_1, \ldots, \omega_n$ to Ω_j, for any j in J.

For any j in $\{1, \ldots, n\}$ and $x = (x_1, \ldots, x_n)$ in the boundary $\partial\Omega$ of Ω, we put $J(j, x) = \{t \in \mathbb{R} : (x_1, \ldots, x_{j-1}, x_j + t, x_{j+1}, \ldots, x_n) \in \Omega\}$. In this paper, for any x and j, we suppose that $J(j, x)$ is a finite union of open intervals. We put:

$$\partial_j\Omega = \{x \in \partial\Omega : 0 \in \overline{J(j,x)} \;\&\; \omega_j(x) \neq 0\},$$
$$\partial_0\Omega = h(\{0\} \times G) \cup \partial_1\Omega \cup \ldots \cup \partial_n\Omega.$$

Sometimes we do not need the smoothness of $\partial\Omega$; then the normal vector may not be defined at some x in $\partial\Omega$. But we require that $\partial\Omega$ is partially smooth in every direction as above. We note, for example, that a cylinder in $\mathbb{R}^3$ is such a domain.

We put:

$$\|u\|_* = \{\sum_{1\leqslant j\leqslant n} \int_\Omega |D_j u|^2 \omega_j \mathrm{d}x\}^{\frac{1}{2}},$$

$$S = \{u = v|_{\overline{\Omega}} \mid v \in C^1(\mathbb{R}^n), \text{ having compact support and } \|u\|_* < \infty\},$$

$$S_0 = \{u \in S \mid (\forall x \in \partial_0\Omega)(u(x) = 0)\},$$

$$S_{00} = \{u \in S \mid (\forall x \in \partial\Omega)(u(x) = 0)\},$$

$$\|u\| = \{\int_\Omega |u(x)|^2 \mathrm{d}x\}^{\frac{1}{2}},$$

$$\|u\|_1 = \{\|u\|^2 + \|u\|_*^2\}^{\frac{1}{2}}.$$

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Let W, W_0 and W_{00} be the completions of $(S, \|\cdot\|_1)$, $(S_0, \|\cdot\|_1)$ and $(S_{00}, \|\cdot\|_1)$, respectively. Note that W_0 and W_{00} can be defined for any domain Ω studied in Lemma 1.2.

We have the following Poincaré inequality:

LEMMA 1.1. *Let Ω be a flow-domain parametrised by $\{(h_j, D_j)\}_{j\in J}$, let ω, $\omega_1, \ldots, \omega_n$ be nonnegative measurable funcitons on Ω. Then for every u in W_0 we have:*

$$\int_\Omega |u(y)|^2 \omega(y)dy \leqslant d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n) \sum_{j\in I(\{h_j\})} \int_\Omega |D_j u(y)|^2 \omega_j(y)dy.$$

PROOF: We can suppose $\{(h_j, D_j)\}_{j\in J}$ consists of a unique element (h, D). Let G be as in Definition 1.1. Let u be in S; then for any given x in G we have:

$$\frac{\partial u}{\partial y}(h(y,x)) = \nabla u(h(y,x)) \cdot \frac{\partial h}{\partial y}(y,x) \qquad \text{a.e. on } [0, b_x).$$

By differential calculus [**28**, p.178] and(1) we get:

$$\begin{aligned} u(h(t,x)) &= \int_0^t \frac{\partial u}{\partial y}(h(y,x))\mathrm{d}y \\ &= \int_0^t \nabla u(h(x,y)) \cdot \frac{\partial h}{\partial y}(y,x)\mathrm{d}y \\ &= \int_0^t \sum_{j\in I(h)} D_j u(h(y,x)) \frac{\partial h_j}{\partial y}(y,x)\mathrm{d}y. \end{aligned}$$

By the Cauchy-Schwarz inequality we have:

$$\begin{aligned} |u(h(t,x))| &\leqslant \int_0^t \left\{ \left[\sum_{j\in I(h)} |D_j u(h(y,x))|^2 \right]^{\frac12} \left[\sum_{j\in I(h)} \left| \frac{\partial h_j}{\partial y}(y,x) \right|^2 \right]^{\frac12} \right\} \mathrm{d}y, \\ &\leqslant \left\{ \int_0^t \sum_{j\in I(h)} |D_j u(h(y,x))|^2 \,\mathrm{d}y \right\}^{\frac12} \left\{ \int_0^t \sum_{j\in I(h)} \left| \frac{\partial h_j}{\partial y}(y,x) \right|^2 \mathrm{d}y \right\}^{\frac12}. \end{aligned}$$

Then by (2) we get:

$$|u(h(t,x))|^2 \leqslant s(h,x,t) \sum_{j\in I(h)} \int_0^t |D_j u(h(y,x))|^2 \,\mathrm{d}y. \tag{6}$$

Arguing as in the proof of Lemma 1 in [7], we get the results. ∎

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Remark 1.1. In Lemma 1.1, ω_j may be identically zero if j is not in $I(h)$. This property is essential in studying degenerate elliptic equations, considered in [**6**].

Let Ω be a bounded open subset of $\mathbf{R}^n$ and δ be a positive real number; we put $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$, where $d(x, \partial\Omega)$ is the distance from x to $\partial\Omega$. We have the following result:

LEMMA 1.2. *Let Ω be a bounded open subset of $\mathbf{R}^n$, and let δ_0 be a sufficiently small positive real number. Let ω, $\omega_1, \ldots, \omega_n$ be nonnegative measurable functions on Ω. We assume that for any δ in $(0, \delta_0]$ there exist an open subset Ω'_δ in $\mathbf{R}^n$ and a positive real number ν_δ such that:*

(i) $\Omega_\delta \subset \Omega'_\delta \subset \overline{\Omega}'_\delta \subset \Omega_{\frac{\delta}{2}}$, *and for any j in $\{1, \ldots, n\}$ and x in Ω'_δ, we have*

$$\nu_\delta^{-1} \leqslant \omega_j(x) \leqslant \nu\delta \qquad \text{and}$$
$$\nu_\delta^{-1} \leqslant \omega(x) \leqslant \nu_\delta;$$

(ii) $\Omega \backslash \Omega'_{\delta_0}$ *is a flow-domain parametrised by $\{(h_j, D_j)\}_{j\in J}$, whose $d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n)$ is finite;*

(iii) $\partial\Omega'_\delta$ *is of class C^2.*

Then there exists a constant $C(\omega; \omega_1, \ldots, \omega_n)$, such that for any u in W_0 we have

$$\int_\Omega |u|^2 \omega(x)dx \leqslant C(\omega; \omega_1, \ldots, \omega_n) \int_\Omega \sum_{j=1}^n |D_j u|^2 \omega_j(x)dx.$$

PROOF: By Lemma 1.1, for any u in W_0 we have:

$$(7) \qquad \int_{\Omega\backslash\Omega'_{\delta_0}} |u|^2 \omega(x)\mathrm{d}x \leqslant d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n) \int_{\Omega\backslash\Omega'_{\delta_0}} \sum_{j\in I(\{h_j\})} |D_j u|^2 \omega_j \mathrm{d}x.$$

Thus by the trace theorem [**21**, p.39] and (i) there exists a constant C_1 such that, for any u in W_0, we get

$$\int_{\partial\Omega'_{\delta_0}} |u|^2 \omega(x)\mathrm{d}x \leqslant C_1 \int_{\Omega'_{\frac{1}{2}\delta_0}\backslash\Omega'_{\delta_0}} [|u|^2 \omega + \sum_{j=1}^n |D_j u|^2 \omega_j]\mathrm{d}x.$$

