AN EMBEDDING THEOREM FOR FREE INVERSE SEMIGROUPS

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In this note it is shown that if S is a free inverse semigroup of rank at least two and if e, f are idempotents of S such that $e > f$ then S can be embedded in the partial semigroup $eSe\backslash fSf$. The proof makes use of Scheiblich's construction for free inverse semigroups [**7**, **8**] and of Reilly's characterisation of a set of free generators in an inverse semigroup [**4**, **5**].

The terminology and notation throughout is that of Howie [**2**]. For any two sets M and N we denote by $M\backslash N$ the set of all elements of M that are not in N.

Let X be a nonempty set and let G denote the free group $\mathscr{F}\mathscr{G}_X$ on X [**1**, Ch. 7]. The elements of G are the 'reduced words' in the alphabet $X \cup X^{-1}$, where X^{-1} denotes the set of formal inverses of the elements of X, and the identity of G is the empty word 1. The *length* (more precisely, the *X-length*) $l(a)$ of $a \in G$ is defined by

$$l(a) = \begin{cases} n & \text{if the reduced form of } a \text{ is } x_1x_2\ldots x_n \ (x_i \in X \cup X^{-1}), \\ 0 & \text{if } a = 1. \end{cases}$$

The cardinal number $|X|$ of X is called the *rank* of G. By [**1**, Theorem 7.3.3], two free groups are isomorphic if and only if they have the same rank.

For all $a \in G$ the set of all initial segments of a (including 1) will be denoted by $\bar{a}$ and for all nonempty subsets A of G we write $\bar{A} = \{\bar{a} : a \in A\}$. We say that A is *left closed* if and only if $A = \bar{A}$.

Scheiblich [**7**, **8**] constructs the free inverse semigroup on X as follows. Let $\mathscr{Y}$ denote the set of all finite left closed subsets of G having at least two elements and let

$$S = \{(A, g) \in \mathscr{Y} \times G : g \in A\}.$$

If $(A, g), (B, h) \in S$ then $A \cup gB \in \mathscr{Y}$, as is easily checked: consequently, a multiplication can be defined on S by the rule that

$$(A, g)(B, h) = (A \cup gB, gh).$$

With respect to this multiplication S is an inverse semigroup in which

$$(\forall (A, g) \in S) \quad (A, g)^{-1} = (g^{-1}A, g^{-1}).$$

Let $\kappa : X \to S$ denote the mapping $x \mapsto (\bar{x}, x)$. Then to each mapping θ from X into an arbitrary inverse semigroup T there corresponds a unique homomorphism $\phi : S \to T$ such that $\theta = \kappa\phi$. Thus S is the free inverse semigroup on X. It will be denoted here by $\mathscr{F}\mathscr{I}_X$.

We call $|X|$ the *rank* of S. It can be shown that two free inverse semigroups are isomorphic if and only if they have the same rank (see [**6**, Corollary 1.5]).

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The semilattice of S is the set

$$E(S)=\{(A,1): A\in\mathcal{Y}\}$$

and its partial ordering is given by the rule:

$$(\forall A, B\in\mathcal{Y})\quad (A,1)\geq(B,1)\Leftrightarrow A\subseteq B.$$

Let $(A,1)$, $(B,1)\in E(S)$. We say that $(A,1)$ *covers* $(B,1)$ if and only if $(A,1)>(B,1)$ and there is no $(C,1)\in E(S)$ such that $(A,1)>(C,1)>(B,1)$. Thus $(A,1)$ covers $(B,1)$ if and only if $A\subset B$ and $|A|+1=|B|$.

