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DIAGRAMS OF AN ABELIAN GROUP

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Abstract

In this paper, we characterize quadratic number fields possessing unique factorization in terms of the power cancellation property of torsion-free rank-two abelian groups, in terms of Σ-unique decomposition, in terms of a pair of point set topological properties of Eilenberg–Mac Lane spaces, and in terms of the sequence of rational primes. We give a complete set of topological invariants of abelian groups, we characterize those abelian groups that have the power cancellation property in the category of abelian groups, and we characterize those abelian groups that have Σ-unique decomposition. Our methods can be used to characterize any direct sum decomposition property of an abelian group.

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1. Introduction

Our references for abelian group theory are [1, 6, 10, 11] and our reference for topology is [12]. The main results of this paper are on algebraic problems, so we use as little topology as possible.

Let G be an abelian group. Then G has the *power cancellation property* if for each integer $n > 0$ and abelian group H, $G^n \cong H^n \Rightarrow G \cong H$. We say that G has Σ-*unique decomposition* if for each integer $n > 0$, there is exactly one direct sum decomposition of G^n up to the order of the indecomposable direct summands and the isomorphism classes of the indecomposable direct summands.

Let **KAb** be the category of $K(A, 1)$-spaces where A ranges over abelian groups A. Let $\sim$ denote *is homotopic to*. Let $Y \in$ **KAb**. We say that Y is K-*indecomposable* if given $U, V \in$ **KAb** such that $Y \sim U \times V$, then U or V contracts to a point. We say that Y has a *unique finite Cartesian K-decomposition* if:

(1) $Y = \prod_{i\in\mathcal{I}} U_i$ for some finite set $\{U_i \mid i \in \mathcal{I}\} \subset$ **KAb** of K-indecomposable spaces; and
(2) $Y = \prod_{j\in\mathcal{J}} V_j$ for some finite set $\{V_j \mid j \in \mathcal{J}\} \subset$ **KAb** of K-indecomposable spaces, then there is a bijection $\Sigma : \mathcal{I} \longrightarrow \mathcal{J}$ such that $U_i \sim V_{\Sigma(i)}$ for each $i \in \mathcal{I}$.

The topological space Y has the *power cancellation property in* **KAb** if for each integer $n > 0$ and topological space $Z \in \mathbf{KAb}$, $Y^n \sim Z^n \Rightarrow Y \sim Z$. We say that Y has a Σ*-unique decomposition in* **KAb** if for each integer $n > 0$, Y^n has a unique finite Cartesian K-decomposition. These direct sum properties appear surprisingly in the study of the unique factorization problem in algebraic number fields.

The sequence $\{L(p)h(G(p)) \mid \text{rational primes } p\}$ is asymptotically equal to the sequence of rational primes. In particular, let $\mathbf{k}$ be a quadratic number field, let $\overline{E}$ be the ring of algebraic integers in $\mathbf{k}$, and for each rational prime p let $E(p) = \mathbb{Z} + p\overline{E}$. Let $G(p)$ be a reduced torsion-free rank-two abelian group such that $\mathrm{End}(G(p)) \cong E(p)$. These groups exist by Butler's theorem [6, Theorem I.2.6]. Let $L(p) = \mathrm{card}(u(\overline{E})/u(E(p)))$ where $u(R)$ is the groups of units in the ring R. For an abelian group H let $h(H)$ be the number of isomorphism classes of groups L that are locally isomorphic to H. Sequences s_n and t_n are *asymptotically equal* if $\lim_{n\to\infty} s_n/t_n = 1$.

The main theorem of this paper is as follows.

THEOREM 1.1. *Let* $\mathbf{k}$ *be a quadratic number field. The following are equivalent.*

(1) *The quadratic number field* $\mathbf{k}$ *has unique factorization.*
(2) *The sequence* $\{L(p)h(G(p)) \mid \text{rational primes } p\}$ *is asymptotically equal to the sequence of rational primes.*
(3) *Each strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$ *has the power cancellation property.*
(4) *Some strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$ *has the power cancellation property.*
(5) *Each strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$ *has Σ-unique decomposition.*
(6) *Some strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$ *has Σ-unique decomposition.*
(7) *Given a strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$*, each $K(G, 1)$-space has the power cancellation property in* **KAb**.
(8) *For some strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$*, each $K(G, 1)$-space has the power cancellation property in* **KAb**.
(9) *Given a strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$*, each $K(G, 1)$-space has Σ-unique decomposition in* **KAb**.
(10) *For some strongly indecomposable reduced torsion-free rank-two abelian group G such that* $\mathrm{End}(G) \cong \overline{E}$*, each $K(G, 1)$-space has Σ-unique decomposition in* **KAb**.

Consider some of the more popular consequences of the properties used above. In Theorem 8.7, for a given integer $h > 0$, we characterize those algebraic number fields $\mathbf{k}$ with class number $h(\mathbf{k}) = h$. Determining the class number of an algebraic number field is a problem that goes back to Gauss. Specifically, one wishes to write a list of the quadratic number fields $\mathbf{k}$ such that $h(\mathbf{k}) = 1$ or, equivalently, such that $\mathbf{k}$ has unique factorization. See Theorem 1.1(1).

Theorem 1.1(2) involves a sequence $\{s_p \mid \text{primes } p\}$ of integers that is asymptotically equal to the sequence of rational primes. Thus, $\{s_p \mid \text{primes } p\}$ is related to the prime number theorem, although we do not explore the relationship in this paper.

Theorem 1.1(3) and (5) are about direct sum decompositions of strongly indecomposable torsion-free rank-two abelian groups. These properties, called power cancellation and Σ-unique decomposition, will be hard because they are related to hard problems in number theory. We also apply our methods to the study of more general direct sum decompositions of abelian groups. Consequently, we give a complete set of topological invariants for abelian groups up to isomorphism, and we show how one would characterize any direct sum property of an abelian group.

Theorem 1.1(7) and (9) are about the classic $K(G, 1)$-spaces from pointset topology. We give decomposition properties of $K(G, 1)$-spaces for which $\mathrm{End}(G) \cong \overline{E}$ provide a topological characterization of the unique factorization property in the quadratic number field $\mathbf{k}$.

To begin our proof of Theorem 1.1 we construct commutative diagrams of categories and category equivalences. With these diagrams we show that abelian groups are at the center of several areas of mathematics, including ring theory, module theory, homology, and point set topology. The first diagram is Figure 1. Opposing arrows denote inverse category equivalences.

Until we reach Section 8, G denotes a fixed abelian group, A and Q denote variable abelian groups, and

$$\mathbf{P}(G) = \{\text{abelian groups } Q \mid Q \oplus Q' \cong \oplus_{\mathcal{I}} G \text{ for some set } \mathcal{I} \\ \text{and some abelian group } Q'\}.$$

Let $X = K(G, 1)$ be a fixed *Eilenberg–Mac Lane space* [12]. Let Y, U, and V denote topological spaces, and let f denote a continuous map or a sequence of continuous maps. These are abstract topological spaces, *not linear topologies*, on groups. For a space X, $\pi_1(X)$ is the fundamental group of X, $\prod_{i\in\mathcal{I}} X_{\text{ß}}$ is a Cartesian product of the spaces $\{X_i \mid i \in \mathcal{I}\}$, $X \sim Y$ means that X is homotopic to Y, $f \sim g$ denotes homotopic continuous maps f and g, $[f]$ denotes the homotopy equivalence class of f, and $f^* = \pi_1(f)$.

2. *G*-plexes and *G*-coplexes

We begin our discussion in earnest. Our discussion of G-plexes and G-coplexes comes from [4, 6, 7]. A G-plex is a complex

$$\mathcal{Q} = \cdots \longrightarrow Q_{k+1} \xrightarrow{\delta_{k+1}} Q_k \xrightarrow{\delta_k} \cdots \xrightarrow{\delta_1} Q_0 \qquad (2.1)$$

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$$
\begin{tikzcd}
 & & G\text{-}\mathbf{Plex} \arrow[dl, shift left, "\widehat{K}(\cdot)"] \arrow[dr, "\mathrm{h}_G"] & \\
 & X\text{-}\mathbf{Plex} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, "\mathrm{h}_X"'] \arrow[dl, "\mathbf{V}"'] \arrow[dr, "\mathbf{U}"] & & \mathbf{Mod}\text{-}\mathrm{End}(G) \arrow[dl, "\mathrm{Tor}^*_E(\cdot,\, G)"] \\
\mathbf{SKAb} \arrow[rr, shift left, "\Pi_1"] & & \mathbf{SAb} \arrow[ll, shift left, "K"] & \\
 & X\text{-}\mathbf{coPlex} \arrow[ul, "\mathbf{V}^*"] \arrow[ur, "\mathbf{U}^*"] \arrow[rr, "\mathrm{h}^X"'] \arrow[dr, shift right, "\widehat{\pi}_1^*"'] & & \mathrm{End}(G)\text{-}\mathbf{Mod} \arrow[ul, "\mathrm{Ext}^*_E(\cdot,\, G)"'] \\
 & & G\text{-}\mathbf{coPlex} \arrow[ul, shift right, "\widehat{\pi}_1^{*-1}"'] \arrow[ur, "\mathrm{h}^G"'] &
\end{tikzcd}
$$

FIGURE 1.

of abelian group homomorphisms δ_k such that:

(1) for each integer $k \geq 0$, $Q_k \in \mathbf{P}(G)$; and

(2) given an integer $k > 0$ and a map $\phi : G \longrightarrow Q_k$ such that $\delta_k \phi = 0$, there is a map $\psi : G \longrightarrow Q_{k+1}$ such that $\delta_{k+1}\psi = \phi$ as in the following commutative diagram:

$$
\begin{tikzcd}
 & G \arrow[dl, "\exists\psi"'] \arrow[d, "\forall\phi"] \arrow[dr, "0"] & \\
Q_{k+1} \arrow[r, "\delta_{k+1}"'] & Q_k \arrow[r, "\delta_k"'] & Q_{k-1}
\end{tikzcd}
$$

Let G-**Plex** denote the category of G-plexes whose maps are homotopy equivalence classes $[f]$ of chain maps $f : \mathcal{Q} \longrightarrow \mathcal{Q}'$ between G-plexes. (The homotopy of chain maps between algebraic complexes is the classic version found in [14].)

