# CONGRUENCE RELATIONS ON ORTHOMODULAR LATTICES

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## 1. Introduction

We denote lattice join and meet by $\vee$ and $\wedge$ respectively and the associated partial order by $\leqq$. A lattice $L$ with 0 and $I$ is said to be orthocomplemented if it admits a dual automorphism $x \to x'$, that is a one-one mapping of $L$ onto itself such that

$$x \leqq y \Leftrightarrow y' \leqq x',$$

which is involutive, so that

$$(x')' = x$$

for each $x$ in $L$ and, further, is such that

$$x \wedge x' = 0, \quad x \vee x' = I$$

for each $x$ in $L$.

Here and elsewhere $\Leftrightarrow$ stands for logical equivalence, or "if and only if", and below $\Rightarrow$ stands for "implies".

We call $x'$ the orthocomplement of $x$. Two elements $x$, $y$ of an orthocomplemented lattice are said to be orthogonal, and we write $x \perp y$, if and only if $x \leqq y'$. Clearly

$$x \perp y \Rightarrow y \perp x$$

and

$$x \perp y \Rightarrow x \wedge y = 0$$

although the converse of this last implication is not generally true. A lattice $L$ is said to be orthomodular if it is orthocomplemented and satisfies the following weak modular law,

(1.1) $$x \perp y \Rightarrow (x \vee y) \wedge x' = y.$$

Orthocomplementation of (1.1) yields the equivalent formulation

(1.2) $$a \leqq b \Rightarrow a \vee (b \wedge a') = b.$$

We remark that in an orthomodular lattice all closed intervals $[0, a]$ are orthocomplemented by the mapping $x \to a \wedge x'$ and with respect to this orthocomplementation each such closed interval is an orthomodular lattice. Any orthocomplemented modular lattice is orthomodular, in particular a Boolean algebra is an orthomodular lattice. A well-known example of an orthomodular lattice is the set of closed (linear) subspaces in Hilbert space partially ordered by set inclusion. Such an orthomodular lattice is modular only in the finite-dimensional case.

A congruence relation $\theta$ on a lattice is an equivalence relation such that

$$x_i \equiv y_i(\theta) \qquad i = 1, 2$$

implies

$$x_1 \vee x_2 \equiv y_1 \vee y_2(\theta)$$

and

$$x_1 \wedge x_2 \equiv y_1 \wedge y_2(\theta)$$

If $\theta$ is a congruence relation on a lattice $L$ with 0 we denote by $J_\theta$ the set of its elements which are congruent to 0. It is wellknown, Birkhoff [1, page 23], that $J_\theta$ is an ideal, that is, (i) $x \in J_\theta$ and $y \in J_\theta$ together imply that $x \vee y \in J_\theta$ and (ii) $x \in J_\theta$ and $y < x$ together imply that $y \in J_\theta$.

Let $L$ be a lattice and let $S$ be the set of its ideals and $C$ the set of congruence relations on $L$. The following results are well-known.

(a) If the lattice has a 0 then there is a natural mapping of $C$ onto a subset of $S$. (For any congruence $\theta$, $J_\theta$ is an ideal.)

(b) If all closed intervals [0, a] of $L$ are complemented then there is a one-one correspondence of $C$ with a subset of $S$.

(c) In a Boolean algebra there is a one-one correspondence of $C$ with the whole of $S$ (not just a subset).

These results will be found in Birkhoff [1]; for (a) and (b) see Theorem 3, page 23, and for (c) Theorem 8, page 159.

The following result is established in the next section.

THEOREM 1. *If $L$ is an orthomodular lattice and if $S_1$ is the subset of $S$ consisting of those ideals $J$ for which*

$$a \in J \Rightarrow (x \vee a) \wedge x' \in J \tag{1.3}$$

*for each $x$ in $L$, then there is a one-one mapping of the subset $S_1$ onto the whole of $C$.*

Note that in a Boolean algebra $S_1 = S$, for in the presence of distributivity

$$(x \vee a) \wedge x' = a \wedge x' \leqq a \in J.$$

Thus Theorem 1 contains (c) as a special case and to that extent may be considered a generalisation of the known result for Bolean algebras. Note

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that the difference between the known result (b) and Theorem 1 is that the theorem specifies the subset $S_1$ of $S$ whereas the result (b) does not give any criterion whether or not a given ideal is, for some congruence $\theta$, the ideal of elements congruent to 0 modulo $\theta$.