The mapping $\pi: S\to G$ defined by

$$(A,g)\pi=g$$

is evidently a surjective homomorphism (and $\pi\circ\pi^{-1}$ is the least group congruence on S [**2**, Ch. V, § 3]). Let us denote the $\mathcal{Y}$-component of a typical element a of S by $\mathcal{S}(a)$. This provides us with the following notation for the elements of S:

$$(\forall a\in S)\quad a=(\mathcal{S}(a), a\pi).$$

Let K be a nonempty subset of S. Then the inverse subsemigroup of S generated by K is the subsemigroup $\langle K\cup K^{-1}\rangle$ generated by $K\cup K^{-1}$, where K^{-1} denotes $\{k^{-1}: k\in K\}$. We say that K is *a set of free generators* of $\langle K\cup K^{-1}\rangle$ if and only if to each mapping θ from K into an arbitrary inverse semigroup T there corresponds a unique homomorphism $\phi:\langle K\cup K^{-1}\rangle\to T$ such that $\theta=\iota\phi$, where ι denotes the inclusion mapping $K\to\langle K\cup K^{-1}\rangle$ [**4, 5**]. If K is a set of free generators of $\langle K\cup K^{-1}\rangle$ then $\langle K\cup K^{-1}\rangle$ is a free inverse subsemigroup of S of rank $|K|$. Note, in particular, that $\{(\bar{x}, x): x\in X\}$ is a set of free generators of S.

Reilly has provided a useful criterion for a nonempty subset K of S to be a set of free generators of $\langle K\cup K^{-1}\rangle$ [**4**, Theorem 2.2 and remark on p. 417; **5**]. We state his result as

Lemma 1. *Let $S=\mathcal{F}\mathcal{I}_X$ and let K be a nonempty subset of S. Then K is a set of free generators of $\langle K\cup K^{-1}\rangle$ if and only if it satisfies the following two conditions*:

(i) $K\cap K^{-1}=\varnothing$,

(ii) *if $v\in K\cup K^{-1}$ is such that*

$$\mathcal{S}(v)\subseteq\bigcup_{i=1}^{n}\mathcal{S}(v_{i,1}v_{i,2}\dots v_{i,n_i}),$$

where each $v_{i,j}\in K\cup K^{-1}$ and $v_{i,j}^{-1}\neq v_{i,j+1}$ $(j=1,2,\dots,n_i-1;\ i=1,2,\dots,n)$, then $v=v_{i,1}$ for some i.

Our main theorem follows below.

Theorem 1. *Let $S=\mathcal{F}\mathcal{I}_X$, where $|X|\geq 2$, and let $e, f\in E(S)$ be such that $e>f$. Then S can be embedded in the partial semigroup $eSe\backslash fSf$.*

Proof. Choose $g\in E(S)$ such that $e>g\geq f$ and e covers g. Since $gSg\supseteq fSf$ it is enough to prove that S can be embedded in the partial semigroup $eSe\backslash gSg$.

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Let $e=(A,1)$. Then $g=(A\cup\{ua\},1)$ for some $u\in A$ and $a\in X\cup X^{-1}$, where the final letter in the reduced form of u is not a^{-1} and $ua\notin A$. Write

$$r=\max\{l(w)+1:w\in A\},$$

choose $b\in X\backslash\{a,a^{-1}\}$ (this is possible since $|X|\geq 2$) and define a subset Y of G $(=\mathcal{FG}_X)$ by

$$Y=\{b^{-r}xb^{r}:x\in X\backslash\{b\}\}\cup\{a^{-r}ba^{r}\}.$$

Every element of Y has length $2r+1$ (being reduced as written). Further, $Y\cap Y^{-1}=\varnothing$, since $X\cap X^{-1}=\varnothing$. Each element of $Y\cup Y^{-1}$ is fully determined by its initial segment of length $r+1$, and if $p,q\in Y\cup Y^{-1}$ are such that $p\neq q^{-1}$ then the cancellation in the product pq does not reach the central factor of either p or q. Thus the elements of Y have central significant factors.