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Let $H_0(\cdot)$ denote the zeroth homology functor acting on complexes of abelian groups, and let

$$\mathrm{h}_G(\cdot) = H_0 \circ \mathrm{Hom}(G, \cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}(G).$$

We note that $\mathrm{h}_G(\mathcal{Q}) = \operatorname{coker} \mathrm{Hom}(G, \delta_1)$ is an object in $\mathbf{Mod}\text{-}\mathrm{End}(G)$.

Let us say that G is *self-small* if for each cardinal c the natural map

$$\oplus_c \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}(G, \oplus_c G)$$

is an isomorphism of right $\mathrm{End}(G)$-modules. Finitely generated modules are self-small $*$-modules and tilting modules are self-small lattices over a $\mathbb{Z}$-order in a finite-dimensional $\mathbb{Q}$-algebra are self-small and torsion-free finite rank modules over an integral domain are self-small. The *Arnold–Lady–Murley theorem* [2, 7] states that G is self-small if and only if the functor

$$\mathrm{Hom}(G, \cdot) : \mathbf{P}(G) \longrightarrow \mathbf{P}(\mathrm{End}(G))$$

is a category equivalence. From [7, Theorem 2.1.11] we see that if G is self-small, then the functor

$$\mathrm{h}_G(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}(G)$$

is a category equivalence. In this case, the inverse of $\mathrm{h}_G(\cdot)$ is a functor

$$\mathrm{t}_G(\cdot) : \mathbf{Mod}\text{-}\mathrm{End}(G) \longrightarrow G\text{-}\mathbf{Plex}$$

defined as follows. For each $M \in \mathbf{Mod}\text{-}\mathrm{End}(G)$ fix a projective resolution $\mathcal{P}(M)$ of M. Given a map $f : M \to M'$ in $\mathbf{Mod}\text{-}\mathrm{End}(G)$ lift f to a chain map $f^{\#} : \mathcal{P}(M) \longrightarrow \mathcal{P}(M')$. Then the assignments

$$\mathrm{t}_G(M) = \mathcal{P}(M) \otimes_{\mathrm{End}(G)} G$$
$$\mathrm{t}_G(f) = f^{\#} \otimes_{\mathrm{End}(G)} G$$

define $\mathrm{t}_G(\cdot) : \mathbf{Mod}\text{-}\mathrm{End}(G) \longrightarrow G\text{-}\mathbf{Plex}$.

We dualize the category G-**Plex** and the functor $\mathrm{h}_G(\cdot)$. Let $\mathbf{coP}(G) = \{$abelian groups $W \mid W \oplus W' = \prod_c G$ for some cardinal c and some group $W'\}$. A *G-coplex* is a complex

$$\mathcal{W} = W_0 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_k} W_k \xrightarrow{\sigma_{k+1}} W_{k+1} \longrightarrow \cdots$$

of abelian groups such that:

(1) for each integer $k \geq 0$, $W_k \in \mathbf{coP}(G)$; and
(2) given an integer $k > 0$, and given a map $\phi : W_k \longrightarrow G$ such that $\phi\sigma_k = 0$ then there is a map $\psi : W_{k+1} \longrightarrow G$ such that $\psi\sigma_{k+1} = \phi$ as in the following commutative diagram:

$$\begin{tikzcd} W_{k-1} \arrow[r, "\sigma_k"] \arrow[dr, "0"'] & W_k \arrow[r, "\sigma_{k+1}"] \arrow[d, "\forall\phi"] & W_{k+1} \arrow[dl, "\exists\psi"] \\ & G & \end{tikzcd}$$

Let G-**coPlex** denote the category whose objects are the G-coplexes and whose maps are the homotopy equivalence classes $[f]$ of chain maps $f : \mathcal{W} \longrightarrow \mathcal{W}'$ of G-coplexes.

We wish to dualize and characterize completely the left modules over the endomorphism ring as in [4]. To do this we must make a set theoretic assumption.

The dual to the Arnold–Lady–Murley theorem is a result due to Huber and Warfield [13]. We say that G is *self-slender* if for each *nonmeasurable cardinal c* (see [3, 10, 11]), the natural map

$$\oplus_c \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}\Big(\prod_c G, G\Big)$$

is an isomorphism. Since we wish to find the most complete dualization of the Arnold–Lady–Murley theorem we assume the axiom.

(μ) **Measurable cardinals do not exist**.

This axiom is not so bold as it is known [3] that measurable cardinals do not exist under Gödel's constructibility hypothesis $V = L$.

For our purposes the Huber–Warfield theorem [13] states that if (μ) is true and if G is self-slender, then the point evaluation homomorphism

$$\Psi_W : W \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathrm{End}(G)}(W, G), G)$$

is an isomorphism for each $W \in \mathbf{coP}(G)$. This is the last we have to say about set theory except to state that we are assuming the (μ) holds.

Let

$$\mathrm{h}^G(\cdot) = H_0 \circ \mathrm{Hom}(\cdot, G) : G\textbf{-coPlex} \longrightarrow \mathrm{End}(G)\textbf{-Mod}$$

be defined by $\mathrm{h}^G(\mathcal{W}) = \mathrm{coker}\,\mathrm{Hom}(\sigma_1, G)$. Then by [4], if ($\mu$) is true and if G is self-slender, then $\mathrm{h}^G(\cdot)$ is a duality.

Consequently, we have characterized **Mod**-End(G) and End(G)-**Mod** in terms of categories G-**Plex** and G-**coPlex**. In G-**Plex**, G is a small projective generator [7, 9]. In G-**coPlex**, G is a slender injective cogenerator [4, 7].

3. Some algebraic topology

Our reference for topological issues is [12]. We have fixed an abelian group G, and a point set topological space $X = K(G, 1)$. For topological spaces U and V let

$$\mathrm{Hom}_{\mathbf{C}}(U, V) = \{[f] \mid f : U \longrightarrow V \text{is a continuous map}\}$$

and let $\mathrm{End}_{\mathbf{C}}(X) = \mathrm{Hom}_{\mathbf{C}}(X, X)$. Our choice of X possesses a lifting property.

THEOREM 3.1 [12, Proposition 1B.9]. *Let $U = K(A, 1)$ and $V = K(B, 1)$ for some abelian groups A and B. A group homomorphism $\phi : A \longrightarrow B$ is induced by a continuous map $f : U \longrightarrow V$ that is unique up to homotopy equivalence.*

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COROLLARY 3.2. *Let G be an abelian group and let $X = K(G, 1)$.*

(1) *Then* $\mathrm{End}_{\mathbf{C}}(X)$ *possesses a natural ring structure. The natural map* $\rho : \mathrm{End}_{\mathbf{C}}(X) \longrightarrow \mathrm{End}(G)$ *taking* $[f]$ *to* f^* *is a ring isomorphism.*

(2) *Let* $U = K(A, 1)$ *for some abelian group* A. *Then* $\mathrm{Hom}_{\mathbf{C}}(X, U)$ *is a right* $\mathrm{End}_{\mathbf{C}}(X)$*-module. The natural map* $\alpha : \mathrm{Hom}_{\mathbf{C}}(X, U) \longrightarrow \mathrm{Hom}(G, A)$ *is a right* $\mathrm{End}_{\mathbf{C}}(X)$*-module isomorphism.*

(3) *Let* $U = K(A, 1)$ *for some abelian group* A. *Then* $\mathrm{Hom}_{\mathbf{C}}(U, X)$ *is a left* $\mathrm{End}_{\mathbf{C}}(X)$*-module. The natural map* $\beta : \mathrm{Hom}_{\mathbf{C}}(U, X) \longrightarrow \mathrm{Hom}(A, G)$ *is an isomorphism of left* $\mathrm{End}_{\mathbf{C}}(X)$*-modules.*

We identify $\mathrm{End}_{\mathbf{C}}(X)$ and $\mathrm{End}(G)$ via the isomorphism in Corollary 3.2(a).

We are motivated by the category G-**Plex** and the category equivalence $\mathrm{h}_G(\cdot)$ from [9] defined above to study $\mathrm{End}_{\mathbf{C}}(X)$ in a similar manner.

An X-*plex* is a complex

$$\mathcal{C} = \cdots \longrightarrow X_{k+1} \xrightarrow{\lambda_{k+1}} X_k \xrightarrow{\lambda_k} \cdots \xrightarrow{\lambda_1} X_0 \tag{3.1}$$

of continuous maps λ_k such that:

(1) for each integer $k \geq 0$, X_k is a $K(Q_k, 1)$ space for some $Q_k \in \mathbf{P}(G)$;

(2) for each integer $k \geq 0$, $\lambda_k \lambda_{k+1}$ is a null homotopic map;

(3) for each integer $k > 0$ and for each continuous map $f : X \longrightarrow X_k$ such that $\lambda_k f$ is a null homotopic map, there is a map $g : X \longrightarrow X_{k+1}$ such that $f \sim \lambda_{k+1} g$ as in the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "\exists g"'] \arrow[d, "\forall f"] \arrow[dr, "\text{null homotopic map}"] & \\ X_{k+1} \arrow[r, "\lambda_{k+1}"'] & X_k \arrow[r, "\lambda_k"'] & X_{k-1} \end{tikzcd}$$

In this context, commutativity of the diagram means that $\lambda_{k+1} g \sim f$. A *ladder map* $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is a sequence $f = (\ldots, f_2, f_1, f_0)$ of continuous maps such that

$$f_{k-1}\lambda_k \sim \lambda'_k f_k \quad \text{for each integer } k > 0.$$

The way to remember this is by the following familiar commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r, "\lambda_{k+1}"] & X_k \arrow[r, "\lambda_k"] \arrow[d, "f_k"] & X_{k-1} \arrow[r, "\lambda_{k-1}"] \arrow[d, "f_{k-1}"] & \cdots \\ \cdots \arrow[r, "\lambda'_{k+1}"'] & X'_k \arrow[r, "\lambda'_k"'] & X'_{k-1} \arrow[r, "\lambda'_{k-1}"'] & \cdots \end{tikzcd}$$

We define *the homotopy equivalence of ladder maps* between X-plexes. Let f and $g : \mathcal{C} \longrightarrow \mathcal{C}'$ be ladder maps. As $a^* = b^*$ whenever $a \sim b$, we see that

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$f^* = (f_k^* \mid k \geq 1) : \pi_1(\mathcal{C}) \longrightarrow \pi_1(\mathcal{C}')$ is a chain map between complexes of abelian groups. The classic homotopy equivalence class, $[f^*]$, is then defined. We say that *f is homotopic to g*, and we write $f \sim g$, if $[f^*] = [g^*]$. It follows that *is homotopic to* is an equivalence relation on the set of ladder maps between X-plexes $\mathcal{C}$ and $\mathcal{C}'$.

