On the Turbulence in the Protoplanetary Cloud

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THE problem of turbulence in the protoplanetary cloud is of importance for planetary cosmogony. Chaotic macroscopic motions probably existed in the cloud during its formation. Further evolution of the cloud depended to a great extent upon whether these original motions damped in a short time, or turbulence supported by some source of energy existed during planet formation. According to Kuiper and Fessenkov's hypotheses, massive protoplanets formed as a result of gravitational instability and turned into planets after the dissipation of light elements. Large-scale turbulent motions with mean velocities exceeding the thermal velocities of atoms and molecules would prevent, however, gravitational instability in the cloud, even if its mass was of the order of the mass of the sun. According to Edgeworth and to Gurevitch and Lebedinsky the planets grew gradually from small condensations formed in a flattened dust disk with a mass equal to that of the present planetary system. But even small scale turbulent motions would prevent extreme flattening of the disk necessary in this case for gravitational instability. The problem of turbulence is also connected with the problem of present distribution of angular momentum between the sun and planets, as large-scale turbulence produces redistribution of matter and of angular momentum in the cloud.

The hypothesis of the presence of large-scale turbulence in the protoplanetary cloud was introduced by von Weizsäcker.[1] But Weizsäcker's arguments do not prove its existence. The Reynolds number is very large for the cloud (about 10^{10}). But for a rotating medium the Reynolds number cannot be considered as a criterion of turbulence. Weizsäcker regards turbulence as a result of convective instability. But he uses the criterion of convection for nonrotating media, which is inapplicable in the case of the rotating cloud. The problem therefore needs further study.

In order to reveal the main features of motions in a flat protoplanetary cloud, one can use the results of investigations of fluid motion between two rotating coaxial cylinders. Rayleigh,[2] Taylor,[3] and Synge[4] proved that such a motion of an incompressible fluid is stable if the angular momentum increases outwards: $d(\omega r^2)/dr>0$. This condition had to be satisfied for the protoplanetary cloud. If we neglect the pressure gradient in the cloud and its own gravitation as compared with the gravitation of the sun, the angular momentum will be proportional to $\sqrt{r}$. Then this condition becomes identical with the condition of stability of circular orbits well known in stellar dynamics. But the condition was obtained for an incompressible fluid and does not take into account the possibility of convection. On the other hand, Weizsäcker, using the criterion of convection, left out of account the condition of stability of circular orbits. These two conditions were combined in the paper by Safronov and E. L. Rouscol.[5] The condition of convection for a flat rotating cloud (cylindrical rotation) was found to be:

$$\omega r^2\frac{d(\omega r^2)}{dr}<\frac{r^3}{2\rho^2}\frac{dp}{dr}\left[\left(\frac{d\rho}{dr}\right)_{ad}-\frac{d\rho}{dr}\right]$$
$$=\frac{r^3}{2\gamma\rho T}\frac{dp}{dr}\left[\frac{dT}{dr}-(\gamma-1)\frac{T}{\rho}\frac{d\rho}{dr}\right], \quad (1)$$

and $(\omega r^2)^2=GMr+(r^3dp/\rho dr)$. When considering small disturbances, it is possible to approximate smooth functions ρ and T for small intervals of r by power functions

$$\rho\sim r^{-a_1}, \quad T\sim r^{-a_2}. \quad (2)$$

The condition of convection is then reduced to

$$2-(\gamma-1/\gamma)(a_1+a_2)>GM/rRT. \quad (3)$$

The protoplanetary cloud being largely an HI region, one can take as maximum value of T on the right-hand side of the inequality (3) the temperature of a blackbody in a transparent cloud, $T_0\approx 300(r_{ae})^{-\frac{1}{2}}$, where r_{ae} is the distance from the sun in a.u. Then

$$(GM/RTr)>350r_{ae}^{-\frac{1}{2}}, \quad (4)$$

and the condition of convection (3) is not satisfied for any acceptable values of a_1 and a_2. Hence, the undisturbed protoplanetary cloud is stable with respect to small disturbances and convection could not arise in it at any admissible values of temperature and of density gradients.

The possibility of large-scale turbulence over a long time scale is open to serious objections from energetic considerations. Solar radiation entering the flat cloud is insufficient to support turbulence. Gravitational energy of the parts of the cloud approaching the sun suffices only for a short time. Weizsäcker's value of the mean turbulent velocity of about one tenth of the orbital velocity leads to a time of disintegration of the cloud of about 10^3 years, while 10^8 years are needed for the planet formation according to Weizsäcker himself.

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[5] V. S. Safronov and E. L. Rouscol, Compt. rend. acad. sci. U.R.S.S. **108**, 413 (1956); Problems Cosmogony (Moscow) **5**, 22 (1957).