Thus, by (7), for any u in W_0 we have:

$$(8) \qquad \int_{\partial\Omega'_{\delta_0}} |u|^2 \omega(x)\mathrm{d}x \leqslant C_1(1 + d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n)) \int_{\Omega\backslash\Omega'_{\delta_0}} \sum_{j=1}^n |D_j u|^2 \omega_j \mathrm{d}x.$$

By the Poincaré inequality ([**29**, p.355]) amd (i) there exists a positive real number C_2 such that for every u in W_0 we have

$$(9) \qquad \int_{\Omega'_{\delta_0}} |u|^2 \omega dx \leqslant C_2 [\int_{\Omega'_{\delta_0}} \sum_{j=1}^{n} |D_j u|^2 \omega_j dx + \int_{\partial\Omega'_{\delta_0}} |u|^2 \omega ds].$$

From (7), (8) and (9) we get the lemma with $C(\omega;\omega_1,\ldots,\omega_n)$, which is equal to

$$\max\{C_2, d(\{h_j\},\omega;\omega_1,\ldots,\omega_n) + C_1C_2(1 + d(\{h_j\},\omega;\omega_1,\ldots,\omega_n))\}.$$

∎

Now consider a concrete example. Let G be an open set in R^{n-1} and let $x \mapsto c(x)$ be a mapping from G into $(0,\infty]$. Put

$$\Omega = \{(x,s) \mid x \in G \text{ and } 0 < s < c(x)\}.$$

We have the following result:

LEMMA 1.3. *Let G, Ω be as above, let ϕ be a positive measurable function on G, and let γ be in the interval $\left(\frac{3}{2},2\right)$. For any $x = (x_1,\ldots,x_n)$ in Ω we put $\omega(x) = x_n^{-\gamma}\phi(x_1,\ldots,x_{n-1})$, $\omega_n(x) = \phi(x_1,\ldots,x_{n-1})$ and $\omega_1(x) = \ldots = \omega_{n-1}(x) = 0$. Then for any u in W_0 we have:*

If $\gamma = 2$, then

$$(10) \qquad \int_\Omega |u|^2 \omega(x) dx \leqslant 4 \int_\Omega |D_n u|^2 \omega_n(x) dx;$$

If γ belongs to $\left(\frac{3}{2},2\right)$, and there exists a real number C such that, for any x in G, $c(x) \leqslant C$, then

$$(11) \qquad \int_\Omega |u(x)|^2 \omega(x) dx \leqslant C_\gamma C^{2-\gamma} \int_\Omega |D_n u(x)|^2 \omega_n(x) dx,$$

where $C_\gamma = 4(4-2\gamma)^{\frac{4-2\gamma}{2\gamma-3}}$.

PROOF: Put:

$$\begin{aligned} b_x &= c(x)^{3/2} \qquad \forall x \in G, \\ D &= \{(s,x) \mid x \in G \;\&\; 0 < s < b_x\}, \\ h(s,x) &= \left(x, s^{2/3}\right) \qquad \forall (s,x) \in D. \end{aligned}$$

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Then Ω is a flow-domain parametrised by (h,D). For every x in G and y, t in $(0,b_x)$ we have:

$$\begin{aligned}
\frac{\partial h}{\partial y}(y,x) &= \left(0,\ldots,\frac{2}{3}y^{-\frac{1}{3}}\right),\\
s(h,x,t) &= \frac{4}{9}\int_0^t \xi^{\frac{2}{3}}\mathrm{d}\xi = \frac{4}{3}t^{-\frac{1}{3}},\\
|J_h(x,t)| &= \frac{2}{3}t^{\frac{1}{3}},\\
d(h,\omega;\omega_1) &= \frac{4}{3}\sup_{(x,y)\in D}\int_y^{b_x}\frac{t^{\frac{1}{3}-\frac{2\gamma}{3}-\frac{1}{3}}}{y^{-\frac{1}{3}}}\mathrm{d}t = \frac{4}{2\gamma-3}[y^{\frac{4-2\gamma}{3}} - y^{\frac{1}{3}}b_x^{\frac{3-2\gamma}{3}}].
\end{aligned}$$

Therefore if $\gamma = 2$, we have $d(h,\omega;\omega_1,\ldots,\omega_n) \leqslant 4$. By Lemma 1.1, we have (10). In the case of (ii), by calculation, for any y in $(0,b_x)$ we have

$$\frac{4}{2\gamma-3}\left[y^{\frac{4-2\gamma}{3}} - y^{\frac{1}{3}}b_x^{\frac{3-2\gamma}{3}}\right] \leqslant C_\gamma b_x^{\frac{4-2\gamma}{3}} = C_\gamma c(x)^{2-\gamma}.$$

Using Lemma 1.1 again, we have the proof. ∎

Remark 1.2. Galdi and Rionero showed that (10) is the best estimate (see [**15**, p.16]). Our method is also applicable to other cases in [**26**].

Using the foregoing results, we define a class of domains in $\mathbb{R}^n$, which is useful in the third section. Let Ω be a domain in $\mathbb{R}^n$. For any positive real number δ we put $\Omega_\delta = \{y\in\Omega \mid d(y,\partial\Omega) > \delta\}$, where $d(y,\partial\Omega)$ is the distance from y to $\partial\Omega$.

DEFINITION 1.3: Let Ω be a domain in $\mathbb{R}^n$, and γ, ε be nonnegative real numbers. We put $\omega = d(\cdot,\partial\Omega)^{-\gamma}$ and $\omega_1 = \cdots = \omega_n \equiv 1$. We say

- (i) Ω is a *$C(\gamma,\varepsilon)$-domain*, if, for any $\varepsilon' > \varepsilon$ there exists a positive real number δ and an open subset Ω'_δ of Ω, such that $\Omega'_\delta \subset \Omega_\delta$, $\partial\Omega'_\delta$ is of class C^2, $\Omega\backslash\Omega'_\delta$ is a flow-domain parametrised by a family $\{(h_j,D_j)\}_{j\in J}$ and $d(\{h_j\},\omega;\omega_1,\ldots,\omega_n) \leqslant \varepsilon'$;
- (ii) Ω is a *$C(\gamma)$-domain*, if it is a $C(\gamma,\varepsilon_0)$-domain for some ε_0;
- (iii) Ω is a *$C_*(\gamma)$-domain*, if it is a $C(\gamma)$-domain and $C(\gamma',0)$-domain for any γ' in the interval $(0,\gamma)$.

We have the following example:

LEMMA 1.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, such that $\partial\Omega$ is of class C^2. Then Ω is a $C_*(2)$-domain.*

PROOF: Since $\partial\Omega$ is compact and of class C^2, there is a finite family $\{(\Gamma_j,G_j,\phi_j,k_j)\}_{j\in J}$, such that for any j in J, x in G and sufficiently small positive real number δ, we have: Γ_j is open in $\partial\Omega$; $\bigcup_{j\in J}\Gamma_j = \partial\Omega$; G_j is a bounded open

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subset of $\mathbb{R}^{n-1}$; ϕ_j is a C^2 function from G_j to $\mathbb{R}$; $\|\nabla\phi_j(x)\| \geqslant c > 0$; $\Gamma_j = \Psi_j(G_j)$ and

$$\{(z_1,\ldots,z_n) \mid y = \left(z_1,\ldots,z_{k_j-1},z_{k_j+1},\ldots,z_n\right) \in G_j,\\ \phi_j(y) < z_{k_j} < \phi_j(y)+c\} \subset \Omega\backslash\Omega_\delta,$$

where c is a positive constant, and $\Psi_j(x) = \left(x_1,\ldots,x_{k_j-1},\phi_j(x),x_{k_j+1},\ldots,x_n\right)$, for any $x = \left(x_1,\ldots,x_{k_j-1}.x_{k_j+1},\ldots,x_n\right)$ in G.