Let X-**Plex** denote the category whose objects are X-plexes, and whose maps $[f] : \mathcal{C} \longrightarrow \mathcal{C}'$ are the homotopy equivalence classes of ladder maps $f : \mathcal{C} \longrightarrow \mathcal{C}'$.

LEMMA 3.3. *Let G be an abelian group and let $X = K(G, 1)$. Then $\pi_1(\cdot)$ lifts to a functor*

$$\widehat{\pi}_1(\cdot) : X\textbf{-Plex} \longrightarrow G\textbf{-Plex}$$

defined by

$$\widehat{\pi}_1(\mathcal{C}) = \cdots \longrightarrow \pi_1(X_{k+1}) \xrightarrow{\lambda_{k+1}^*} \pi_1(X_k) \xrightarrow{\lambda_k^*} \cdots \xrightarrow{\lambda_1^*} \pi_k(X_0) \tag{3.2}$$

for X-plexes $\mathcal{C}$ and

$$\widehat{\pi}_1([f]) = [(\ldots, f_2^*, f_1^*, f_0^*)]$$

for ladder maps $f = (\ldots, f_2, f_1, f_0) : \mathcal{C} \longrightarrow \mathcal{C}'$ in X-plex.

PROOF. It is readily shown that $\widehat{\pi}_1$ is well defined, thus proving the lemma. □

THEOREM 3.4. *Let G be an abelian group and let $X = K(G, 1)$. Then*

$$\widehat{\pi}_1(\cdot) : X\textbf{-Plex} \longrightarrow G\textbf{-Plex}$$

is a category equivalence.

PROOF. Let $\mathcal{Q}$ be a G-plex with terms Q_k and connecting maps δ_k. For each integer $k \geq 0$ let $X_k = K(Q_k, 1)$, so that $\pi_1(X_k) = Q_k \in \mathbf{P}(G)$. Since $X_k = K(Q_k, 1)$, Theorem 3.1 states that for integers $k > 0$ there are continuous maps $\lambda_k : X_k \longrightarrow X_{k-1}$ such that $\lambda_k^* = \delta_k$. Since

$$0 = \delta_k \delta_{k+1} = (\lambda_k \lambda_{k+1})^*$$

for each integer $k \geq 1$, and since the lifting is unique up to homotopy equivalence, $\lambda_k \lambda_{k+1}$ is null homotopic. We have constructed a complex $\mathcal{C}$ as given in (3.1) such that $\pi_1(\mathcal{C}) = \mathcal{Q}$.

Let $k > 0$ be an integer and let $f : X \longrightarrow X_k$ be a continuous map such that $\lambda_k f$ is null homotopic. Then $\lambda_k^* f^* = (\lambda_k f)^* = 0$ and $f^* : G \longrightarrow \pi_1(X_k) = Q_k$. By the lifting property of G over the G-plex $\mathcal{Q}$, there is a map $\psi : G \longrightarrow Q_{k+1}$ such that $f^* = \delta_{k+1}\psi = \lambda_{k+1}^*\psi$. Theorem 3.1 states that $\psi = g^*$ for some continuous map $g : X \longrightarrow X_{k+1}$. Then $f^* = \lambda_{k+1}^* g^* = (\lambda_{k+1} g)^*$. By Theorem 3.1 these liftings are unique up to homotopy equivalence, so $f \sim \lambda_{k+1} g$. Thus, $\mathcal{C}$ is an X-plex such that $\widehat{\pi}_1(\mathcal{C}) = \mathcal{Q}$.

If $\widehat{\pi}_1([f]) = \widehat{\pi}_1([g])$ for some ladder maps f, g in X-**Plex**, then $[f^*] = [g^*]$ which by definition implies that $f \sim g$. Therefore, $\widehat{\pi}_1(\cdot) : X\textbf{-Plex} \longrightarrow G\textbf{-Plex}$ is a faithful functor.

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Let $\phi = (\phi_k \mid k \geq 0) : \mathcal{Q} \longrightarrow \mathcal{Q}'$ be a chain map between G-plexes. By the above construction, there are complexes $\mathcal{C}$ and $\mathcal{C}'$ such that $\pi_1(\mathcal{C}) = \mathcal{Q}$ and $\pi_1(\mathcal{C}') = \mathcal{Q}'$. By Theorem 3.1 for each integer $k \geq 0$ there is a map g_k, unique up to homotopy, such that $g_k^* = \phi_k$. By the fact that $\phi : \pi_1(\mathcal{C}) \longrightarrow \pi_1(\mathcal{C}')$ is a chain map $g_k^* \lambda_{k+1}^* = \lambda'^*_{k+1} g^*_{k+1}$. Again the uniqueness of the lifting in Theorem 3.1 implies that $g_k \lambda_{k+1} \sim \lambda'_{k+1} g_{k+1}$. We have constructed the ladder map $g = (g_k \mid k \geq 0) : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $g^* = \phi$. Thus, $\widehat{\pi}_1(\cdot) : X$-**Plex** $\longrightarrow G$-**Plex** is a full functor and, therefore, $\widehat{\pi}_1(\cdot) : X$-**Plex** $\longrightarrow G$-**Plex** is a category equivalence. □

COROLLARY 3.5. *The inverse of* $\widehat{\pi}_1 : X$-**Plex** $\longrightarrow G$-**Plex** *is the functor*

$$\widehat{K}(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow X\text{-}\mathbf{Plex}$$

given by $\widehat{K}(A) = K(A, 1)$.

PROOF. The construction of $K(A, 1)$ is functorial in A by [12] so $\widehat{K}(\cdot)$ is a functor. By its construction $\pi_1(K(A, 1)) \cong A$ so, by Theorem 3.4, $\widehat{K}(\cdot)$ is the inverse of $\widehat{\pi}_1$. This proves the corollary. □

We dualize the above discussion. An X-*coplex* is a complex

$$\mathcal{U} = U_0 \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_k} U_k \xrightarrow{\mu_{k+1}} U_{k+1} \longrightarrow \cdots$$

such that:

(1) for each integer $k \geq 0$, U_k is a $K(W_k, 1)$ space for some $W_k \in \mathbf{coP}(G)$;
(2) for each $k \geq 0$, $\mu_{k+1}\mu_k$ is a null homotopic map; and
(3) for a given integer $k > 0$, and given a map $f : U_k \longrightarrow X$ such that $f\mu_k$ is null homotopic, then there is a map $g : U_{k+1} \longrightarrow X$ such that $g\mu_{k+1} \sim f$ as in the accompanying commutative diagram:

$$\begin{tikzcd} U_{k-1} \arrow[r, "\mu_k"] \arrow[dr, "\text{null homotopic map}"'] & U_k \arrow[r, "\mu_{k+1}"] \arrow[d, "\forall f"] & U_{k+1} \arrow[dl, "\exists g"] \\ & X & \end{tikzcd}$$

Let X-**coPlex** denote the category whose objects are the X-coplexes and whose maps are the homotopy equivalence classes $[f]$ of ladder maps $f : \mathcal{U} \longrightarrow \mathcal{U}'$.

Dual to Theorem 3.4 is Theorem 3.6.

THEOREM 3.6. *Let G be an abelian group and let* $X = K(G, 1)$. *Then*

$$\widehat{\pi}_1^*(\cdot) : X\text{-}\mathbf{coPlex} \longrightarrow G\text{-}\mathbf{coPlex}$$

is a category equivalence with inverse

$$\widehat{K}^*(\cdot) : G\text{-}\mathbf{coPlex} \longrightarrow X\text{-}\mathbf{coPlex}$$

given by $\widehat{K}^*(\cdot) = K(\cdot, 1)$.

PROOF. Dualize the proof of Theorem 3.4. □

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$$\begin{tikzcd}
& G\text{-}\mathbf{Plex} \arrow[dl, shift left, "\widehat{K}(\cdot)"] \arrow[dr, "\mathrm{h}_G"] & \\
X\text{-}\mathbf{Plex} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, "\mathrm{h}_X"'] & & \mathbf{Mod}\text{-}\mathrm{End}(G)
\end{tikzcd}$$

FIGURE 2.

4. Commutative triangles

We use the category equivalences as a basis for the construction of the diagrams in Figures 2, 1, and 7. These diagrams provide a functorial connection between the topological structure of X, the homological structure of G, and the modules over $\mathrm{End}(G)$.

