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CLASS TWO NILPOTENT CAPABLE GROUPS

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We give a bound for the number of generators for groups of exponent p depending on the rank of the centre when centre and commutator subgroup coincide, provided that the group is isomorphic to the quotient group modulo the centre of some group.

1. INTRODUCTION

Let p be an odd prime and G be a p-group. Beyl and Tappe [**1**, p.142] called G *capable* if and only if there is a group H such that $G = H/Z(H)$. It has been known for a long time [**1**, Corollary 4.16, p.223] that the only capable extra-special p-groups are those of order p^3, and it was proved by the first named author in [**2**] by considering pairs of skew symmetric bilinear forms, that capable p-groups G, satisfying $G' = Z(G)$, with $Z(G)$ elementary Abelian of rank 2, have order at most p^7. The main part of this note is devoted to the proof of a generalisation of this statement. The question we consider is what could be said about the rank of G/G' if $G' = Z(G)$ is of given rank k and $exp(G) = p$. We found an upper bound of that rank as a function of k, namely $2k + \binom{k}{2}$. The bound of that rank is best possible, for this see the example in [**2**, p.248].

We shall begin with the case $G = H/Z(H)$, where $Z(H)$ is cyclic (Lemma) before we use this as the initial step of an induction proof (Theorem). Finally, we shall prove (Proposition), that certain central products are not capable (using a more general concept of central product, compared to that of Beyl, Tappe [**1**, p.222]). This result can be considered as a generalisation of [**1**, Theorem 4.15, p.221] with the weaker property of being non-capable, instead of unicentral.

2.

In what follows we shall consider p to be an odd prime.

LEMMA 1. *Assume that $G = H/Z(H)$, where $Z(H)$ is cyclic, G is of exponent p, and $G' = Z(G)$. If $Z(G)$ is of rank k, then the rank of G/G' is at most $2k + \binom{k}{2}$.*

PROOF: By construction, $G' = H'Z(H)/Z(H)$ and so $H'Z(H) = Z_2(H)$. We deduce that H' is Abelian. H is nilpotent of class 3 and $H/Z(H)$ is of exponent p. This implies that H_3 is of exponent p and, since $p \neq 2$, H' is of exponent p.

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The conditions of the Lemma imply that H' is of rank $k+1$ and H_3 is cyclic. G induces by conjugation in H' a group of automorphisms which fixes H_3 and the quotient group H'/H_3 elementwise. This group of automorphisms is induced precisely by $H/C(H')$ and is elementary Abelian of rank k. Choose $a_1, a_2, \ldots, a_k$ such that

$$H = \langle a_1, a_2, \ldots, a_k, C(H') \rangle ,$$

and put

$$T = \langle a_1, a_2, \ldots, a_k, H' \rangle .$$

So, $TC(H') = H$ and $C(T) = C(TZ(H)) = C(TH'Z(H)) = C(TC(H')) = Z(H)$.

Now, every element $u \in C(H') \backslash (T \cap C(H'))$ operates by conjugation on the elements a_i such that

$$u^{-1} a_i u = a_i b_i c_i,$$

where $c_i \in H_3$ and b_i is a representative of a coset $b_i H_3$ in H'.

Since $u \in C(H')$, we obtain

$$u^{-1}[a_i, a_j]u = [a_i, a_j]$$

and therefore $[a_i, a_j] = [a_i b_i, a_j b_j] = [a_i, a_j][a_i, b_j][b_i, a_j]$, since b_i, b_j are elements of H' and H is nilpotent of class 3.

We deduce

$$[a_i, b_j] = [a_j, b_i] \quad \text{for all } i, j.$$

We want to find the maximum of possible choices for $b_1, b_2, \ldots, b_k$. Assume that we have fixed b_1, then we find for b_2 the condition $[a_1, b_2] = [a_2, b_1]$, so $b_2 \in H'$ is fixed modulo $C(a_1)$. For b_1 we had at most p^k choices, but for b_2 only p^{k-1} choices at most. Assume that $b_1, b_2, \ldots, b_{i-1}$ are chosen, then for b_i we have the restrictions

$$\begin{gathered} [a_i, b_1] = [a_1, b_i] \\ [a_i, b_2] = [a_2, b_i] \\ \ldots \\ [a_i, b_{i-1}] = [a_{i-1}, b_i], \end{gathered}$$

so b_i is fixed in H' modulo $C(\langle a_1, a_2, \ldots, a_{i-1} \rangle)$, giving p^{k-i+1} many choices at most.

We deduce by induction on k that there are at most p^m choices of collections of $(b_1, b_2, \ldots, b_k)$, where $m = k + \binom{k}{2}$.

By multiplication of u with a suitable element of H' it is always possible to have $c_i = 1$ for all i; in particular, it is impossible by construction to have $b_i = 1$ for all i.

So, $|C(H')\backslash T \cap C(H')| \leqslant p^m$, and $T \cap C(H') = H'$. Now

$$|H/Z_2(H)| \cdot |TC(H')/C(H')| \cdot |C(H')/Z_2(H)| = |TC(H')/C(H')| \cdot |C(H')/Z(H)H'| \leqslant p^k p^m.$$

and the rank of $H/Z_2(H) = G/G'$ is at most $2k + \binom{k}{2}$. □

REMARK. The bound for the rank is best possible, for this see the example in [**2**, p.248].