We can now suppose that $\Psi_1(x) = (x_1,\ldots,x_{n-1},\phi_1(x))$ for any $x = (x_1,\ldots,x_{n-1})$ in G_1. We put $D_1 = G_1 \times (0,\delta)$. For any x in G_1, any y in $\mathbb{R}$, we put $\Phi(x,y) = y - \phi_1(x)$ and

$$h_1(x,t) = (x,\phi_1(x)) + t^{2/3}\frac{\nabla\Phi(x,\phi_1(x))}{\|\nabla\Phi(x,\phi_1(x))\|}.$$

We know that $-\dfrac{\nabla\Phi(x,\phi_1(x))}{\|\nabla\Phi(x,\phi_1(x))\|}$ is nothing but the outward normal vector $\nu(x,\phi_1(x))$ of $\partial\Omega$, at $(x,\phi_1(x))$. By calculation, we have

$$\left|\frac{\partial h_1}{\partial t}(x,t)\right| = \frac{2}{3}t^{-1/3}, \tag{12}$$

$$\lim_{t\to 0}\left(\frac{2}{3}t^{-1/3}\right)^{-1}|J_{h_1}(x,t)| = 1. \tag{13}$$

Arguing as in [**5**, **23**], we can find a sufficiently small positive real number δ_0, such that, for any δ in the interval $(0,\delta_0]$, the set Ω_δ is a region with a boundary $\partial\Omega_\delta$ of class C^2.

Moreover, for each x in $\partial\Omega$ there exists a unique point x_δ in $\partial\Omega_\delta$, such that $x_\delta = x - \delta\nu(x)$, where $\nu(x)$ is the outward normal vector of $\partial\Omega$ at x. The map $x \mapsto x_\delta$ is one-to-one, and there exists a positive constant γ, such that, for any δ in $(0,\delta_0]$,

$$\gamma^{-1}d(x_\delta,\partial\Omega) \leqslant \|x_\delta - x\| = \delta \leqslant \gamma d(x_\delta,\partial\Omega) \qquad \forall x \in \partial\Omega. \tag{14}$$

Therefore $\Omega\backslash\Omega_\delta$ is a flow-domain for any sufficiently small δ. Furthermore by (12), (13), (14) and the proof of Lemma 1.3, we have the lemma. ∎

Remark 1.3. The boundary of a $C_\star(2)$-domain may not be in C^2. For example, let $\Omega = (0,1)\times(0,1)$. Let F be the union of diagonals in Ω. Then $\Omega\backslash F$ is a flow-domain, and the distance function $d(\cdot,\partial\Omega)$ is of class $C^\infty(\Omega\backslash F)$. It is easy to see that Ω is a $C_\star(2)$-domain.

2. $(\omega_1, \ldots, \omega_n)$-ELLIPTIC EQUATIONS

Let a_{ij}, b_j and c be measurable complex functions on $\overline{\Omega}$. Suppose that for every i, j in $\{1, \ldots, n\}$ we have $a_{ij} = a_{ji}$. Consider the following partial differential operator,

$$Au = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j u) + \sum_{j=1}^{n} b_j D_j u + cu.$$

In this section we assume that there exist a constant B, and nonnegative measureable functions $\omega_1, \ldots, \omega_n$, defined on Ω, such that for every x in Ω and $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n$,

$$\sum_{j=1}^{n} \omega_j(x)\xi_j^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant B\sum_{j=1}^{n} \omega_j(x)\xi_j^2. \tag{15}$$

We say A is an $(\omega_1, \ldots, \omega_n)$-*elliptic operator*. Hereafter we consider A as a linear operator defined on a linear subspace $D(A)$ of $L_2(\Omega)$ as follows: $Au = f$ if and only if

$$\int_\Omega \{\sum_{i,j=1}^{n} a_{ij}D_j u D_i v + \sum_{j=1}^{n} b_j v D_j u + cuv\}dx = \int_\Omega v(x)f(x)dx \qquad \forall v \in S_{00}. \tag{16}$$

DEFINITION 2.1: Let ω be a nonnegative real number defined on $\overline{\Omega}$; we write

$$K(\omega; \omega_1, \ldots, \omega_n) = \begin{cases} d(\{h_j\}, \omega; \omega_1, \ldots, \omega_n) & \text{if } \Omega \text{ is as in Lemma 1.1,} \\ C(\omega; \omega_1, \ldots, \omega_n) & \text{if } \Omega \text{ is as in Lemma 1.2.} \end{cases}$$

Then we have the following estimates.

LEMMA 2.1. *Let Ω be as in Lemma 1.1 or Lemma 1.2. For any u in W_0 and any v in W we have:*

$$\|u\|_*^2 \leqslant \sum_{i,j=1}^{n} \int_\Omega a_{ij}D_i u D_j \overline{u} dx, \tag{17}$$

$$\left| \int_\Omega \sum_{i,j=1}^{n} a_{ij} D_i u D_j \overline{v} dx \right| \leqslant B \|u\|_* \|v\|_*, \tag{18}$$

$$\left| \int_\Omega \sum_{j=1}^{n} b_j \overline{u} D_j v dx \right| \leqslant K_1 \|u\|_* \|v\| \star, \tag{19}$$

$$\int_\Omega |cu\overline{v}|\, dx \leqslant K_0 \|u\|_* \|v\|_*,$$

where

$$K_1 = \{n \sup_{1\leqslant j\leqslant n} K\Big(|b_j|^2\,\omega_j^{-1};\omega_1,\dots,\omega_n\Big)\}^{\frac{1}{2}}, \tag{21}$$

$$K_0 = K(|c|\,;\omega_1,\dots,\omega_n). \tag{22}$$

PROOF: From (15) we have (17). Let u and v be in S. By the condition on Ω, the Cauchy-Schwarz's inequality for a nonnegative definite hermitian sesquilinear form [**16**, Section 36, no. 10] and by (15) we have:

$$\begin{aligned}
&\left|\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \bar{v}\,\mathrm{d}x\right| \\
&\leqslant \left|\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \bar{u}\,\mathrm{d}x\right|^{\frac{1}{2}} \left|\int_\Omega \sum_{i,j=1}^n a_{ij} D_i v D_j \bar{v}\,\mathrm{d}x\right|^{\frac{1}{2}} \\
&\leqslant B\,\|u\|_\star\,\|v\|_\star\,.
\end{aligned}$$

On the other hand we have by Lemma 1.1 and Lemma 1.2

$$\begin{aligned}
\left|\int_\Omega \sum_{j=1}^n b_j \bar{u} D_j v\,\mathrm{d}x\right| &\leqslant \{\int_\Omega \sum_{j=1}^n |b_j|^2\,\omega_j^{-1}\,|u|^2\,\mathrm{d}x\}^{\frac{1}{2}}\{\int_\Omega \sum_{j=1}^n |D_j v|^2\,\omega_j \mathrm{d}x\}^{\frac{1}{2}} \\
&\leqslant \{n \sup_{1\leqslant n} K\Big(|b_j|^2\,\omega_j^{-1};\omega_1,\dots,\omega_n\Big)\}^{\frac{1}{2}}\,\|u\|_\star\,\|v\|_\star\,.
\end{aligned}$$