Let H_0 denote the zeroth homology functor, and define a functor

$$\mathrm{h}_X(\cdot) : X\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}_{\mathbf{C}}(X)$$

by

$$\mathrm{h}_X(\mathcal{C}) = H_0 \circ \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{C})$$

for X-plexes $\mathcal{C}$ and

$$\mathrm{h}_X([f]) = H_0 \circ \mathrm{Hom}_{\mathbf{C}}(X, f).$$

THEOREM 4.1. *Let G be an abelian group and let $X = K(G, 1)$. There is a commutative triangle (see Figure 2) in which opposing arrows denote inverse category equivalences.*

PROOF. Let $\mathcal{C} \in X\text{-}\mathbf{Plex}$. The terms X_k in $\mathcal{C}$ satisfy $\widehat{\pi}_1(X_k) \cong Q_k$ for some $Q_k \in \mathbf{P}(G)$. Also the maps $\lambda_k : X_k \longrightarrow X_{k-1}$ in $\mathcal{C}$ satisfy $\widehat{\pi}_1(\lambda_k) = \delta_k : Q_k \longrightarrow Q_{k-1}$. So, by Corollary 3.2,

$$\mathrm{h}_G \circ \widehat{\pi}_1(\lambda_k) = H_0 \circ \mathrm{Hom}(G, \lambda_k^*) \cong H_0 \circ \mathrm{Hom}_{\mathbf{C}}(X, \lambda_k) = \mathrm{h}_X(\lambda_k).$$

Thus, $\mathrm{h}_X(\mathcal{C}) \cong \mathrm{h}_G \circ \widehat{\pi}_1(\mathcal{C})$. Similarly, $\mathrm{h}_X([f]) = \mathrm{h}_G \circ \widehat{\pi}_1([f])$, for ladder maps f, so the triangle commutes. □

THEOREM 4.2. *Let G be a self-small abelian group and let $X = K(G, 1)$. There is a commutative triangle (see Figure 3) in which opposing arrows represent inverse functors.*

PROOF. The functors $\widehat{\pi}_1(\cdot)$ and $\widehat{K}(\cdot)$ are inverse functors by Theorem 3.4. The functors $\mathrm{h}_G(\cdot)$ and $\mathrm{t}_G(\cdot)$ are inverse functors by [7, Theorem 2.1.12]. Also, by Theorem 4.1, $\mathrm{h}_X(\cdot)$ is a category equivalence. This completes the proof. □

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$$\begin{tikzcd}
 & G\text{-}\mathbf{Plex} \arrow[dl, shift left, "\widehat{K}"] \arrow[dr, shift left, "\mathrm{h}_G"] & \\
X\text{-}\mathbf{Plex} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, shift right, "\mathrm{h}_X"'] & & \mathbf{Mod}\text{-}\mathrm{End}(G) \arrow[ul, shift left, "\mathrm{t}_G"] \arrow[ll, shift right, "\mathrm{h}_X^{-1}"']
\end{tikzcd}$$

FIGURE 3.

$$\begin{tikzcd}
 & G\text{-}\mathbf{coPlex} \arrow[dl, shift left, "\widehat{K}^*"] \arrow[dr, "\mathrm{h}^G"] & \\
X\text{-}\mathbf{coPlex} \arrow[ur, shift left, "\widehat{\pi}_1^*"] \arrow[rr, "\mathrm{h}^X"'] & & \mathrm{End}(G)\text{-}\mathbf{Mod}
\end{tikzcd}$$

FIGURE 4.

By dualizing the above arguments one can prove the following results. Define a functor

$$\mathrm{h}^X(\cdot) : X\text{-}\mathbf{coPlex} \longrightarrow \mathrm{End}(G)\text{-}\mathbf{Mod}$$

by

$$\mathrm{h}^X(\mathcal{W}) = H_0 \circ \mathrm{Hom}_{\mathbf{C}}(\mathcal{W}, X)$$
$$\mathrm{h}^X([f]) = H_0 \circ \mathrm{Hom}_{\mathbf{C}}(f, X).$$

THEOREM 4.3. *Let G be an abelian group and let $X = K(G, 1)$. There is a commutative diagram (Figure 4) in which opposing arrows represent inverse dualities.*

THEOREM 4.4. *Assume that measurable cardinals do not exist, let G be a self-slender abelian group, and let $X = K(G, 1)$. There is a commutative diagram (Figure 5) in which opposing arrows represent inverse dualities.*

REMARK 4.5. It is worth noting that the vertices of these triangles represent categories from ring theory, abelian groups, homology theory, and topology.

REMARK 4.6. A right R-module G is self-small if it is finitely generated as an R-module or if its additive structure $(G, +)$ is a self-small abelian group. At the time of writing the only known examples of self-slender right R-modules G are those G whose additive structure $(G, +)$ is a self-slender abelian group. The *torsion-free* abelian

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$$\begin{tikzcd}
 & G\text{-}\mathbf{coPlex} \arrow[dl, shift left, "\widehat{K}^*"] \arrow[dr, shift left, "\mathrm{h}^G"] & \\
X\text{-}\mathbf{coPlex} \arrow[ur, shift left, "\widehat{\pi}_1^*"] \arrow[rr, shift right, "\mathrm{h}^X"'] & & \mathrm{End}(G)\text{-}\mathbf{Mod} \arrow[ul, shift left, "\mathrm{h}^G"] \arrow[ll, shift right, "(\mathrm{h}^X)^{-1}"']
\end{tikzcd}$$

FIGURE 5.

group G is self-slender if and only if G does not contain copies of $\mathbb{Q}$, $\prod_{\mathbb{N}} \mathbb{Z}$, or the p-adic integers $\widehat{\mathbb{Z}}_p$ for some rational prime p (see [10] and [11, Theorem 95.3]).

5. Three diamonds

The goal of this section is to construct the diagram shown in Figure 1.

We have constructed the commutative triangles at the top and bottom of Figure 1 as Figures 2 and 4.

The objects in the category **SAb** are the sequences $S = (A_k \mid k > 0)$ of abelian groups and the maps $\phi : S \longrightarrow S'$ in **SAb** are the sequences $\phi = (\phi_k \mid k > 0)$ of abelian group homomorphisms $\phi_k : A_k \longrightarrow A'_k$ for each integer $k > 0$.

Let

$$\mathrm{Tor}^*_E(\cdot, G) = (\mathrm{Tor}^k_{\mathrm{End}(G)}(\cdot, G) \mid \text{ integers } k > 0)$$
$$\mathrm{Ext}^*_E(\cdot, G) = (\mathrm{Ext}^k_{\mathrm{End}(G)}(\cdot, G) \mid \text{ integers } k > 0).$$

The functors $\mathrm{Tor}^*_E(\cdot, G)$ and $\mathrm{Ext}^*_E(\cdot, G)$ have image in **SAb**.

Define the functors $\mathbf{U}$ and $\mathbf{U}^*$ in the only manner possible that makes the diagram in Figure 1 commute:

$$\mathbf{U}(\cdot) = \mathrm{Tor}^*_E(\mathrm{h}_X(\cdot), G) : X\text{-}\mathbf{Plex} \longrightarrow \mathbf{SAb} \tag{5.1}$$
$$\mathbf{U}^*(\cdot) = \mathrm{Ext}^*_E(\mathrm{h}^X(\cdot), G) : X\text{-}\mathbf{coPlex} \longrightarrow \mathbf{SAb}.$$

Now to define the middle diamond in Figure 1.

The objects in the category **SKAb** are sequences $(U_k \mid \text{ integers } k > 0)$ where for each integer $k > 0$ there is an abelian group A_k such that $U_k = K(A_k, 1)$. Maps in **SKAb** are sequences

$$([f_k] \mid \text{ integers } k > 0) : (U_k \mid \text{ integers } k > 0) \longrightarrow (V_k \mid \text{ integers } k > 0)$$

where for each integer $k > 0$, $[f_k]$ is the homotopy equivalence class of a continuous map $f_k : U_k \longrightarrow V_k$.

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The fundamental group functor $\pi_1(\cdot)$ induces a functor

$$\Pi_1(\cdot) : \mathbf{SKAb} \longrightarrow \mathbf{SAb}$$

such that $\Pi_1(U_k \mid \text{ integers } k > 0) = (\pi_1(U_k) \mid \text{ integers } k > 0)$.

Let A be an abelian group. The construction of the topological space $K(A, 1)$ induces a functor

$$K(\cdot) : \mathbf{SAb} \longrightarrow \mathbf{SKAb}$$

given by $K(S) = (K(A_k, 1) \mid \text{ integers } k > 0)$ (see [12]).

PROPOSITION 5.1. *The equivalences $\Pi_1(\cdot)$ and $K(\cdot)$ are inverse category equivalences.*

PROOF. Since $\pi_1(K(A, 1)) \cong A$ for an abelian group A, given $\mathcal{S} \in \mathbf{SAb}$ we have

$$\Pi_1(K(\mathcal{S})) \cong \mathcal{S}. \tag{5.2}$$

Furthermore, since $K(\pi_1(X), 1) \sim X$ for an Eilenberg–Mac Lane space X, given $\mathcal{X} \in \mathbf{SKAb}$ we have

$$K(\Pi_1(\mathcal{X})) \sim \mathcal{X}. \tag{5.3}$$

This proves the proposition. $\square$

Now to define the functors $\mathbf{V}$ and $\mathbf{V}^*$ in the only manner possible that makes the diagram in Figure 1 commute:

$$\begin{gathered} \mathbf{V}(\cdot) = K \circ \mathbf{U}(\cdot) : X\text{-}\mathbf{Plex} \longrightarrow \mathbf{SKAb} \\ \mathbf{V}^*(\cdot) = K \circ \mathbf{U}^*(\cdot) : X\text{-}\mathbf{coPlex} \longrightarrow \mathbf{SKAb}. \end{gathered} \tag{5.4}$$

THEOREM 5.2. *Let G be an abelian group and let $X = K(G, 1)$. There is a commutative diagram, Figure 1, of additive categories and functors, where opposing arrows denote inverse category equivalences.*

REMARK 5.3. We can fill in a blank place in Figure 1 as follows. Let **Bim** be the category of $\mathrm{End}(G)$–$\mathrm{End}(G)$-bimodules M such that

$$\mathrm{Tor}_E^*(M, G) \cong \mathrm{Ext}_E^*(M, G).$$

The $\mathrm{End}(G)$–$\mathrm{End}(G)$-bimodule $\mathrm{End}(G)$ is in **Bim** so **Bim** is not a trivial category. The category **Bim** and the canonical inclusion functors $\mathbf{Bim} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}(G)$ and $\mathbf{Bim} \longrightarrow \mathrm{End}(G)\text{-}\mathbf{Mod}$ embed in Figure 1 in a way that makes Figure 1 commute. It is interesting that such an unnatural isomorphism should arise in such a natural context.

If we assume that (μ) holds and assume that G is self-small and self-slender, then we produce the more detailed diagram, Figure 6. We construct Figure 6 under these assumptions.