For all $y\in Y\cup Y^{-1}$, write

$$B_y=A\cup\bar{y}\cup yA.$$

Thus, for all $y\in Y\cup Y^{-1}$,

$$(B_y,y)=(A,1)(\bar{y},y)(A,1)\in eSe$$

and

$$(B_y,y)^{-1}=(B_{y^{-1}},y^{-1}).$$

Let K denote $\{(B_y,y):y\in Y\}$ and write $T=\langle K\cup K^{-1}\rangle$. Evidently $T\subseteq eSe$ and $|K|=|Y|=|X|$. To establish the theorem it suffices to show that K is a set of free generators of T and that $T\cap gSg=\varnothing$.

Since $Y\cap Y^{-1}=\varnothing$ it follows that $K\cap K^{-1}=\varnothing$; that is, K satisfies condition (i) of Lemma 1. We prove next that K also satisfies condition (ii). Let $v\in K\cup K^{-1}$ be such that

$$\mathcal{S}(v)\subseteq\bigcup_{i=1}^{n}\mathcal{S}(v_{i,1}v_{i,2}\ldots v_{i,n_i}),$$

where each $v_{i,j}\in K\cup K^{-1}$ and $v_{i,j}^{-1}\neq v_{i,j+1}$ $(j=1,2,\ldots,n_i-1;\ i=1,2,\ldots,n)$. Now $v\pi\in\mathcal{S}(v)$. Hence there exists $k\in\{1,2,\ldots,n\}$ such that

$$v\pi\in\mathcal{S}(v_{k,1}v_{k,2}\ldots v_{k,n_k}).$$

For $j=1,2,\ldots,n_k$ let us write $y_j=v_{k,j}\pi$ and $B_j=B_{y_j}$: thus

$$v_{k,j}=(B_j,y_j)\quad(j=1,2,\ldots,n_k).$$

Hence

$$v\pi\in B_1\cup y_1B_2\cup(y_1y_2)B_3\cup\ldots\cup(y_1y_2\ldots y_{n_k-1})B_{n_k}.$$

Suppose that $v\pi\notin B_1$. Then there exists $t\in\{1,2,\ldots,n_k-1\}$ such that $v\pi\in(y_1y_2\ldots y_t)B_{t+1}$ and so

$$y_t^{-1}y_{t-1}^{-1}\ldots y_1^{-1}(v\pi)\in B_{t+1}.$$

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Now $y_j^{-1} \neq y_{j+1}$ $(j = 1, 2, \ldots, t-1)$ (for otherwise $v_{k,j}^{-1} = v_{k,j+1}$ for some j, which is a contradiction); also $y_1 \neq v\pi$, since $y_1 \in B_1$ while $v\pi \notin B_1$. Hence the initial segment of $y_t^{-1}y_{t-1}^{-1} \ldots y_1^{-1}(v\pi)$ of length $r+1$ is just the initial segment of y_t^{-1} of length $r+1$. Thus the initial segment of y_t^{-1} of length $r+1$ lies in B_{t+1}, since B_{t+1} is left closed.

Now $B_{t+1} = A \cup \bar{y}_{t+1} \cup y_{t+1}A$. Since no element of A has length exceeding $r-1$, the only element of B_{t+1} of length precisely $r+1$ is the initial segment of y_{t+1} of this length. Hence y_t^{-1} and y_{t+1} have the same initial segment of length $r+1$ and so $y_t^{-1} = y_{t+1}$, which is a contradiction. Consequently $v\pi \in B_1$.

Since $v\pi \in Y \cup Y^{-1}$ and $B_1 = A \cup \bar{y}_1 \cup y_1 A$, a similar argument to that above then shows that $v\pi = y_1$. Hence $v = (B_1, y_1) = v_{k,1}$. We have thus proved that K satisfies condition (ii) of Lemma 1. Thus T is free of rank $|K|$ and so $S \cong T$.