Therefore we obtain (19). Analogously we have (20). ∎

We put

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x.$$

By Lemma 1.1, 1.2 and 2.1 we have the following Garding inequality:

LEMMA 2.2. *Let Ω be as in Lemma 1.1 or Lemma 1.2. Let A, K_0 and K_1 be defined as in (16), (21) and (22). If $K_1 + K_0 < 1$, we have*

$$(Au,u) \geqslant (1 - K_1 - K_0)\,\|u\|_\star^2 \qquad \forall u \in W_0.$$

Using Garding's inequality for strongly elliptic operators [**25**, p.209], we get estimates in the interior part of Ω. By Lemmas 1.1 and 1.2 we obtain estimates near the boundary of Ω. Then we have another Garding inequality as follows:

LEMMA 2.3. *Let Ω and $\{h_j\}_j$ be as in Lemma 1.2, and A be defined as in (16). Assume*

$$K_1' + K_0' < 1,$$

where

$$K_1' = \left\{ n \sup_{i \leqslant j \leqslant n} d\Big(\{h_j\}, |b_j|^2 \omega_j^{-1}; \omega_1, \ldots, \omega_n\Big) \right\}^{\frac{1}{2}},$$
$$K_0' = d(\{h_j\}, |c|; \omega_1, \ldots, \omega_n).$$

Then there exist a positive constant C and a real number λ_0, such that, for any u in W_0 and any $\lambda \geqslant \lambda_0$, we have

$$(Au + \lambda u, u) \geqslant C \|u\|_*^2 .$$

DEFINITION 2.2: Let ω be a positive measurable function on Ω such that $K(\omega; \omega_1, \ldots, \omega_n)$ is finite (see Definition 2.1). Put

$$L_{2,\omega}(\Omega) = \{u \mid \omega u \text{ or } u \in L_2(\Omega)\}.$$

Let φ be in W, f be in $L_{2,\omega}(\Omega)$ and λ be a real number; we consider the following boundary problems:

$$(P_{\varphi,0}) \begin{cases} Au + \lambda u &= f \qquad \text{in } \Omega, \\ u &= \varphi \qquad \text{on } \partial_0 \Omega. \end{cases}$$

$$(P_{\varphi,00}) \begin{cases} Au + \lambda u &= f \qquad \text{in } \Omega, \\ u &= \varphi \qquad \text{on } \partial \Omega. \end{cases}$$

Remark 2.1. If A is not degenerate, then $\partial_0 \Omega = \partial \Omega$ (see the definitions in Section 1), and $(P_{\varphi,0})$ and $(P_{\varphi,00})$ are the same. But, in general, $(P_{\varphi,0})$ and $(P_{\varphi,00})$ are different (see problem (P_2) in [7]).

DEFINITION 2.3: Let A, φ, f and λ be as above. Let u be in W. We say u is *a solution of* $(P_{\varphi,0})$ (respectively, $(P_{\varphi,00})$) if $u - \varphi \in W_0$ (respectively, $u - \varphi \in W_{00}$) and

$$Au + \lambda u = f \qquad \text{in } \Omega.$$

Using the Lax-Milgram theorem and Lemmas 2.2 and 2.3, we have:

THEOREM 2.1. *Let A, K_0, K_1, K'_0 and K'_1 be as in Lemma 2.1, or Lemma 2.3. Then we have:*

(i) *if Ω is as in Lemma 1.1, or Lemma 1.2, and $K_1 + K_0 < 1$, then for each f in $L_{2,\omega}(\Omega)$, each φ in W and each nonnegative real number λ, $(P_{\varphi,0})$ has a unique solution u, which is also the unique solution of $(P_{\varphi,00})$;*

(ii) *if Ω is as in Lemma 1.2, and if $K'_0 + K'_1 < 1$, then there exists a real number λ_0 such that, for each f in $L_{2,\omega}(\Omega)$, each φ in W and each real number $\lambda \geqslant \lambda_0$, $(P_{\varphi,0})$ has a unique solution u, which is also the unique solution of $(P_{\varphi,00})$;*

(iii) *in both cases, there exists a constant K such that*

$$\|u\|_1 \leqslant K\{\|\varphi\|_1 + \min\left(\|f\|, \left\|f\omega^{-\frac{1}{2}}\right\|\right)\},$$

where K is independent of f and φ.

PROOF: By Lemmas 2.2 and 2.3, we have the uniqueness of the solution of $(P_{\varphi,0})$. If u is a solution of $(P_{\varphi,00})$, then it is a solution of $(P_{\varphi,0})$. Therefore it is sufficient to show that $(P_{\varphi,00})$ has a solution.

By Lemmas 1.1 and 2.1, the maps $v \mapsto (A\varphi, v)$ and $v \mapsto (f, v)$ are continuous on W_{00}. We remark that the norm of the second mapping is less than $\min\left(\|f\|, \left\|f\omega^{-\frac{1}{2}}\right\|\right)$. Using the Lax-Milgram theorem and Lemma 2.2, we can find w in W_0 such that

$$(Lw, v) = (f - A\varphi, v) \qquad \forall v \in W_{00},$$
$$\|w\|_1 \leqslant M\{\|\varphi\|_1 + \min\left(\|f\|, \left\|f\omega^{-\frac{1}{2}}\right\|\right)\},$$

where M is independent of f and φ.

Therefore, we get (i) with $u = w + \varphi$. Analogously, we obtain (ii) and (iii). ∎

Remark 2.2. If a_{ij} is continuously differentiable on Ω, by Lemma 2 in [**7**] and the directional smoothness of $\partial\Omega$ (see Section 1), we see that the solutions in Theorem 2.1 are the generalised solutions in the usual sense.

Remark 2.3. The foregoing theorems are applicable for an unbounded domain Ω. If A is uniformly elliptic, Mäulen and Janssen [**17, 18, 19, 22**] have proved these results for some special wheighted functions. If Ω is a flow-domain, (i) of Theorem 2.1 is proved in [**7**]. We note that our results are valid also for a degenerate or singular operator A.

Remark 2.4. If $\omega_1 = \cdots = \omega_n = \Phi$, then $(\omega_1, \ldots, \omega_n)$-elliptic operators are studied in [**2**]. If $c \equiv 0$, and if $b_j \equiv 0$ for any j in $\{1, \ldots, n\}$, these operators are considered in [**1, 12, 13, 27, 33**].