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$$\begin{tikzcd}
 & & G\text{-}\mathbf{Plex} \arrow[dl, shift left, "\widehat{K}"] \arrow[dr, shift left, "\mathrm{h}_G"] \arrow[dd, "H_*^P"] & \\
 & X\text{-}\mathbf{Plex} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, shift right, "\mathrm{h}_X"'] \arrow[dl, "\mathbf{V}"'] \arrow[dr, "\mathbf{U}"] & & \mathbf{Mod}\text{-}\mathrm{End}(G) \arrow[ul, shift left, "\mathrm{t}_G"] \arrow[ll, shift right, "\mathrm{h}_X^{-1}"'] \arrow[dl, "\mathrm{Tor}_E^*(\cdot,\, G)"] \\
\mathbf{SKAb} \arrow[rr, shift left, "\Pi_1"] & & \mathbf{SAb} \arrow[ll, shift left, "K"] & \\
 & X\text{-}\mathbf{coPlex} \arrow[ul, "\mathbf{V}^*"] \arrow[ur, "\mathbf{U}^*"'] \arrow[rr, shift right, "\mathrm{h}^X"'] \arrow[dr, shift left, "\widehat{\pi}_1^*"] & & \mathrm{End}(G)\text{-}\mathbf{Mod} \arrow[ll, shift right, "(\mathrm{h}^X)^{-1}"'] \arrow[ul, "\mathrm{Ext}_E^*(\cdot,\, G)"'] \arrow[dl, shift left, "\mathrm{h}^G"] \\
 & & G\text{-}\mathbf{coPlex} \arrow[ul, shift left, "\widehat{K}^*"] \arrow[ur, shift left, "\mathrm{h}^G"] \arrow[uu, "H_*^c"'] &
\end{tikzcd}$$

FIGURE 6.

Since G is self-small, there is a commutative triangle Figure 3 of categories and category equivalences. This triangle replaces the top triangle in Figure 1 to form the top triangle in Figure 6. Similarly, assuming that (μ) holds and assuming that G is self-slender, Figure 5 replaces the bottom triangle in Figure 1 to form the bottom triangle of Figure 6.

The vertical maps in Figure 6 are the homology functors

$$H_*^P(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{SAb} \quad \text{and}$$
$$H_*^c(\cdot) : G\text{-}\mathbf{coPlex} \longrightarrow \mathbf{SAb}$$

of algebraic complexes. If G is self-small, then [7, Lemma 11.3.1] states that for each integer $k > 0$

$$\mathrm{Tor}^k_{\mathrm{End}(G)}(\mathrm{h}_G(\cdot),\, G) = H_k^P(\cdot) \tag{5.5}$$

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so that the insertion of the homology functor $H_*^P(\cdot)$ into Figure 6 preserves the commutativity of the diagram in Figure 6.

Assuming that (μ) holds, and assuming that G is self-slender, the commutative diagram [7, Diagram (11.16)] shows us that

$$\mathrm{Ext}_E^k(\mathrm{h}^G(\cdot), G) = H_k^c(\cdot) \tag{5.6}$$

for each integer $k > 0$. Thus, we can insert the vertical homology functor $H_*^c(\cdot)$ into the bottom diamond of Figure 6 and preserve the commutativity of the diagram in Figure 6.

THEOREM 5.4. *Assume that measurable cardinals do not exist, and let G be a self-small and a self-sender abelian group. There is a commutative diagram (Figure 6) of additive categories and functors, in which opposing arrows represent inverse category equivalences.*

REMARK 5.5. All of the arrows in the diagrams in Figures 1 and 6 are functors. This property does not apply to the diagram constructed in [8], where some of the arrows are not functors.

REMARK 5.6. Assume that measurable cardinals do not exist. The properties self-small and self-slender occur naturally when studying $\mathrm{End}(G)$ and G. For instance, in [2, 7] it was shown that self-small modules occur naturally when studying **Mod**-$\mathrm{End}(G)$, while in [4, 13] it was shown that self-slender modules occur naturally when studying $\mathrm{End}(G)$-**Mod**. Reduced torsion-free finite rank abelian groups are both self-small and self-slender.

COROLLARY 5.7. *Assume that measurable cardinals do not exist, and assume that G is a reduced torsion-free finite rank abelian group. There is a commutative diagram (Figure 6) of additive categories and functors, in which opposing arrows represent inverse category equivalences.*

6. Prism diagrams

We construct several commutative diagrams whose purpose is to further emphasize the relationship between the abelian group G and the $K(G, 1)$ space X.

We continue to use the fixed abelian group G and $X = K(G, 1)$. We make no other assumptions until we state them explicitly. We first construct the diagram shown in Figure 7.

We have defined $\widehat{\pi}_1(\cdot)$, $\widehat{K}(\cdot)$, $\mathrm{h}_X(\cdot)$, $\mathrm{h}_G(\cdot)$, Π_1, and $K(\cdot)$ when we constructed the diagrams in Figures 2 and 1.

Define the arrow χ on the right-hand side of Figure 7 so that the triangle commutes:

$$\chi(\cdot) = \mathrm{Tor}_E^*(\cdot, G) \circ \mathrm{h}_G(\cdot).$$

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$$
\begin{tikzcd}
X\text{-}\mathbf{Plex} \arrow[rr, shift left, "\widehat{\pi}_1"] \arrow[dr, "\mathrm{h}_X"] \arrow[dd, "\sigma"'] & & G\text{-}\mathbf{Plex} \arrow[ll, shift left, "\widehat{K}"] \arrow[dl, "\mathrm{h}_G"'] \arrow[dd, "\chi"] \\
& \mathbf{Mod}\text{-}\mathrm{End}(G) \arrow[dl, "\tau"] \arrow[dr, "\mathrm{Tor}^*_E(\cdot,\, G)"'] & \\
\mathbf{SKAb} \arrow[rr, shift left, "\Pi_1"] & & \mathbf{SAb} \arrow[ll, shift left, "K"]
\end{tikzcd}
$$

FIGURE 7.

By the commutativity of the diagram in Figure 2, the upper triangle of Figure 7 is commutative. Define $\tau(\cdot)$ so that the diagram of Figure 7 commutes:

$$\tau(\cdot) = K \circ \mathrm{Tor}^*_E(\cdot,\, G).$$

Then define $\sigma(\cdot)$ so that the diagram of Figure 7 commutes:

$$\sigma(\cdot) = \tau \circ \mathrm{h}_X(\cdot).$$

THEOREM 6.1. *Let G be an abelian group. There is a commutative diagram (Diagram 7), of categories and functors, in which opposing arrows represent inverse category equivalences.*

By dualizing the above process we construct the diagram shown in Figure 8.

THEOREM 6.2. *Let G be a group. There is a commutative diagram (Figure 8), of categories and functors, in which opposing arrows represent inverse category equivalences.*

In the case where G is self-small or self-slender we can be specific about χ and χ^*.

PROPOSITION 6.3. *Let G be an abelian group and let $X = K(G, 1)$.*

(1) *Assume that G is self-small. Then $\chi(\cdot) = H^P_*(\cdot)$, the homology functor.*
(2) *Assume that measurable cardinals do not exist, and assume that G is self-slender. Then $\chi^*(\cdot) = H^c_*(\cdot)$.*

PROOF. Parts (1) and (2) follow from Figures 3 and 5, and the identities (5.5), (5.6). This completes the proof. □

REMARK 6.4. One of the first observations we can make about Figure 7 is that the triangle with vertices X-**Plex**, **Mod**-End(G) = **Mod**-End$_{\mathbf{C}}(X)$, and **SKAb** is one of topology, while the triangle with vertices G-**Plex**, **Mod**-End(G), and **SAb** is one of

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$$
\begin{tikzcd}
X\text{-}\mathbf{coPlex} \arrow[rr, shift left, "\widehat{\pi}_1^*"] \arrow[dr, "\mathrm{h}^X"] \arrow[dd, "\sigma^*"'] & & G\text{-}\mathbf{coPlex} \arrow[ll, shift left, "\widehat{K}^*"] \arrow[dl, "\mathrm{h}^G"'] \arrow[dd, "\chi^*"] \\
& \mathrm{End}(G)\text{-}\mathbf{Mod} \arrow[dl, "\tau^*"] \arrow[dr, "\mathrm{Ext}^*_E(\cdot, G)"'] & \\
\mathbf{SKAb} \arrow[rr, shift left, "\Pi_1"] & & \mathbf{SAb} \arrow[ll, shift left, "K"]
\end{tikzcd}
$$

FIGURE 8.

algebra. As the horizontal arrows in Figure 7 represent inverse functors we see that the triangle of topology is the same as the triangle of algebra. The same can be said of the triangles in Figure 8.

7. Coherent complexes

Let us examine which parts of Figures 3, 6, and 8 are retained when we remove the self-small and self-slender assumptions from the abelian group G.

The right $\mathrm{End}(G)$-module M is *coherent* if it possesses a projective resolution whose terms are finitely generated projective right $\mathrm{End}(G)$-modules. Let $\mathbf{Coh}\text{-}\mathrm{End}(G)$ denote the category of coherent right $\mathrm{End}(G)$-modules. The category of coherent left $\mathrm{End}(G)$-modules is denoted by $\mathrm{End}(G)\text{-}\mathbf{Coh}$.

A G-plex $\mathcal{Q}$ as in (2.1) is *coherent* if each term Q_k in $\mathcal{Q}$ is a direct summand of G^n for some integer $n > 0$. Let G-**CohPlx** be the full subcategory of G-**Plex** whose objects are coherent G-plexes. Dually define a *coherent G-coplex* as a G-coplex $\mathcal{W}$ whose terms W_k are direct summand of G^n for some integer $n > 0$. Let G-**CohCoplex** denote the full subcategory of G**-coPlex** whose objects are coherent G-coplexes.

An X-plex $\mathcal{C}$ as in (3.1) is *coherent* if each term X_k of $\mathcal{C}$ is a $K(Q_k, 1)$ space for some direct summand Q_k of G^n for some integer $n = n(\mathcal{C}, k)$. The category X-**CohPlx** is the full subcategory of X**-Plex** whose objects are coherent X-plexes. We say that the X-coplex $\mathcal{W}$ is a *coherent X-coplex* if each term W_k of $\mathcal{W}$ is a $K(Q_k, 1)$ space for some direct summand Q_k of G^n for some integer $n = n(\mathcal{W}, k)$. The full subcategory of X**-coPlex** whose objects are coherent G-coplexes is denoted by X-**CohCoplex**.