Finally, we show that $T \cap gSg = \varnothing$. Suppose that the conclusion is false and hence that there exists $(C, h) \in T \cap gSg$. As remarked earlier, $g = (D, 1)$, where $D = A \cup \{ua\}$. Since $(C, h) \in gSg$,

$$(C, h) = g(C, h)g = (D \cup C \cup hD, h).$$

Thus $C = (D \cup hD) \cup C$ and so $D \cup hD \subseteq C$. In particular, $ua \in C$.

Since $(C, h) \in T$, there exist $y_1, y_2, \ldots, y_n \in Y \cup Y^{-1}$ (not to be confused with the y_i in the first part of the proof) such that

$$(C, h) = (B_1, y_1)(B_2, y_2) \ldots (B_n, y_n),$$

where, for each i, we have written B_i for B_{y_i}. But $ua \in C$. Hence

$$ua \in B_1 \cup y_1 B_2 \cup (y_1 y_2) B_3 \cup \ldots \cup (y_1 y_2 \ldots y_{n-1}) B_n.$$

Suppose first that $ua \in B_1 = A \cup \bar{y}_1 \cup y_1 A$. Since $ua \notin A$ we have that $ua \in \bar{y}_1 \cup y_1 A$. But since $l(ua) \leq r$, this implies that ua is an initial segment of y_1 and hence is of the form a^{-s} or b^{-s} for some positive integer s. Thus $a = a^{-1}$ or $a = b^{-1}$ and in either case we have a contradiction.

Similarly, if $ua \in (y_1 y_2 \ldots y_t) B_{t+1}$ and $y_1 y_2 \ldots y_t = 1$ for some $t \in \{1, 2, \ldots, n-1\}$, we again obtain a contradiction. Thus, for some $t \in \{1, 2, \ldots, n-1\}$, $ua \in (y_1 y_2 \ldots y_t) B_{t+1}$, where $y_1 y_2 \ldots y_t \neq 1$. Now we can write $y_1 y_2 \ldots y_t$ as a reduced word in $Y \cup Y^{-1}$: thus

$$y_1 y_2 \ldots y_t = z_1 z_2 \ldots z_k,$$

where each $z_i \in Y \cup Y^{-1}$ and $z_i^{-1} \neq z_{i+1}$ $(i = 1, 2, \ldots, k-1)$. Hence

$$ua \in z_1 z_2 \ldots z_k B_{t+1}.$$

Let us write

$$w = z_k^{-1} z_{k-1}^{-1} \ldots z_1^{-1} ua. \tag{1}$$

Then $w \in B_{t+1}$. Now cancellation on the right-hand side of (1) cannot reach the central factor of any z_i^{-1}, since $z_i^{-1} \neq z_{i+1}$ $(i = 1, 2, \ldots, k-1)$ and $l(u) < r$. Also it cannot reach the final letter a since $l(u) < r$ and the final segment of z_1^{-1} of length r is either a^r or b^r.

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Hence

$$l(w) \geq r+3 \tag{2}$$

and the last letter of the reduced form of w is a.

Since $w \in B_{t+1} = A \cup \bar{y}_{t+1} \cup y_{t+1}A$ and no word in A has length exceeding $r-1$, (2) shows that

$$w \in \bar{y}_{t+1} \cup y_{t+1}A. \tag{3}$$

Hence the initial segments of w and y_{t+1} of length $r+2$ must coincide; also

$$l(w) \leq (2r+1)+(r-1) = 3r. \tag{4}$$

Suppose that $k>1$. If there exist i, j such that $z_i^{-1} = a^{-r}ba^r$ and $z_j^{-1} = b^{-r}xb^r$ for some $x \in (X \cup X^{-1}) \backslash \{b, b^{-1}\}$ then

$$l(w) \geq (r+1)+2r = 3r+1,$$

which contradicts (4). Thus either

$$z_i^{-1} = b^{-r}x_ib^r \quad \text{for some} \quad x_i \in (X \cup X^{-1}) \backslash \{b, b^{-1}\} \quad (i=1,2,\ldots,k)$$

or

$$z_i^{-1} = a^{-r}ca^r \quad (i=1,2,\ldots,k), \quad \text{where} \quad c \in \{b, b^{-1}\}.$$