Interest in orthomodular lattices is stimulated by their use as a model for the logic of quantum mechanics, see, for example, Mackey [4] section (2.2), and the consequent need to develop probability theory on such structures. Further details of such a development will be given in a later paper, Finch [2].

Finally in section 3 below we consider in some detail a class of ideals in the lattice of closed subspaces of Hilbert space which satisfy the condition (1.3) and which, therefore, generate congruence relations. This example shows that there are orthomodular lattices other than Boolean algebras which possess congruence relations.

## 2. Proof of Theorem 1

It is convenient to break up the proof of theorem 1 into a number of contributory results.

PROPOSITION (2.1). *If $\theta$ is a congruence on an orthomodular lattice $L$ then*

$$a \in J_\theta \Rightarrow (x \vee a) \wedge x' \in J_\theta$$

*for each $x$ in $L$.*

PROOF. Since $a = 0(\theta)$

$$(x \vee a) \wedge x' \equiv x \wedge x' = 0(\theta).$$

In virtue of the proposition it remains only to show that each ideal in $S_1$ determines a unique congruence in $C$. To do so we establish

PROPOSITION (2.2). *Let $L$ be an orthomodular lattice and let $J$ be an ideal of $L$ such that* (1.3) *holds. Write $x \sim y$ in $L$ if and only if there are elements $a$ and $b$ in $J$ such that*

$$x \vee a = y \vee b.$$

*Then $x \sim y$ is a congruence relation on $L$, and*

$$J = \{x : x \sim 0\}. \tag{2.1}$$

Before proceeding to the proof of this proposition we establish the preliminary

LEMMA (2.1). *Under the hypothesis of proposition* (2.2), *$x \sim y$ implies that there exist elements $\alpha$, $\beta$ in $J$ such that $\alpha \perp x$, $\beta \perp y$ and*

$$x \vee \alpha = y \vee \beta.$$

Proof. By hypothesis there are elements $a$ and $b$ in $J$ such that

$$x \vee a = y \vee b.$$

Take

$$\alpha = (x \vee a) \wedge x',$$

and

$$\beta = (y \vee b) \wedge b'.$$

Then $\alpha \perp x$, $\beta \perp y$ and each of $\alpha$ and $\beta$ belong to $J$ because of (1.3). Finally using (1.2) gives

$$x \vee \alpha = x \vee a = y \vee b = y \vee \beta.$$

An immediate consequence of this lemma is

Lemma (2.2). *Under the hypothesis of proposition* (2.2)

$$x \sim y \Leftrightarrow x' \sim y'.$$

Proof. Suppose $x \sim y$ and let $\alpha$, $\beta$ be defined as in the proof of Lemma (2.1). Using (1.1) we have

$$x = (x \vee \alpha) \wedge \alpha' = (y \vee \beta) \wedge \alpha'$$

and so

$$x' = (y' \wedge \beta') \vee \alpha.$$

Similarly

$$y' = (x' \wedge \alpha') \vee \beta.$$

Thus

$$x' \vee \beta = x' \vee (x' \wedge \alpha') \vee \beta = x' \vee y'$$

and

$$y' \vee \alpha = y' \vee (y' \wedge \beta') \vee \alpha = y' \vee x'.$$

Hence $x' \vee \beta = y' \vee \alpha$ and since $\alpha$ and $\beta$ belong to $J$ it follows that $x' \sim y'$. The reverse implication is immediate since orthocomplementation is involutive.

Proof of Proposition (2.2). Note first that (2.1) is an immediate consequence of the definitions since $x \in J$ implies that $x \sim 0$ and conversely if $x \sim 0$ then there is an element $a$ in $J$ such that $x \vee a$ belongs to $J$. Since $x \leqq x \vee a$ and $J$ is an ideal it follows that $x$ belongs to $J$.

The fact that the relation $\sim$ is reflexive and symmetric is obvious and transitivity follows from the fact that if $x \sim y \sim z$ then there are elements $a$, $b$, $c$, and $d$ in $J$ such that

$$x \vee a = y \vee b$$

and

$$y \vee c = z \vee d.$$

Thus

$$x \vee (a \vee c) = z \vee (b \vee d).$$

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Since $J$ is an ideal $a \vee c$ and $b \vee d$ each belong to $J$ and this establishes that $x \sim z$.