Now we consider a concrete example. Let G be an open set in $\mathbb{R}^{n-1}$ and $x \mapsto c(x)$ be a mapping from G into $(0,\infty]$. Put

$$\Omega = \{(x,s) \mid x \in G \ \& \ 0 < s < c(x)\}.$$

Suppose $a_{ij} = a_{ji}$, for any i, j in $\{1,\dots,n\}$. Suppose that there exist a positive continuous function ϕ on D and positive real numbers C_1, C_2, C_3 and C_4 such that, for any $(x_1,\dots,x_n)$ in Ω and any $\xi = (\xi_1,\dots,\xi_n)$ in $\mathbb{R}^n$, we have:

$$C_1\phi(x_1,\dots,x_{n-1})\sum_{j=1}^{n}\xi_j^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant C_2\phi(x_1,\dots,x_n)\sum_{j=1}^{n}\xi_j^2,$$
$$|b_j(x)| \leqslant C_3 x_n^{-1}\phi(x_1,\dots,x_{n-1}),$$

and

$$|c(x)| \leqslant C_4 x_n^{-2}\phi(x_1,\dots,x_{n-1}).$$

Under these conditions we have the following result:

THEOREM 2.2. *Put $\omega_j(x) = C_1\phi(x_1,\dots,x_n)$ for any $x = (x_1,\dots,x_n)$ in Ω, and $1 \leqslant j \leqslant n$. Suppose that*

$$2n^{\frac{1}{2}}C_3 + 4C_4 < C_1. \tag{23}$$

Put $\omega(x) = x_n$ for any $x = (x_1,\dots,x_n)$ in Ω. Let f be in $L_{2,\omega}(\Omega)$, let φ be in W and let λ be a nonnegative real number. Then there exists a unique solution v in W_0 of the problem $(P_{\varphi,0})$.

PROOF: By Lemma 1.3, Ω is a flow-domain parametrised by an (h,D), and we get:

$$d\Big(h, |b_j|^2\,\omega_j^{-1}; \omega_1,\dots,\omega_n\Big) \leqslant 4C_3^2C_1^{-2} \qquad \forall j,$$
$$d(h, |c|; \omega_1,\dots,\omega_n) \leqslant 4C_4C_1^{-1}.$$

Therefore by applying Theorem 2.1 we have the proof. ∎

Remark 2.5. In this theorem we see that c and b_j may be nonsmooth and unbounded, and Ω may also be unbounded and nonsmooth. If $b \equiv 0$, Theorem 2.2 is proved in [7].

3. Singular elliptic equations

In this section we consider the case in which c and b_j may be unbounded. We need the following definition:

Definition 3.1: Let Ω be a $C_*(2)$-domain in $\mathbb{R}^n$, V be a subset of $\partial\Omega$, and C_1, C_2, β, and γ be positive real numbers. We say A is *$(V, C_1, C_2, \beta, \gamma)$-singular on Ω* if it satisfies the following conditions:

(i) For any positive real number δ and any j in $\{1, \dots, n\}$, b_j and c are bounded on Ω_δ, and $D_j b_j$ belongs to $L_\infty(\Omega_\delta)$.

(ii) There exists a positive real number δ such that, for any j in $\{1, \dots, n\}$, x in $\Omega\backslash\Omega_\delta$ we have: $|b_j(x)| \leqslant C_1 d(x, V)^{-\beta}$ and $|c(x)| \leqslant C_2 d(x, V)^{-\gamma}$.

We have the following result:

Theorem 3.1. *Let Ω be a $C_*(2)$-domain in $\mathbb{R}^n$, $\omega_j \equiv 1$ for any j, and ω be $d(\cdot, \partial\Omega)$. Let A be an $(\omega_1, \dots, \omega_n)$-elliptic operator, defined as in (16). Assume that A is $(\partial\Omega, C_1, C_2, \beta, \gamma)$-singular on Ω and one of the following conditions is satisfied:*

(i) *β and $\frac{\gamma}{2}$ belong to the open interval $(0,1)$;*

(ii) *$\beta = 1$, γ is in $(0,2)$ and C_1 is sufficiently small;*

(iii) *β is in $(0,1)$, $\gamma = 2$ and C_2 is sufficiently small;*

(iv) *$\beta = 1$, $\gamma = 2$ and C_1, C_2 are sufficiently small.*

Then there exists a real number λ_0, such that the problem $(P_{\varphi,00})$ has a unique solution for any φ in W, any f in $L_{2,\omega}(\Omega)$ and any real number $\lambda \geqslant \lambda_0$.

Proof: Assume (i) holds. Then by definition 1.3 we see that for any sufficiently small δ_0, we have $K_0' + K_1' < 1$, where K_0' and K_1' are defined in Lemma 2.3. Applying Theorem 2.1, we get the result. Analogously, we have the theorem for the cases of (ii), (iii) and (iv). ∎

Remark 3.1. If Ω is a bounded domain having a C^2-boundary, $|b_j|^2$ and $|c|$ are in $L^r(\Omega)$, $\frac{n}{2} < r \leqslant \infty$, then Theorem 3.1 is proved by Stampacchia in [**30**, **31**]. The result of Stampaccia is extended by Chabrowski and Thompson in [**5**] for an L_2-boundary condition. Arguing as in the proof of Lemma 3 in [**23**], our result answers an open problem posed by Chabrowski in [**4**, p.88].

If A is nonsingular, f is in $L_2(\Omega)$ and φ is in $W^{2,2}(\Omega)$, it is well-known that the solution u of $(P_{\varphi,00})$ belongs to $W^{2,2}(\Omega)$ (see Theorem 8.12 in [**14**]). We shall establish the regularity of solutions in Theorem 3.1 in the case in which f and φ are in $L^2_{\mathrm{loc}}(\Omega)$ and $W^{2,2}_{\mathrm{loc}}$ respectively.

Let Ω, A, φ and f be as in Theorem 3.1, such that $\partial\Omega$ is of class C^2. Let δ_0 be a sufficiently small real number, such that $d(\cdot, \partial\Omega)$ is of class $C^2(\Omega\backslash\Omega_{\delta_0})$. For any δ in the interval $\left(0, \frac{\delta_0}{2}\right)$, we can find a function η in $C^2\left(\overline{\Omega}\right)$, having the following properties:

(i) $(\forall x \in \Omega_{\delta_0})\left(\frac{3}{4}\delta_0 \leqslant \eta(x)\right)$;

(ii) $(\forall x \in \Omega_\delta\backslash\Omega_{\delta_0})(\eta(x) = d(x, \partial\Omega))$;

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(iii) $(\forall x \in \Omega \backslash \Omega_{\delta/2})(\eta(x) = 0)$;
(iv) $(\forall x \in \Omega)(\eta(x) \leqslant 2d(x, \partial\Omega))$ and
(v) $(\forall x \in \Omega)(|\Delta \eta(x)| \leqslant 4)$.

Let k be in $\{1, \dots, n\}$, h be a real number and v be in $C^1(\Omega)$, such that the support of v is contained in Ω_δ and $0 < |h| < \delta$. We put

$$\Delta^h v(x) = \frac{v(x + he_k) - v(x)}{h},$$

where $x \in \Omega$ and $e_k = \left(\delta_k^j\right)_{1 \leqslant j \leqslant n}$.