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It seems probable that the ratio of the mean turbulent velocity to the orbital velocity, and the ratio of the mixing length to the distance from the sun, are of the same order of magnitude. Chandrasekhar and ter Haar[6] have obtained $l=0.62r$ from the law of planetary distances and take the value of the turbulent velocity to be slightly higher than one-half of the orbital velocity. Karman's formula[7] for the mixing length in a rotating medium leads to a still higher value, namely, $l=2kr\approx 0.8r$. Under these conditions the time of disintegration of the cloud is less than 10^2 years and formation of the planets is impossible. Large-scale turbulent motions, if such existed at the initial stage of the evolution, had to damp rapidly. According to the energetic considerations only motions of a scale a thousand times less than follows from Karman's formula could exist for a long time.

It is of interest to investigate the problem of the transfer of matter and angular momentum during the existence of turbulence in the cloud. According to Weizsäcker, turbulent friction diminished the angular momentum of the rapidly rotating inner parts of the cloud, which therefore approached the sun. The outer parts acquired the momentum and went away from the sun. Weizsäcker uses shearing stresses depending on the gradient of angular velocity:

$$\tau_{r\varphi}'=\eta r(d\omega/dr). \tag{5}$$

But this tensor of molecular viscosity stresses is valid, strictly speaking, only for the case of small free paths and is unfit for large-scale turbulent motions. Prandtl found another expression for the stresses as a function of the gradient of angular momentum:

$$\tau_{r\varphi}'=\eta r^{-1}(d/dr)(\omega r^2). \tag{6}$$

Karman[7] gives the same expression (6) without any comment on Weizsäcker's using expression (5). In the solar system, angular velocity decreases with the distance from the sun, while the angular momentum increases. Hence the direction of the transfer of matter and angular momentum in the cloud according to Prandtl's and Weizsäcker's formulas are opposite.

Taylor[8] believes that the steady value of angular momentum in the central region of turbulent flow (inner cylinder rotating) found experimentally by him and Wattendorf,[9] contradicts Prandtl's formula, as the latter in this case gives zero shearing stresses and would make impossible the transport of angular momentum. However, the equilization of angular momentum in the main part of the flow agrees with Prandtl's expression. The accuracy of the experiment is not sufficient to state that the derivative of angular momentum is exactly zero. We can only say that the derivative is very small, but this conclusion follows just from Prandtl's formula, if the turbulent viscosity is great. The same takes place in the rectilinear flow in tubes. The almost flat velocity profile far from the walls of the tube, and its sharp bending near the walls, can be explained if we suppose that turbulent viscosity is high far from the walls and decreases rapidly when approaching the walls (as the first or the second power of the distance from the walls, for example). A similar suggestion about turbulent viscosity in a rotating flow permits one to explain, by using Prandtl's formula, the almost constant value of angular momentum far from the walls and its sharp fall near the walls. Neither the relation (5) resulting from the molecular viscosity tensor, nor Taylor's suggestion of vorticity conservation explains this peculiarity of turbulent rotational motion.

Probably Prandtl's formula is not quite accurate, because of the semiempirical character of turbulence theory. On the ground of a new interpretation of the mixing length, Wasiutynsky[10] has obtained an expression for stresses in a more general form. For the case of cylindrical rotation, he gives

$$\tau_{r\varphi}'=\frac{\rho K_r{}^r}{r}\frac{d(\omega r^2)}{dr}-2\rho K_\varphi{}^\varphi\omega. \tag{7}$$

When $K_\phi{}^\phi=0$ (purely radial exchange) one obtains Prandtl's formula; with $K_\phi{}^\phi=K_r{}^r$ (isotropy) one obtains a formula of molecular viscous stresses with the exception that turbulent viscosity enters instead of the molecular. He found as condition of nondecreasing turbulence for incompressible ideal fluid

$$\left[2K_\varphi{}^\varphi\omega r-K_r{}^r\frac{d(\omega r^2)}{dr}\right]\frac{d\omega}{dr}\leqslant 0. \tag{8}$$

It is not clear whether this generalization is only formal, or characterizes the turbulent motion more exactly. Nor is it clear which values of the ratio $K_\phi{}^\phi/K_r{}^r$ are more probable in the actual turbulent flow. One may think that for the rotating system around the gravitating center $K_\varphi{}^\varphi<K_r{}^r$. It is well known, for example, that pecular velocities of stars in radial direction are higher than in the direction of rotation. For the solar system ($\omega\sim r^{-\frac{3}{2}}$) the turbulence would decrease according to this formula, if $K_\phi{}^\phi<\frac{1}{4}K_r{}^r$. The sign of the stresses is then given by Prandtl's formula and the transfer of matter and of angular momentum is opposite to that found by Weizsäcker. According to energetic considerations it seems probable that this situation occurred for large-scale turbulence. It might be believed that small-scale turbulence would be more isotropic. But small-scale turbulence would be inconsistent with the theoretical value of mixing length found by Karman for a rotating system. It is not clear whether such turbulent motions are possible.

Being only an astronomer the author should like to know the opinions of specialists on turbulence about these questions.

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