We prove next that the relation is a congruence. In the first place lattice joins are preserved, for suppose that $x_i \sim y_i$, $i = 1, 2$, and that

$$x_i \vee a_i = y_i \vee b_i, \qquad i = 1, 2$$

where $a_i$ and $b_i$ belong to $J$. Then $(a_1 \vee a_2)$ and $(b_1 \vee b_2)$ belong to $J$ also and since

$$(x_1 \vee x_2) \vee (a_1 \vee a_2) = (y_1 \vee y_2) \vee (b_1 \vee b_2)$$

we have

$$x_1 \vee x_2 \sim y_1 \vee y_2.$$

It follows from the preservation of lattice joins and Lemma (2.2) that lattice meets are also preserved. To exhibit this note that if $x_i \sim y_i$ then $x'_i \sim y'_i$ and so

$$(x_1 \wedge x_2)' = x'_1 \vee x'_2 \sim y'_1 \vee y'_2 = (y_1 \wedge y_2)'$$

and so

$$x_1 \wedge x_2 \sim y_1 \wedge y_2.$$

This completes the proof of Proposition (2.2).

Combining Propositions (2.1) and (2.2) we see that Theorem 1 will be established if we show that for any congruence $\theta$ on $L$ the construction of Proposition (2.2) applied to the ideal $J_\theta$ leads back to the congruence $\theta$. This fact however follows from the known result (statement (b) of section 1) that two distinct congruences cannot give the same ideal of elements congruent to zero. It follows that the congruence of Proposition (2.2) must be equivalent to the known one, namely given $J_\theta$, $x \equiv y(\theta)$ means the existence of an element $c$ in $J_\theta$ such that $(x \wedge y) \wedge c = 0$ and $(x \wedge y) \vee c = x \vee y$. We now establish this equivalence directly in

Proposition (2.3). *Under the hypothesis of proposition* (2.2) $x \sim y$ *if and only if there is an element* $c$ *in* $J$ *such that*

$$(x \wedge y) \wedge c = 0 \tag{2.2}$$

and

$$(x \wedge y) \vee c = (x \vee y). \tag{2.3}$$

Proof. If $x \sim y$ then

$$x \wedge y \sim x \wedge x = x = x \vee x \sim x \vee y.$$

Thus if

$$c = (x \vee y) \wedge (x \wedge y)'$$

then

$$c \sim (x \vee y) \wedge (x \vee y)' = 0,$$

that is $c$ belongs to $J$. With this choice of $c$ the equation (2.2) is clearly true and equation (2.3) holds because of orthomodularity in $L$ by using equation (1.2).

Conversely if equation (2.3) holds with $c$ in $J$ then

$$x \wedge y \sim x \vee y$$

and consequently

$$x = x \vee (x \wedge y) \sim x \vee (x \vee y) = x \vee y.$$

Similarly $y \sim x \vee y$ and so $x \sim y$. This completes the proof of the proposition.

Note that in Proposition (2.3) it is sufficient to require only that equation (2.3) hold with $c$ in $J$, since that equation then implies the existence of a $c_1$ in $J$ which satisfies each of the equations (2.2) and (2.3). To see this write

$$c_1 = \{(x \wedge y) \vee c\} \wedge (x \wedge y)' = (x \vee y) \wedge (x \wedge y)'$$

Then by (1.3) $c_1$ belongs to $J$, clearly

$$(x \wedge y) \wedge c_1 = 0$$

and also, by (1.2),

$$(x \wedge y) \vee c_1 = (x \wedge y) \vee c = x \vee y.$$

We state without proof the following result

PROPOSITION (2.4). *If $L$ is an orthomodular lattice and $\theta$ is a congruence relation on $L$ then the factor lattice $L/J_\theta$ is orthomodular.*

In fact if $\theta(x)$ denotes the congruence class containing element $x$ in $L$ then the mapping

$$\theta(x) \to (\theta(x))' = \theta(x')$$

is an orthocomplementation of $L/J_\theta$ and with respect to this orthocomplementation the factor lattice is easily shown to be orthomodular.

## 3. An example

In this section we give an example of an orthomodular lattice which is not a Boolean algebra and which possesses an ideal $J$ such that (1.3) holds. We prove

THEOREM 2. *Let $L$ be the lattice of closed subspaces of infinite dimensional Hilbert space $H$; then $L$ is an orthomodular lattice. Let dim $x$ be the dimension of $x \in L$ and let $K \leqq$ dim $H$ be a fixed but arbitrary infinite cardinal. Let $J(K)$ be the set of all closed subspaces in $H$ with dim $x < K$. Then $J(K)$ is an ideal of $L$ and*