We have the following lemma:

LEMMA 3.1. *Assume a_{ij} and $b_j D_j a_{ij}$ belong to $L_\infty(\Omega)$ for any i, j in $\{1, \dots, n\}$. Let δ, h and η be as above. Then there exists a positive real number M, such that, for any i, j in $\{1, \dots, n\}$, any u and w_1 in W_0, any w in W, any f in $L_{2,\omega}(\Omega)$ and any positive real number ε, we have:*

(i) $$\left| \int_{\Omega_\delta} f \Delta^{-h} (\eta^2 \Delta^h u) \, dx \right| \leqslant M \varepsilon^{-1} \{ \|f\omega\|^2 + \|u\|_*^2 \} + \varepsilon \|u\|_{**\delta}^2,$$

(ii) $$\left| \int_{\Omega_\delta} b_j D_j w \Delta^{-1} (\eta^2 \Delta^h u) \, dx \right| \leqslant M \varepsilon^{-1} \{ \|u\|_*^2 + \|\omega\|_*^2 \} + \varepsilon \|u\|_{**\delta}^2,$$

(iii) $$\left| \int_{\Omega_\delta} c w_1 \Delta^{-h} (\eta^2 \Delta^h u) \, dx \right| \leqslant M \varepsilon^{-1} \left(\|u\|_*^2 + \|w\|_*^2 \right) + \varepsilon \|u\|_{**\delta}^2,$$

(iv)
$$\left| \int_{\Omega_\delta} D_i (a_{ij} D_j w) \Delta^{-h} (\eta^2 \Delta^h u) \, dx \right| \leqslant M \varepsilon^{-1} \left(\|w\|_*^2 \|w\|_{**\delta}^2 + \|u\|_*^2 \right) + \varepsilon \|u\|_{**\delta}^2,$$

where $\|\varphi\|_{**\delta} = \{ \int_{\Omega_\delta} \sum_{i,j=1}^n |\omega D_i D_j \varphi|^2 \, dx \}^{\frac{1}{2}}$ *and* $\omega = d(\cdot, \partial\Omega)$.

PROOF: By the Cauchy-Schwarz inequality, the Young inequality of Lemma 7.23

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in [14] and by the properties of A, we have:

$$\left|\int_{\Omega_\delta} f\Delta^{-h}(\eta^2\Delta^h u)\mathrm{d}x\right|$$

$$\leqslant \left|2\int_{\Omega_\delta} f\eta(\Delta^{-h}\eta)\Delta^h u\mathrm{d}x + \int_{\Omega_\delta} f\eta^2\Delta^{-h}(\Delta^h u)\mathrm{d}x\right|,$$

$$\leqslant 8\int_{\Omega_\delta} |\eta f|^2\,\mathrm{d}x + 8\int_{\Omega_\delta} \left|\Delta^h u\right|^2\mathrm{d}x + 2\varepsilon^{-1}\int_{\Omega_\delta} |\eta f|^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_{\Omega_\delta} \left|\eta\Delta^{-h}(\Delta^h u)\right|^2\mathrm{d}x$$

$$\leqslant \left(8+\frac{2}{\varepsilon}\right)\|\eta f\|^2 + 8\|u\|_*^2 + \frac{2}{\varepsilon}\|u\|_{**\delta}^2;$$

and

$$\left|\int_{\Omega_\delta} b_j(D_j w)\Delta^{-h}(\eta^2\Delta^h u)\mathrm{d}x\right|$$

$$= |2\int_{\Omega_\delta} b_j(D_j w)\eta(\Delta^{-h}\eta)\Delta^h u\mathrm{d}x + \int_{\Omega_\delta} b_j(D_j w)\eta^2\Delta^{-h}(\Delta^h u)\mathrm{d}x|,$$

$$\leqslant \left(8+\frac{2}{\varepsilon}\right)\int_{\Omega_\delta} |b_j\eta D_j w|^2\,\mathrm{d}x + 8\int_{\Omega_\delta}\left|\Delta^h u\right|^2\mathrm{d}x + \frac{\varepsilon}{2}\int_{\Omega_\delta}\left|\eta\Delta^{-h}(\Delta^h u)\right|^2\mathrm{d}x,$$

$$\leqslant C_1\left(8+\frac{2}{\varepsilon}\right)\|w\|_*^2 + 8\|u\|_*^2 + \varepsilon\|u\|_{**\delta}^2;$$

and

$$\left|\int_{\Omega_\delta} cw_1\Delta_h(\eta^2\Delta^h u)\mathrm{d}x\right| \leqslant \left|8\int_{\Omega_\delta} cw_1\eta\Delta^h u\mathrm{d}x + \int_{\Omega_\delta} cw_1\eta^2\Delta^{-h}(\Delta^h u)\mathrm{d}x\right|$$

$$\leqslant 8C_2^2\int_{\Omega_\delta}\left|w_1\eta^{-1}\right|^2\mathrm{d}x + 8\int_{\Omega_\delta}\left|\Delta^h u\right|^2\mathrm{d}x$$

$$+\frac{C_2^2}{\varepsilon}\int_{\Omega_\delta}\left|w_1\eta^{-1}\right|^2\mathrm{d}x + \frac{\varepsilon}{2}\int_{\Omega_\delta}\left|\eta\Delta^{-h}(\Delta^h u)\right|^2\mathrm{d}x,$$

$$\leqslant C_2^2\left(8+\frac{2}{\varepsilon}\right)\left\|w_1\eta^{-1}\right\|^2 + 8\|u\|_*^2 + \varepsilon\|u\|_{**\delta}^2.$$

By Lemma 1.2, we can choose M, satisfying (i) and (ii). Now we have:

$$\int_{\Omega_\delta} D_i(a_{ij}D_j w)\Delta^{-h}(\eta^2\Delta^h u)\mathrm{d}x$$

$$= \int_{\Omega_\delta}(D_j a_{ij})(D_j w)\Delta^{-h}(\eta^2\Delta^h u)\mathrm{d}x + \int_{\Omega_\delta} a_{ij}(D_i D_j w)\Delta^{-h}(\eta^2\Delta^h u)\mathrm{d}x.$$

Therefore, by the conditions of a_{ij}, by (i) and (ii), we can find M such that M satisfies (iv) also. ∎

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THEOREM 3.2. *Let Ω, A and ω be as in Theorem 3.1, such that $\partial\Omega$ is of class C^2. Assume a_{ij} and $b_j^{-1}D_j a_{ij}$ belong to $L_\infty(\Omega)$, i and j in $\{1,\dots,n\}$. Then there exists a constant $C(\Omega, A)$ such that, for any f in $L_{2,\omega}(\Omega)$ and any φ in W such that $\|\varphi\|_{**}$ is finite, the solution u in Theorem 3.1 belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$ and we have*

$$\|u\|_{**}^2 \leqslant C(\Omega, A)\Big(\|\varphi\|_1^2 + \|\varphi\|_{**}^2 + \|f\omega\|^2\Big),$$

where $\|w\|_{**} = \{\int_\Omega \sum_{i,j=1}^n |D_i D_j w|^2\,\omega^2 dx\}^{\frac{1}{2}}$.

PROOF: The proof is similar to that of Theorem 8.8 in [**14**] and therefore we omit some details. Let u be the solution of $(P_{\varphi,0})$. Let δ, k, η and h be as in the proof of Lemma 3.1. Put $v = \eta^2\Delta^h u$. We have (see [**14**, p.184]):

$$\sum_{i,j=1}^n \int_{\Omega_\delta} a_{ij}(x+he_k)D_j\Delta^h u D_i v = -\int_{\Omega_\delta}\{\sum_{i,j=1}^n \Delta^h a_{ij} D_j u D_i v + g\Delta^{-h} v\}dx,$$

where

$$g = -\sum_{i,j=1}^n D_i(a_{ij}D_j\varphi) - \sum_{j=1}^n b_j D_j(u-\varphi) - c(u-\varphi) + f.$$

By Lemma 3.1, for any positive real number ε, there exists a constant $M_1(\Omega, A, \varepsilon)$ such that

$$\sum_{i,j=1}^n \int_{\Omega_\delta} a_{ij}(x+he_k)D_j\Delta^h u D_i v$$
$$\leqslant M_1(\Omega, A, \varepsilon)\Big(\|u\|_*^2 + \|\varphi\|_1^2 + \|\varphi\|_{**}^2 + \|f\omega\|^2\Big) + \varepsilon\,\|u\|_{**\delta}^2.$$