THEOREM 7.1. *Let G be an abelian group and let $X = K(G, 1)$. There is a commutative triangle (see Figure 9) of categories and category equivalences.*

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$$\begin{tikzcd}
& G\text{-}\mathbf{CohPlx} \arrow[dl, shift left, "\widehat{K}"] \arrow[dr, shift left, "\mathrm{h}^G"] & \\
X\text{-}\mathbf{CohPlx} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, shift right, "\mathrm{h}^X"'] & & \mathbf{Coh}\text{-}\mathrm{End}(G) \arrow[ul, shift left, "\mathrm{h}^G"] \arrow[ll, shift right, "(\mathrm{h}^X)^{-1}"']
\end{tikzcd}$$

FIGURE 9.

$$\begin{tikzcd}
& G\text{-}\mathbf{CohCoplex} \arrow[dl, shift left, "\widehat{K}^*"] \arrow[dr, shift left, "\mathrm{h}^G"] & \\
X\text{-}\mathbf{CohCoplex} \arrow[ur, shift left, "\widehat{\pi}_1^*"] \arrow[rr, shift right, "\mathrm{h}^X"'] & & \mathrm{End}(G)\text{-}\mathbf{Coh} \arrow[ul, shift left, "\mathrm{h}^G"] \arrow[ll, shift right, "(\mathrm{h}^X)^{-1}"']
\end{tikzcd}$$

FIGURE 10.

PROOF. One readily verifies that

$$\widehat{\pi}_1(X\text{-}\mathbf{CohPlx}) \subset G\text{-}\mathbf{CohPlx}$$
$$\widehat{K}(G\text{-}\mathbf{CohPlx}) \subset X\text{-}\mathbf{CohPlx}$$

so by Theorem 3.4 there is a category equivalence

$$\widehat{\pi}_1(\cdot) : X\text{-}\mathbf{CohPlx} \longrightarrow G\text{-}\mathbf{CohPlx}$$

and its inverse $\widehat{K}$ as in Figure 9.

By [7, Theorem 2.1.13], there are inverse category equivalences

$$\mathrm{h}_G(\cdot) : G\text{-}\mathbf{CohPlx} \longrightarrow \mathbf{Coh}\text{-}\mathrm{End}(G)$$
$$\mathrm{t}_G(\cdot) : \mathbf{Coh}\text{-}\mathrm{End}(G) \longrightarrow G\text{-}\mathbf{CohPlx}$$

as in Figure 9. By the commutativity of the diagram in Figure 2 the functor $\mathrm{h}_X(\cdot) : X\text{-}\mathbf{CohPlx} \longrightarrow \mathbf{Coh}\text{-}\mathrm{End}(G)$ at the bottom of Figure 9 makes the diagram commute. Hence, $\mathrm{h}_X(\cdot)$ is a category equivalence. This completes the proof. $\square$

Dualizing the above process yields a different commutative triangle.

THEOREM 7.2. *Let G be an abelian group and let $X = K(G, 1)$. There is a commutative triangle (see Figure 10) of categories and category equivalences.*

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PROOF. Dualize the proof of Theorem 7.1. □

THEOREM 7.3. *Let G be an abelian group. There is a commutative diagram, Figure 11, of additive categories and functors, in which opposing arrows represent inverse functors.*

PROOF. Substitute the commutative diagrams in Theorems 7.1 and 7.2 into the commutative diagram Figure 1. The result is a commutative diagram. Include the vertical functors H_*^P and H_*^c as in Figure 6. We leave it as an exercise for the reader to show that the inclusion of these vertical functors yields a commutative diagram. This completes the proof. □

8. Applications

In this section, G and A denote variable abelian groups, X is a $K(G, 1)$-space, and Y is a $K(A, 1)$-space. As in [8], we apply Figures 3 and 5 to the uniqueness of decomposition of abelian groups.

Let **KAb** denote the category of $K(A, 1)$ spaces where A ranges over the category **Ab** of abelian groups. Let **A** be a category in which $\cong$ is an equivalence relation, and let **T** be a category in which $\sim$ is an equivalence relation. We say that $\mathbf{T}/\sim$ is *a complete set of topological invariants for* **A** *modulo* $\cong$ if there is a bijective map $\Sigma : \mathbf{A}/\cong \longrightarrow \mathbf{T}/\sim$.

Let A be an abelian group and identify $K(A, 0, 0, \ldots) = K(A, 1)$.

THEOREM 8.1. *We say that* $\mathbf{KAb}/\sim$ *is a complete set of topological invariants for* **Ab** *modulo isomorphism.*

PROOF. The functors $K(\cdot)$ and $\Pi_1(\cdot)$ are inverses by Proposition 5.1, so $K(\cdot)$ induces a bijection $\Sigma : \mathbf{Ab}/\cong \longrightarrow \mathbf{KAb}/\sim$. This completes the proof. □

Let $Y \in \mathbf{KAb}$. We say that Y is *K-indecomposable* if given $U, V \in \mathbf{KAb}$ such that $Y \sim U \times V$, then U or V contracts to a point. We say that Y has a *unique finite Cartesian K-decomposition* if:

(1) $Y = \prod_{i\in\mathcal{I}} U_i$ for some finite set $\{U_i \mid i \in \mathcal{I}\} \subset \mathbf{KAb}$ of K-indecomposable spaces; and
(2) $Y = \prod_{j\in\mathcal{J}} V_j$ for some finite set $\{V_j \mid j \in \mathcal{J}\} \subset \mathbf{KAb}$ of K-indecomposable spaces, then there is a bijection $\Sigma : \mathcal{I} \longrightarrow \mathcal{J}$ such that $U_i \sim V_{\Sigma(i)}$ for each $i \in \mathcal{I}$.

We say that A has a *unique finite direct sum decomposition* if:

(1) $A = \bigoplus_{i\in\mathcal{I}} B_i$ for some finite set $\{B_i \mid i \in \mathcal{I}\}$ of indecomposable abelian groups; and
(2) $A = \bigoplus_{j\in\mathcal{J}} C_j$ for some finite set $\{C_j \mid j \in \mathcal{J}\}$ indecomposable abelian groups, then there is a bijection $\Sigma : \mathcal{I} \longrightarrow \mathcal{J}$ such that $B_i \cong C_{\Sigma(i)}$ for each $i \in \mathcal{I}$.

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$$
\begin{tikzcd}
 & & G\text{-}\mathbf{CohPlx} \arrow[dl, shift left, "\widehat{K}"] \arrow[dr, shift left, "\mathrm{h}_G"] \arrow[dd, "H_*^P"] & \\
 & X\text{-}\mathbf{CohPlx} \arrow[ur, shift left, "\widehat{\pi}_1"] \arrow[rr, shift right, "\mathrm{h}_X"'] \arrow[dl, "\mathbf{V}"'] \arrow[dr, "\mathbf{U}"] & & \mathbf{Coh}\text{-}\mathrm{End}(G) \arrow[ul, shift left, "\mathrm{t}_G"] \arrow[ll, shift right, "\mathrm{h}_X^{-1}"'] \arrow[dl, "\mathrm{Tor}_E^*(\cdot,\, G)"] \\
\mathbf{SKAb} \arrow[rr, shift left, "\Pi_1"] & & \mathbf{SAb} \arrow[ll, shift left, "K"] & \\
 & X\text{-}\mathbf{CohCoplex} \arrow[ul, "\mathbf{V}^*"] \arrow[ur, "\mathbf{U}^*"'] \arrow[rr, shift right, "\mathrm{h}^X"'] \arrow[dr, shift left, "\widehat{K}^*"] & & \mathrm{End}(G)\text{-}\mathbf{Coh} \arrow[ul, "\mathrm{Ext}_E^*(\cdot,\, G)"'] \arrow[ll, shift right, "(\mathrm{h}^X)^{-1}"'] \arrow[dl, shift left, "\mathrm{h}^G"] \\
 & & G\text{-}\mathbf{CohCoplex} \arrow[ul, shift left, "\widehat{\pi}_1^*"] \arrow[ur, shift left, "\mathrm{h}^G"] \arrow[uu, "H_*^c"'] &
\end{tikzcd}
$$

FIGURE 11.

LEMMA 8.2. *Let A be an abelian group and let $Y = K(A, 1)$.*

(1) *If $A = B \oplus C$, then $Y \sim K(B, 1) \times K(C, 1)$.*
(2) *If $Y \sim U \times V$ for some $U, V \in \mathbf{KAb}$, then $A \cong \pi_1(U) \oplus \pi_1(V)$.*
(3) *A is indecomposable if and only if Y is K-indecomposable.*

PROOF. Parts (1) and (2) follow directly from [12, Theorem 1B.5].

Part (3) follows from parts (1) and (2). This proves the lemma. □

THEOREM 8.3. *Let A be an abelian group and let $Y = K(A, 1)$. Then A has a unique finite direct sum decomposition if and only if Y has a unique finite Cartesian K-decomposition.*

PROOF. One proceeds as in [8, Theorem 7.2]. We sketch a proof. Suppose that A has a unique finite direct sum decomposition $\bigoplus_{i\in\mathcal{I}} B_i$ for some finite set $\{B_i \mid i \in \mathcal{I}\}$ of indecomposable abelian groups. Let $U_i = K(B_i, 1)$. Then by Lemma 8.2(3), U_i is K-indecomposable, and

$$Y = K(A, 1) = K\left(\bigoplus_{\mathcal{I}} B_i, 1\right) \sim \prod_{\mathcal{I}} K(B_i, 1) = \prod_{\mathcal{I}} U_i.$$

Another K-decomposition $Y \sim \prod_{j \in \mathcal{J}} V_j$ corresponds under the equivalence $K(\cdot)$ to a decomposition $\bigoplus_{\mathcal{J}} C_j$ of A. That is, $V_j \sim K(C_j) = K(C_j, 1)$. Since A has the unique finite direct sum decomposition $\bigoplus_{\mathcal{I}} B_i$ we may assume without loss of generality that $\mathcal{J} = \mathcal{I}$ and that $C_i = B_i$ for $i \in \mathcal{I}$. Then $V_i \sim K(C_i) = K(B_i) \sim U_i$, hence Y has a unique finite Cartesian K-decomposition. The proof of the converse is found by reversing the above argument in the obvious way. This completes the proof. □

We say that the abelian group A has the *cancellation property* if $A \oplus B \cong A \oplus C$ for some abelian groups B and C implies that $B \cong C$. A space $Y \in \mathbf{KAb}$ has the *cancellation property in* **KAb** if $Y \times U \sim Y \times V$ for some $U, V \in \mathbf{KAb}$ implies that $U \sim V$.