(3.1) $$a \in J(K) \Rightarrow (x \vee a) \wedge x' \in J(K)$$

*for each $x$ in $L$.*

For the proofs of the results in Hilbert space theory which we use below we refer to the standard texts such as Halmos [3] and von Neumann [5]. If $M$ is a subset of $H$ we denote by $[M]$ the closed subspace generated by $M$. If $x$, $y$ are two closed subspaces of $H$ we denote their possibly non-closed vector sum by $x+y$, this sum is not in general an element of $L$. If $x$, $y$ are two closed subspaces in $H$ we denote by $P_x y$ the projection of $y$ onto $x$. Theorem 2 is a consequence of the following

LEMMA. $$[P_{x'} y] = (x \vee y) \wedge x'.$$

For the fact that $L$ is orthomodular is well-known and it is easily verified that $J(K)$ is an ideal of $L$; however using the lemma we obtain

$$\dim \{(x \vee a) \wedge x'\} = \dim [P_{x'} a] \leqq \dim a$$ [1]

from which (3.1) readily follows.

To prove the lemma we establish firstly that

(3.2) $$P_{x'} y = (x+y) \cap x'$$

where the right-hand side is the set intersection of the closed subspace $x'$ with the possibly non-closed vector sum $x+y$. To do so note that if

$$\zeta \in (x+y) \cap x'$$

then there are vectors $\xi$, $\eta$ in $x$ and $y$ respectively such that

$$\zeta = \xi+\eta = \xi+P_x\eta+P_{x'}\eta.$$

Since $\zeta$ is in $x'$ we must have

$$\xi+P_x\eta = \theta$$

where $\theta$ is the zero vector in $H$. Thus $\zeta = P_{x'}\eta$ belongs to $P_{x'}y$. Conversely if $\zeta$ does belong to $P_{x'}y$ then

$$\zeta = P_{x'}\eta$$

for some $\eta$ in $y$ and writing $\xi = -P_x\eta$ we have

$$\zeta = \xi+\eta,$$

that is $\zeta$ belongs to $(x+y) \cap x'$. This proves (3.2).

We may note in passing that the known orthomodularity of $L$ is a consequence of (3.2), for $x \perp y$ implies that $x+y$ is closed, that is,

[1] This last inequality is not immediately obvious when $\dim a$ is not finite. It may be readily proved however and we omit the details.

$$x \perp y \Rightarrow x+y = x \vee y.$$

But also $x \perp y$ implies $y \leqq x'$ so that $P_{x'}y = y$. Thus from (3.2),

$$x \perp y \Rightarrow (x \vee y) \wedge x' = y.$$

Returning to the proof of the lemma, and the case when $y$ is not necessarily orthogonal to $x$, it follows from (3.2) that

$$[P_{x'}y] \leqq (x \vee y) \wedge x'$$

since $[x+y] = x \vee y$. Thus to establish the lemma we need only to show that to each $\zeta$ in $(x \vee y) \wedge x'$ there is a sequence $\{\eta_n^*\}$ of elements of $P_{x'}y$ such that

$$\eta_n^* \rightarrow \zeta.$$

To do so note that for each vector $\zeta$ in $(x \vee y) \wedge x'$ there exist sequences $\{\xi_n\}$, $\{\eta_n\}$ of vectors in $x$ and $y$ respectively such that

$$\xi_n + \eta_n \rightarrow \zeta.$$

Write

$$\zeta_n = \xi_n + \eta_n = \xi_n + P_x \eta_n + P_{x'} \eta_n. \tag{3.3}$$

Since $\zeta$ belongs to $x'$ we have

$$(P_{x'}\eta_n, \zeta) = (\zeta_n, \zeta) \rightarrow ||\zeta||^2$$

and

$$(\zeta, P_{x'}\eta_n) = (\zeta, \zeta_n) \rightarrow ||\zeta||^2.$$

Thus

$$\limsup ||\zeta - P_{x'}\eta_n||^2 \leqq \{\limsup_{n\rightarrow\infty} ||P_{x'}\eta_n||^2\} - ||\zeta||^2. \tag{3.6}$$

But from (3.3)

$$(\zeta_n, P_{x'}\eta_n) = ||P_{x'}\eta_n||^2,$$

and so, by the Schwartz inequality, either

$$||P_{x'}\eta_n|| = 0 \quad \text{or} \quad 0 < ||P_{x'}\eta_n|| \leqq ||\zeta_n||.$$

Thus

$$\limsup_{n\rightarrow\infty} ||P_{x'}\eta_n|| \leqq ||\zeta||,$$

and hence, from (3.6)

$$\lim ||\zeta - P_{x'}\eta_n|| = 0.$$

Writing $\eta_n^* = P_{x'}\eta_n$ we obtain the desired result. This concludes the proof of the lemma.'

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