In the first case

$$w = b^{-r}x_1x_2\ldots x_kb^rua,$$

where $x_i^{-1} \neq x_{i+1}$ $(i=1,2,\ldots,k-1)$, and so the initial segment of w of length $r+2$ is $b^{-r}x_1x_2$. But the initial segments of w and y_{t+1} of length $r+2$ are the same. This gives a contradiction since $x_2 \neq b$. In the second case, $w = a^{-r}c^ka^rua$ and in the same way we again get a contradiction. Thus $k=1$.

It follows that the initial segments of y_{t+1} and z_1^{-1} of length $r+1$ coincide. Hence $y_{t+1} = z_1^{-1}$ and so

$$w = y_{t+1}ua. \tag{5}$$

Suppose that $w \in y_{t+1}A$. Then, from (5), $ua \in A$, which is false. Hence, from (3), $w \in \bar{y}_{t+1}$. Thus, since the last letter of w (in reduced form) is a, we see from (2) that $w = a^{-r}ca^s$ and $y_{t+1} = a^{-r}ca^r$, where $c \in \{b, b^{-1}\}$ and $2 \leq s \leq r$. Hence, from (5),

$$a^{-r}ca^s = a^{-r}ca^rua$$

and so $u = a^{-(r+1-s)}$. Since $s \leq r$ this contradicts the fact that the last letter in the reduced form of u is not a^{-1}.

Consequently $T \cap gSg = \varnothing$ and the proof is complete.

Remark 1. The result fails for the case $|X|=1$: for if $S = \mathcal{F}\mathcal{I}_X$, where $|X|=1$, and $e, f \in E(S)$ are such that e covers f then the set of idempotents in $eSe \backslash fSf$ is totally ordered, whereas $E(S)$ is not totally ordered.

REMARK 2. Let S be a free inverse semigroup of rank 1 and let $e \in E(S)$. Then $eSe \not\subseteq E(S)$. Choose a non-idempotent element $a \in eSe$. By [**4**, Corollary 2.5], $\langle \{a, a^{-1}\} \rangle$ is a free inverse subsemigroup of S of rank 1. This shows that S can be embedded in eSe.

REMARK 3. From Theorem 1 and Remark 2 we see that if M is an ideal of a free inverse semigroup S of arbitrary rank then S can be embedded in M.

To conclude, we extend Theorem 1 with the aid of a result of O'Carroll's [**3**, p. 19], which we state below as Lemma 2. Recall that, by Schreier's theorem [**1**, Theorem 7.2.1], every subgroup of a free group is itself free.

LEMMA 2. *Let $S = \mathcal{FI}_X$ and $G = \mathcal{FG}_X$, where X is a nonempty set. Let H be a non-trivial subgroup of G. Then there exists a free inverse subsemigroup T of S such that rank of T = rank of H.*

Now let c and d be cardinal numbers such that $c \geq 2$ and $0 < d \leq \max\{\aleph_0, c\}$. It can be shown that every free group of rank c contains a (free) subgroup of rank d. Hence, by Lemma 2, every free inverse semigroup of rank c contains a free inverse subsemigroup of rank d. From Theorem 1 we then immediately obtain

THEOREM 2. *Let S be a free inverse semigroup of rank c, where $c \geq 2$, and let d be any cardinal number such that $0 < d \leq \max\{\aleph_0, c\}$. Let $e, f \in E(S)$ be such that $e > f$. Then the partial semigroup $eSe \backslash fSf$ contains a free inverse subsemigroup of S of rank d.*

In particular, if S is a free inverse semigroup of rank 2 and $e, f \in E(S)$ are such that $e > f$ then $eSe \backslash fSf$ contains a free inverse subsemigroup of S of rank $\aleph_0$. This strengthens [**4**, Corollary 2.7].

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