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By (15) and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}
\sum_j \int_{\Omega_\delta} |\eta D_j \Delta^h u|^2
&\leqslant \sum_{i,j=1}^n \int_{\Omega_\delta} \eta^2 a_{ij}(x+he_k)\Delta^h D_i u \Delta^h D_j u \\
&= \sum_{i,j=1}^n \int_{\Omega_\delta} a_{ij}(x+he_k) D_j \Delta^h u[D_i v - 2\eta(\Delta^h u) D_i \eta]\mathrm{d}x \\
&\leqslant M_1(\Omega, A, \varepsilon)\left(\|u\|_*^2 + \|\varphi\|_1^2 + \|\varphi\|_{**}^2 + \|f\omega\|^2\right) \\
&\qquad + \varepsilon \|u\|_{**\delta}^2 + 64\epsilon^{-1} \sum_{i,j=1}^n \|a_{ij}\|_\infty \left\|\Delta^h u\right\|^2 + \varepsilon \sum_{i,j=1}^n \int_{\Omega_\delta} \left|\eta D_j \Delta^h u\right|^2 \mathrm{d}x, \\
&\leqslant \left(M_1(\Omega, A, \varepsilon) + 64\varepsilon^{-1} \sum_{i,j=1}^n \|a\|_\infty \right) \\
&\qquad \{\|u\|_*^2 + \|\varphi\|_1^2 + \|\varphi\|_{**}^2 + \|f\omega\|^2\} + \varepsilon \|u\|_{**\delta}^2 .
\end{aligned}$$

Therefore, arguing as in the proof of Theorem 8.8 in [**14**], and using (iii) of Theorem 2.1, we get the result. ∎

THEOREM 3.3. *Let Ω be a bounded domain in $\mathbf{R}^n$, whose boundary is of class C^2. Let $\omega_j \equiv 1$ for any j. Let A be an $(\omega_1, \dots, \omega_n)$-elliptic and $(\partial\Omega, C_1, C_2, \beta, \gamma)$-singular operator on Ω, where β and $\frac{\gamma}{2}$ are in the interval $[0,1]$. Assume a_{ij} and $b_j^{-1} D_j a_{ij}$ belong to $L_\infty(\Omega)$ for any i, j in $\{1, \dots, n\}$. Let θ, ν be positive real numbers in $[0,2]$, f be in $L_{2,\omega^{\theta/2}}(\Omega)$ and φ be in W, such that $\int_\Omega |D_i D_j \varphi|^2 \omega^\nu dx < \infty$ for any i and j in $\{1, \dots, n\}$, where $\omega \equiv d(\cdot, \partial\Omega)$. Let u be a solution of $(P_{\varphi,0})$.*

Then u belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega |\omega^\alpha D_i D_j u|^2 \, dx \leqslant C(\Omega, A, \beta, \gamma, \theta, \nu)$$
$$\{\|\varphi\|_*^2 + \sum_{i,j=1}^n \int_\Omega |D_i D_j \varphi|^2 \omega^\nu dx + \left\|\omega^{\theta/2} f\right\|^2\},$$

where $\alpha = \max\{\beta, \frac{\gamma}{2}, \frac{\theta}{2}, \frac{\nu}{2}\}$, and i, $j \in \{1, \dots, n\}$.

PROOF: By Theorem 3.2, it is sufficient to estimate $\int_{\Omega\setminus\Omega_{\delta_0}} |\omega^\alpha D_i D_j u|^2 \, dx$, where δ_0 is a small positive real number. Therefore we can apply the method in the proof of Theorem 8.12 in [**14**], and we shall consider $\Omega\setminus\Omega_{\delta_0}$ as the flow-domain studied in

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Lemma 1.3. We can also suppose that $\eta(x) = x_n$ for any $x = (x_1, \ldots, x_n)$ in $\Omega \backslash \Omega_{\delta_0}$, where $\eta = \omega$.

For any k in $\{1, \ldots, n-1\}$ and any sufficiently small real number h, we put Δ^h, as in Lemma 3.1. We see that $\Delta^h \eta^{2\alpha} = 0$. Using $\eta^{2\alpha}$ instead of η^2, by the proof of Lemma 3.1, for any δ in $(0, \delta_0)$, we get:

$$\begin{aligned}\left|\int_{\Omega_\delta \backslash \Omega_{\delta_0}} f \Delta^{-h}(\eta^{2\alpha}\Delta^h u)\mathrm{d}x\right| &\leqslant \frac{2}{\varepsilon}\int_{\Omega_\delta \backslash \Omega_{\delta_0}} |\eta^\alpha f|^2 \,\mathrm{d}x + \frac{\varepsilon}{2}\int_{\Omega_\delta \backslash \Omega_{\delta_0}} |\eta^\alpha \Delta^{-h}(\Delta^h u)|^2 \,\mathrm{d}x \\ &\leqslant \frac{2}{\varepsilon}\int_{\Omega_\delta \backslash \Omega_{\delta_0}} |f|^2 \eta^\theta \mathrm{d}x + \frac{\varepsilon}{2}\int_{\Omega_\delta \backslash \Omega_{\delta_0}} |\eta^\alpha \Delta^{-h}(\Delta^h u)|^2 \,\mathrm{d}x.\end{aligned}$$

Considering the other inequalities in Lemma 3.1 in a similar way, we get the desired inequality for $\int_{\Omega \backslash \Omega_{\delta_0}} |\omega^\alpha D_i D_j u|^2 \,\mathrm{d}x$ for any $(i,j) \neq (n,n)$.

On the other hand, we have:

$$\begin{aligned}&\omega^\alpha a_{nn} D_n D_n u \\ &\quad = \sum_{(i,j)\neq(n,n)} a_{ij}\omega^\alpha D_i D_j u + \sum_{j=1}^{n} \omega^\alpha \left(\sum_{i=1}^{n} D_i a_{ij} + b_j\right) D_j u - c\omega^\alpha u + \omega^\alpha f.\end{aligned}$$

Because $\omega_n \equiv 1$, we have $a_{nn} \geqslant 1$. Since the functions in the righthand side of the above inequality are in $L_2(\Omega)$, we get the theorem. ∎

Remark 3.2. If $c \equiv 0$ and $b_j \equiv 0$ for any j, the regularity of a degenerate $(\omega_1, \ldots, \omega_n)$-elliptic operator is studied in [**12**, **13**].