The group A has the *power cancellation property* if for each integer $n > 0$ and abelian group B, $A^n \cong B^n$ implies that $A \cong B$. We say that A has a Σ-*unique decomposition* if for each integer $n > 0$, A^n has a unique finite direct sum decomposition.

The topological space Y has the *power cancellation property in* **KAb** if for each integer $n > 0$ and topological space $Z \in \mathbf{KAb}$, $Y^n \sim Z^n$ implies that $Y \sim Z$. We say that Y has Σ-*unique decomposition in* **KAb** if for each integer $n > 0$, Y^n has unique finite Cartesian K-decomposition.

Then the proof of Theorem 8.3 can be used to prove the following three results.

THEOREM 8.4. *Let A be an abelian group and let $Y = K(A, 1)$. Then A has the cancellation property if and only if Y has the cancellation property in* **KAb**.

THEOREM 8.5. *Let A be an abelian group and let $Y = K(A, 1)$. Then A has the power cancellation property if and only if Y has the power cancellation property in* **KAb**.

THEOREM 8.6. *Let A be an abelian group and let $Y = K(A, 1)$. Then A has a Σ-unique decomposition if and only if Y has a Σ-unique decomposition in* **KAb**.

These applications and the work in [5] can be combined to give a nice classification of the unique factorization property in algebraic number fields.

Let G and H be abelian groups. We say that G is *locally isomorphic to H* if given an integer $n > 0$ there is an integer $m > 0$ relatively prime to n, and maps $f : G \to H$ and $g : H \to G$ such that $fg = gf = m \cdot 1$. See [6] for properties of *locally isomorphic* abelian groups.

Let $\mathbf{k}$ be an algebraic number field, let $\overline{E}$ be the ring of algebraic integers in $\mathbf{k}$, and let

$$\Omega(\overline{E}) = \{\text{abelian groups } G \mid \mathrm{End}(G) \cong \overline{E}\}.$$

The *class number of* $\mathbf{k}$, denoted by $h(\mathbf{k})$ is the number of isomorphism classes of fractional right ideals of $\overline{E}$. This is a classic idea from number theory [15].

Given an abelian group G the *class number of* G is the number of isomorphism classes of groups H that are locally isomorphic to G. The class number of G is denoted by $h(G)$.

The spaces X and Y are *power homotopic* if and only if there is an integer $n > 0$ such that $X^n \sim Y^n$. Given a topological space X the *class number of* X *in* **KAb** is the number of homotopy classes of topological spaces Y that are power isomorphic to X in **KAb**. The class number of X is denoted by $h(X)$.

The abelian group G has $\Sigma(h)$-unique decomposition if $h > 0$ is the smallest integer that has the following property. There is a finite set $\{G_1, \ldots, G_h\}$ of indecomposable groups G_i such that for each $n > 0$, if $G^n = H_1 \oplus \cdots \oplus H_t$ for some integer t and indecomposable abelian groups $H_1, \ldots, H_t$, then for each $i = 1, \ldots, t$ there is an integer $1 \le j(i) \le h$ such that $H_i \cong G_{j(i)}$.

The topological space X has $\Sigma(h)$*-unique decomposition in* **KAb** if $h > 0$ is the smallest integer with the following property. There is a finite set $\{X_1, \ldots, X_h\}$ of K-indecomposable spaces X_i such that for each $n > 0$, if $X^n \sim Y_1 \oplus \cdots \oplus Y_t$ for some integer t and K-indecomposable spaces $Y_1, \ldots, Y_t$ in **KAb**, then for each $i = 1, \ldots, t$ there is an integer $1 \le j(i) \le h$ such that $Y_i \sim X_{j(i)}$.

THEOREM 8.7. *Let* $\mathbf{k}$ *be an algebraic number field and let* $h > 0$ *be an integer. The following are equivalent:*

(1) $h(\mathbf{k}) = h$;
(2) $h(G) = h$ *for each* $G \in \Omega(\overline{E})$;
(3) $h(G) = h$ *for some* $G \in \Omega(\overline{E})$;
(4) *for each* $G \in \Omega(\overline{E})$ *there are exactly* h *isomorphism classes of groups* H *such that* $G^n \cong H^n$ *for some integer* $n > 0$;
(5) *for some* $G \in \Omega(\overline{E})$ *there are exactly* h *isomorphism classes of groups* H *such that* $G^n \cong H^n$ *for some integer* $n > 0$;
(6) *each* $G \in \Omega(\overline{E})$ *has* $\Sigma(h)$*-unique decomposition;*
(7) *some* $G \in \Omega(\overline{E})$ *has* $\Sigma(h)$*-unique decomposition;*
(8) *given* $G \in \Omega(\overline{E})$, $h(X) = h$ *for each* $K(G, 1)$*-space* X;
(9) *for some* $G \in \Omega(\overline{E})$, $h(X) = h$ *for each* $K(G, 1)$*-space* X;
(10 *given* $G \in \Omega(\overline{E})$, *each* $K(G, 1)$*-space has* $\Sigma(h)$*-unique decomposition in* **KAb**;
(11) *for some* $G \in \Omega(\overline{E})$, *each* $K(G, 1)$*-space has* $\Sigma(h)$*-unique decomposition in* **KAb**.

PROOF. (1) if and only if (2). Let $G \in \Omega(\overline{E})$. Then $h(G) = h(\overline{E}) = h(\mathbf{k})$ by [5, Corollary 3.2]. This proves (1) if and only if (2).

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(2) implies (4). Let $G \in \Omega(\overline{E})$. Let $\{(G_1), \ldots, (G_h)\}$ be the set of isomorphism classes of groups G_i that are locally isomorphic to G, and let $\{(H_i) \mid i \in I\}$ be a set of isomorphism classes of groups H such that $G^n \cong H^n$ implies $G \cong H$. By Warfield's theorem [1, Theorem 13.9], for each $(G_i) \in \{(G_1), \ldots, (G_h)\}$ there is an integer n such that $G^n \cong G_i^n$. Then $\{(G_1), \ldots, (G_h)\} \subset \{(H_i) \mid i \in I\}$. On the other hand, let $(H_i) \in \{(H_i) \mid i \in I\}$. Then $G^n \cong H^n$ for some integer $n > 0$. By Warfield's theorem [1, Theorem 13.9], G is locally isomorphic to H_i, so that $\{(G_1), \ldots, (G_h)\} \supset \{(H_i) \mid i \in I\}$. Hence $\{(G_1), \ldots, (G_h)\} = \{(H_i) \mid i \in I\}$ so that h is the number of isomorphism classes of groups H such that $G^n \cong H^n$ for some integer $n > 0$. This proves part (4).

(4) implies (6). Let $G \in \Omega(\overline{E})$. By part (4) there is a set $\{(G_1), \ldots, (G_h)\}$ of isomorphism classes of groups H such that $G^n \cong H^n$ for some integer $n > 0$. Let $n > 0$ be an integer, and let $G^n \cong H_1 \oplus \cdots \oplus H_t$ for some integer $t > 0$ and some indecomposable groups $H_1, \ldots, H_t$. Since $\overline{E} = \mathrm{End}(G)$ is an integral domain, [5, Corollary 2.6] states that each H_i is locally isomorphic to G, so $H_i \cong G_{j(i)}$ for some $1 \leq j(i) \leq h$. Thus, there is a minimal integer $0 < h' \leq h$ and a set $\{(K_1), \ldots, (K_{h'})\}$ of isomorphism classes (K_i) of indecomposable groups such that for each $i = 1, \ldots, t$, there is an integer $1 \leq j(i) \leq h'$ such that $H_i \cong K_{j(i)}$.

Specifically, given $(H) \in \{(G_1), \ldots, (G_h)\}$, then there is an integer $n > 0$ such that $G^n \cong H^n$. By our choice of $\{(K_1), \ldots, (K_{h'})\}$, $(H) \in \{(K_1), \ldots, (K_{h'})\}$, hence $\{(G_1), \ldots, (G_h)\} \subset \{(K_1), \ldots, (K_{h'})\}$, thus $h \leq h'$. That is, $h = h'$, which proves part (6).

(6) implies (2). Let $G \in \Omega(\overline{E})$. Suppose that G has $\Sigma(h)$-unique decomposition. There is a set $\{(K_1), \ldots, (K_h)\}$ of isomorphism classes (K_i) such that for each integer $n > 0$, if $G^n \cong H_1 \oplus \cdots \oplus H_t$ for some integer m and indecomposable groups $H_1, \ldots, H_t$ then for each $i = 1, \ldots, t$ there exists a $1 \leq j(i) \leq h$ such that $H_i \cong K_{j(i)}$. Let $\{(G_1), \ldots, (G_{h(G)})\}$ be a complete set of isomorphism classes of groups H that are locally isomorphic to G. Suppose that H is locally isomorphic to G. By Warfield's theorem [1, Theorem 13.9] there is an integer $n > 0$ such that $G^n \cong H^n$. Then $H \cong K_j$ for some $1 \leq j \leq h$, so that $\{(G_1), \ldots, (G_{h(G)})\} \subset \{((K_1), \ldots, (K_h)\}$. Subsequently, $h(G) \leq h$.

Let $(K) \in \{(K_1), \ldots, (K_h)\}$. By the minimality of h, there is an integer $n > 0$ and a direct sum decomposition $G^n \cong K \oplus H_2 \oplus \cdots \oplus H_t$. Since $\mathrm{End}(G) = \overline{E}$ is a commutative integral domain, [5, Corollary 2.6(2)] states that K is locally isomorphic to G, so there exists a $1 \leq j \leq h(G)$ such that $K \cong G_j$. It follows that $h \leq h(G)$, and hence that $h = h(G)$. Thus, we have proved that (1) if and only if (2) implies (4) implies (6) implies (2).

(2) implies (3) implies (5) implies (7) implies (2). By Butler's theorem [6, Theorem I.2.6], there is a $G \in \Omega(\overline{E})$. Then part (2) implies part (3). The rest follows as in (2) implies (4) implies (6) implies (2).

(2) implies (8). Let $G \in \Omega(\overline{E})$ and let X be a $K(G, 1)$-space. Recall the functors $\widehat{K}(\cdot)$ and $\widehat{\pi}(\cdot)$ from Theorem 3.4. Let $\{(G_1), \ldots, (G_{h(G)})\}$ be a complete set of isomorphism classes of groups H that are locally isomorphic to G. Let

$\{(X_1), \ldots, (X_{h(X)})\}$ be a complete set of the homotopy classes of topological spaces Y that are power homotopic to X. Let $H \in \{(G_1), \ldots, (G_{h(G)})\}$. By Warfield's theorem [1, Theorem 13.9], there is an integer n such that $G^n \cong H^n$. Then

$$X^n \sim \widehat{K}(G)^n \sim \widehat{K}(G^n) \sim \widehat{K}(H^n) \sim \widehat{K}(H)^n$$

which shows us that X is power isomorphic to $\widehat{K}(H)$. Thus, $\widehat{K}(\cdot) : G\text{-}\mathbf{Plex} \to X\text{-}\mathbf{Plex}$ induces a map $f : \{(G_1), \ldots, (G_{h(G)})\} \to \{(X_1), \ldots, (X_{h(X)})\}$ such that $f(G_i) = (\widehat{K}(G_i))$. Similarly $\widehat{\pi}_1(\cdot) : X\text{-}\mathbf{Plex} \to G\text{-}\mathbf{Plex}$ induces a map $g : \{(X_1), \ldots, (X_{h(X)})\} \to \{(G_1), \ldots, (G_{h(G)})\}$ such that $g(X_i) = (\widehat{\pi}_1(X_i))$. Since $\widehat{K}(\cdot)$ and $\widehat{\pi}_1(\cdot)$ are inverse category equivalences, one shows that f and g are inverse bijections. Thus, $h(G) = h(X)$. This proves part (8).

(8) implies (10). Let $G \in \Omega(\overline{E})$ and let X be a $K(G, 1)$-space. Let $\{(X_1), \ldots, (X_{h(X)})\}$ be a complete set of homotopy classes of topological spaces X_i that are power homotopic to X. Let $n > 0$ be an integer, and suppose that $X^n \sim Y_1 \times \cdots \times Y_t$ for some indecomposable topological spaces $Y_1, \ldots, Y_t$. Then the category equivalence $\widehat{\pi}_1 : X\text{-}\mathbf{Plex} \to G\text{-}\mathbf{Plex}$ (Theorem 3.4) takes the Cartesian product of K-indecomposable spaces Y_i in **KAb** to a direct sum of indecomposable abelian groups

$$G^n \cong \widehat{\pi}_1(X)^n \cong \widehat{\pi}_1(X^n) \cong \widehat{\pi}_1(Y_1) \oplus \cdots \oplus \widehat{\pi}_1(Y_t).$$

It follows from [5, Corollary 2.6(2)] that since $\mathrm{End}(G) \cong \overline{E}$ is a commutative integral domain, and since each $\widehat{\pi}_1(Y_i)$ is indecomposable, each $\widehat{\pi}_1(Y_i)$ is locally isomorphic to G. By Warfield's theorem [1, Theorem 13.9], there is an integer $m > 0$ such that

$$\widehat{\pi}_1(X^m) \cong G^m \cong \widehat{\pi}_1(Y_i)^m \cong \widehat{\pi}_1(Y_i^m),$$

and since $\widehat{\pi}_1(\cdot)$ is a category equivalence,

$$X^m \sim Y_i^m$$

for each $i = 1, \ldots, t$. Then for each $i = 1, \ldots, t$ there is a $1 \leq j(i) \leq h(X)$ such that $Y_i \sim X_{j(i)}$. Thus, X has $\Sigma(h)$-unique decomposition in **KAb** for some integer $0 \leq h \leq h(X)$.

Let $\{(U_1), \ldots, (U_h)\}$ be a set of homotopy classes of topological spaces associated with the definition of $\Sigma(h)$-unique decomposition for X. There are $h(X)$ homotopy classes $(X_1), \ldots, (X_{h(X)})$, and by arguing above with Warfield's theorem [1, Theorem 13.9], there are integers $m, m_1, \ldots, m_{h(X)} > 0$ such that

$$X^m \sim \prod_{i=1}^{h(X)} X_i^{m_i}.$$

Thus, $\{(X_1), \ldots, (X_{h(X)})\} \subset \{(U_1), \ldots, (U_h)\}$, and so $h(X) \leq h$. Hence, $h(X) = h$, which proves part (10).

(10) implies (4). Proceed in the by now familiar pattern, using the category equivalences $\widehat{\pi}_1$ and $\widehat{K}$ to show that the $\Sigma(h)$-unique decomposition of G follows from the $\Sigma(h)$-unique decomposition of the $K(G, 1)$-space X.

This proves (2) implies (8) implies (10) implies (4). The equivalences (8) if and only if (9) and (10) if and only if (11) are clear. This completes the logical cycle.

For the next theorem, let $\mathbf{k}$ be an algebraic number field, let $\overline{E}$ be the ring of algebraic integers in $\mathbf{k}$, and for each rational prime p let $E(p) = \mathbb{Z} + p\overline{E}$. Let $G(p)$ be a reduced torsion-free finite rank abelian group such that $\mathrm{End}(G(p)) \cong E(p)$. These groups exist by Butler's theorem [6, Theorem I.2.6]. Let $L(p) = \mathrm{card}(u(\overline{E})/u(E(p)))$ where $u(R)$ is the groups of units in the ring R. For an abelian group H let $h(H)$ be the number of isomorphism classes of groups K that are locally isomorphic to H. Sequences s_n and t_n are *asymptotically equal* if $\lim_{n\to\infty} s_n/t_n = 1$.

If a group G possesses the $\Sigma(1)$-unique decomposition we say that G has the Σ*-unique decomposition*. A similar statement is made for $K(G, 1)$-spaces that have Σ*-unique decomposition in* **KAb**. If there is exactly one isomorphism class of groups H such that $G^n \cong H^n$ for some integer $n > 0$ then we say that G has the *power cancellation property*. A similar statement is made about $K(G, 1)$-spaces X that have the *power cancellation property in* **KAb**. Thus the following theorem is proved by allowing $h = 1$ in Theorem 8.7. □

THEOREM 8.8. *Let* $\mathbf{k}$ *be an algebraic number field. The following are equivalent.*

(1) *The algebraic number field* $\mathbf{k}$ *has unique factorization.*
(2) *The sequence* $\{L(p)h(G(p)) \mid \text{rational primes } p\}$ *is asymptotically equal to the sequence* $\{p^{f-1} \mid \text{rational primes } p\}$.
(3) *Each* $G \in \Omega(\overline{E})$ *has the power cancellation property.*
(4) *Some* $G \in \Omega(\overline{E})$ *has the power cancellation property.*
(5) *Each* $G \in \Omega(\overline{E})$ *has* Σ*-unique decomposition.*
(6) *Some* $G \in \Omega(\overline{E})$ *has* Σ*-unique decomposition.*
(7) *Given* $G \in \Omega(\overline{E})$*, each* $K(G, 1)$*-space has the power cancellation property in* **KAb**.
(8) *For some* $G \in \Omega(\overline{E})$*, each* $K(G, 1)$*-space has the power cancellation property in* **KAb**.
(9) *Given* $G \in \Omega(\overline{E})$*, each* $K(G, 1)$*-space has* Σ*-unique decomposition in* **KAb**.
(10) *For some* $G \in \Omega(\overline{E})$*, each* $K(G, 1)$*-space has* Σ*-unique decomposition in* **KAb**.

PROOF. (1) if and only if (2) follows form [5, Theorem 9.4]. The proof of the rest of the theorem follows from Theorem 8.7 and the fact that $\mathbf{k}$ has unique factorization if and only if $h(\mathbf{k}) = 1$. This proves the theorem. □

PROOF OF THEOREM 1.1. (1) if and only if (2). Since $\mathbf{k}$ is a quadratic number field, $f = 2$, so that the proof of the theorem follows from Theorem 8.8(2).

(1) if and only if (3). Assume that condition (1) holds, that is, that the quadratic number field $\mathbf{k}$ has unique factorization. By Theorem 8.8(2), each reduced strongly

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indecomposable torsion-free rank-two group G such that $\mathrm{End}(G) \cong \overline{E}$ has the power cancellation property. This proves condition (3).

(3) if and only if (4). Assume that condition (3) holds. By Butler's theorem [6, Theorem I.2.6], there is a reduced torsion-free abelian group G such that $\overline{E} \subset G \subset \mathbb{Q}\overline{E}$ and such that $\mathrm{End}(G) \cong \overline{E}$. Since $\mathbf{k}$ is a quadratic number field, $\overline{E}$ and so G have torsion-free rank two. Moreover, since $\overline{E}$ is an integral domain, the group G is strongly indecomposable [1, Corollary 7.8]. Condition (4) is proved.

(4) implies (1). Assume that condition (4) holds. Theorem 8.8(3) implies that $\mathbf{k}$ has unique factorization.

(1) implies (5) implies (6) implies (1) follow in a manner similar to the proof of (1) implies (3) implies (4) implies (1).

(1) implies (7) implies (8) implies (9) implies (10) implies (1) follows from Theorem 8.8. This completes the proof of Theorem 1.1. $\square$

REMARK 8.9. Owing to the category equivalence between **Ab** and **KAb** via the functor $\widehat{K}(\cdot)$ we can use the approach in Theorem 8.4 to topologically characterize any direct sum decomposition property of a fixed abelian group A.

REMARK 8.10. The equivalence between **Ab** and **KAb** via the functor $\widehat{K}(\cdot)$ allows us to view abstract abelian groups, which do not come equipped with a geometry, as subspaces of quotients of Euclidean k-space, for some integer $k > 0$.

REMARK 8.11. Over the past 35 years certain aspects of abelian group theory have blurred with the study of modules over more general associative rings. Figures 2, 1, and 7 represent an idea in abelian group theory that is not readily extended to modules over more general rings. The key observation to see this is that a homotopy group is not in general a module over a ring $R \neq \mathbb{Z}$. Therefore Figures 2, 1, and 7 seem to represent a concept in abelian group theory that will be unique to abelian groups for some time.

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