Remark 3.3. Let m be a positive integer such that $m < n$. Let Ω be a flow-domain parametrised by (h, D) in $\mathbf{R}^m$, as in Lemma 1.1, and $\mathcal{U}$ be a bounded open subset of $\mathbf{R}^{n-m}$. Put $\mho = \Omega \times \mathcal{U}$; then $\mho$ is a flow-domain parametrised by $(g, D \times \mathcal{U})$, where $g(x,y,t) = (h(x,t), y)$. It is clear that $I(g) = I(h)$ (see Definition 1.1). In this case, for any $j > m$, ω_j may be identically zero. Applying the above results we get the following theorem:

THEOREM 3.4. *Let Ω be a $C_*(2)$-domain in $\mathbf{R}^m$, $\mathcal{U}$ be a bounded open subset in $\mathbf{R}^{n-m}$ and $\mho = \Omega \times \mathcal{U}$. Let $\omega_j \equiv 1$ for any j in $\{1, \ldots, m\}$, and let ω_j be a bounded nonnegative function on Ω for any j in $\{m+1, \ldots, n\}$. Let A be a $(\omega_1, \ldots, \omega_n)$-elliptic operator. Assume that A is $((\partial\Omega) \times \mathcal{U}, C_1, C_2, 1, 2)$-singular on $\mho$, C_1 and C_2 are sufficiently small, and $|b_j| \leqslant C_3 \omega^{-1} \omega_j^{1/2}$ and $|D_j b_j| \leqslant C_3 \omega^{-\gamma}$ for any j in $\{m+1, \ldots, n\}$, where C_3 is a constant, $\gamma \in (0, 2)$ and $\omega = d(., (\partial\Omega) \times \mathcal{U})$.*

Then there exists a real number λ_0 such that $(P_{\varphi,0})$ has a unique solution for any $\lambda \geqslant \lambda_0$, any f in $L_{2,\omega}(\mho)$ and any φ in W.

PROOF: Let u be in W_0, and v be in W. As in the proof of Lemma 2.1, we have:

$$\int_U |b_j u D_j v|\, dx \leqslant \{K\Big(|b_j|^2 \omega_j^{-1}, \omega_1, \dots, \omega_n\Big)\}^{1/2} \|u\|_\star \|v\|_\star \qquad \forall j \in \{1, \dots, n\},$$

$$|\int_U b_j u D_j u dx| \leqslant \int_U |D_j b_j|\, |u|^2\, dx \leqslant C_3 \int_U |u|^2 \omega^{-\gamma} dx \ \forall j \in \{m+1, \dots, n\}.$$

We can find a constant C_4, such that for any u and v in W_0, and any j in $\{m+1, \dots, n\}$, we obtain:

$$\begin{aligned} |\int_U b_j v D_j u dx| &\leqslant C_3 \int_U \left|w_j^{1/2} D_j u\right| |\omega^{-1} v|\, dx \\ &\leqslant C_3 \{\int_U |D_j u|^2 \omega_j dx\}^{1/2} \{\int_U |\omega^{-1} v|^2 dx\}^{1/2} \leqslant C_4 \|u\|_\star \|v\|_\star \end{aligned}$$

Then arguing as in Lemmas 2.2 and Theorem 2.1, we get the theorem. ∎

Remark 3.4. If $\varphi = 0$, Theorem 3.4 is proved in [**6**].

Remark 3.5. Our method in [**7**] is also applicable to strongly degenerate elliptic equations, studied in [**1, 27, 33**].

4. $(\omega_1, \dots, \omega_n)$-PARABOLIC EQUATIONS

Let A be a $(\omega_1, \dots, \omega_n)$-elliptic operator. It is well-known that $-(A + \lambda I)$ is the infinitesimal generator of an analytic semigroup of contractions on $L_2(\Omega)$, whenever Ω is bounded, A is strongly elliptic, $\partial\Omega$ and the coefficient functions are sufficiently smooth and λ is a sufficiently large real number. This property is important in studying the global existence and the uniqueness of solutions of the following nonlinear parabolic equation:

$$\begin{cases} \dfrac{\partial v}{\partial t} + Av + \lambda v = f(v, t), \\ v(\cdot, 0) = v_0. \end{cases}$$

We consider the following question: when is $-A$ the infinitesimal generator of an analytic semigroup of contractions on $L_2(\Omega)$? Actually, if A is the Laplace operator $-\Delta$, then $-A$ is such an operator.

For an $(\omega_1, \dots, \omega_n)$-elliptic operator, we have the following result:

THEOREM 4.1. *Let Ω be as in Lemma 1.1 or 1.2, let A, K_0, K_0', K_1 and K_1' be as in Lemmas 2.1 and 2.3. We have:*

(i) *if $K_1 + K_0 < 1$, then $-A$ is the infinitesimal generator of an analytic semigroup of contractions on $L_2(\Omega)$;*

(ii) *if $K_0' + K_1' < 1$, then $-A$ is the infinitesimal generator of an analytic semigroup of operators on $L_2(\Omega)$.*

PROOF: Put $\mathcal{D}(A) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$. It is clear that $\mathcal{D}(A)$ is dense in $L_2(\Omega)$. Using Theorem 2.1, Lemmas 2.1 and 2.2, by Lumer-Phillip's theorem [**25**, p.14], we see that $-A$ is the infinitesimal generator of a C_0 semigroup of contractions. Arguing as in the proof of Theorem 2.7 in [**25**, p.211], we see that this semigroup is analytic. Then we get (i). Analagously we obtain (ii). ∎

Remark 4.1. In Theorem 4.1, A may be degenerate or singular and Ω may be unbounded.

Hereafter let Ω and A be as in (i) of Theorem 4.1 and let X be $L_2(\Omega)$. For any α in the interval $[0,1]$, we define the operator A^α and the Banach space $(X_\alpha, \|\cdot\|_\alpha)$ as in [**25**, p.195].

DEFINITION 4.1: Let f be a map from $[0,\infty) \times X_\alpha$ into $L_2(\Omega)$. We say f *satisfies condition* (F) if for every (t,x) in $[0,\infty) \times X_\alpha$, there exists a neightbourhood V of (t,x) in $[0,\infty) \times X_\alpha$ and constants $L \geqslant 0$, θ in $(0,1]$ such that

$$\|f(t_1,x_1) - f(t_2,x_2)\| \leqslant L\Big(|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha\Big) \qquad \forall (t_i, x_i) \in V.$$

Applying Theorem 3.3 in [**25**, p.199], we have:

THEOREM 4.2. *Let Ω and A be as in (i) of Theorem 4.1, and let α be in $[0,1]$. Let f be a map from $[0,\infty) \times X_\alpha$ into $L_2(\Omega)$ satisfying (F). Assume there exists a continuous nondecreasing real function k on $[0,\infty)$ such that*

$$\|f(t,x)\| \leqslant k(t)(1 + \|x\|_\alpha) \qquad \forall (t,x) \in [0,\infty) \times X_\alpha.$$

Put $E = C([0,\infty) : L_2(\Omega)) \cap C^1((0,\infty) : L_2(\Omega))$. Then for every v in X_α, there exists a unique u in E such that u is the solution of the following initial value problem:

$$(Q)\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + Au(t,x) = f(t,u(t,x)) & \forall (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0 & \forall (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,\cdot) = v. & \end{cases}$$

Remark 4.2. If A is $(\partial\Omega, C_1, C_2, \beta, \gamma)$-singular on Ω, by the foregoing theorem, we get the global existence of solutions of singular parabolic equations. Using Theorem 3.1 and (ii) of Theorem 4.1, we can get some more results.

Remark 4.3. If W_0 is compactly embedded in $L_2(\Omega)$, then by (iii) of Theorem 2.1, $-A$ is the infinitesimal genarator of an analytic semigroup of compact operators. In this case we have the local existence of solution of (Q), when f is a continuous mapping from $[0,\infty) \times L_2(\Omega)$ into $L_2(\Omega)$. In [**8**] we obtained some results about the compactness of this embedding for unbounded and nonsmooth domain Ω.

Remark 4.4. Our method is also applicable to degenerate evolution equations studied in [**3**]. The details will appear elsewhere.

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International Centre of Theoretical Physics,
P.O. Box 586,
Miramare,
34100 Trieste,
Italy.