3.

THEOREM 1. *Assume that $G = H/Z(H)$, $G^P = 1$, and $Z(G) = G'$. If $Z(G)$ is of rank k, then the rank of G/G' is at most $2k + \binom{k}{2}$.*

PROOF: In the first step we shall reduce the rank of $Z(H)$ in the following manner: We assume that such a group H exists and the rank of $Z(H)$ is bigger than k. Then, there is also a group $H^*/Z(H^*) = G$, generated by elements of order p, such that $r(Z(H)) \leqslant k$. Later on we shall argue by induction on $r(Z(H))$.

In any case, we have as in the Lemma, that H' is of exponent p and Abelian. So, without loss of generality, we may assume, that $Z(H)$ is a p-group.

Assume now, that K is a subgroup of order p in $Z(H)$ and $Z(H/K) \neq Z(H)/K$. Then, H posesses an element $y \in H'\backslash H_3$, such that $\langle [y, H] \rangle = K$.

By counting the cyclic subgroups of H'/H_3 and of H_3 we obtain that for $r(Z(H)) \geqslant r(H'/H_3) = k$ there is a subgroup K of $Z(H)$, such that $Z(H/K) = Z(H)/K$ and so $G = (H/K)/(Z(H)/K) = (H/K)/Z(H/K)$. We assume now, that H is chosen the smallest possible. The preceeding argument shows that $r(Z(H)) \leqslant r(H'/H_3)$.

Choose a basis $h_1, h_2, \ldots, h_l$ of H and assume that some h_i is not of order p. We shall show that there is a group H^* with $H^*/Z(H^*) = G$, generated by h_j^*, and h_i^* is of order p, while all other generators have the same order as in H. For this we take a cyclic group $\langle z \rangle$, having the same order as $\langle h_i \rangle$ and form $H \times \langle z \rangle / \langle h_j^p z^p \rangle$. In the quotient group take the subgroup H^*, generated by $h_1 \langle h_i^p z^p \rangle, h_2 \langle h_i^p z^p \rangle, \ldots, h_i z \langle h_i^p z^p \rangle, h_{i+1} \langle h_i^p z^p \rangle, \ldots$. In this group H only the i-th generator has changed its order, which has become p; the new H^* still satisfies the conditions.

We summarise: If there is a group H as in the Theorem, then we may assume, without loss of generality, that H is generated by a basis consisting of elements of order p and $Z(H) = H_3$ is elementary Abelian of rank not exceeding k.

We shall treat now the case $p \geqslant 5$, since in this case H is of exponent p. Later on we shall point out how to modify the arguments for the case $p = 3$.

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We proceed by induction on the rank of $Z(H)$. If $Z(H)$ is cyclic, the Theorem is true by the Lemma.

Assume that the theorem has been proved for all ranks of $Z(G)$, whenever the rank of $Z(H)$ is smaller than a fixed number t. We want to show that it is also true if $r(Z(H)) = t$. In the course of making the order of H as small as possible, we have already found that we may reduce to the case where $Z(H/K) \neq Z(H)/K$ for all proper subgroups K of $Z(H)$. Among the cyclic subgroups of $Z(H)$ we choose one, so that $|Z(H/K)|$ is smallest possible, and put $p^w = |(Z(H/K))/(Z(H)/K)|$. Choose a normal subgroup L of H which is maximal with respect to the property $L \cap Z(H) = K$.

If x is an element of H such that $[x, H] \subseteq L$, then $x \in L$, since L is maximal and H is of exponent p.

Now H/L satisfies all conditions of the Theorem for a centre of rank $\leqslant t-1$ and $Z((H/L)/Z(H/L))$ of rank $k-w$.

By the induction hypothesis we may apply the Theorem for H/L and we obtain that $H/H'L$ is of rank $2(k-w) + \binom{k-w}{2}$ at most.

We consider now L and choose a cyclic subgroup K^* different from K in $Z(H)$. By choice of K we know that $Z(H/K^*)/(Z(H)/K^*)$ is at least p^w.

If y is such that $yK^* \in Z(H/K^*)$ and $x \in L$, then $[x,y] \in L \cap K^* = 1$. This shows that $|L/L \cap C(H')| \leqslant p^{k-w}$.

Let $u_1 L \cap C(H'), \dots, u_s L \cap C(H')$ be a basis of $L/L \cap C(H')$. We have

$$g^{-1} u_i g = u_i v_i w_i \text{ with } w_i \in K \text{ and } v_i \in L \cap H'$$

As before in the Lemma we obtain that we may disregard the elements w_i, and

$$[u_i, v_j] = [u_j, v_i] \ \ \forall i, j.$$

Now $[u_i, v_j] = 1$ whenever $j > w$, and so, for $j > w$,

$$[u_1, v_j] = [u_2, v_j] = \cdots = [u_w, v_j] = 1$$

and $v_j = 1$.

For the remaining elements v_j we proceed as in the Lemma and we find analogously, that the number of possible choices for the collection $(v_1, v_2, \dots, v_w)$ is at most $w + \binom{w}{2}$.

If we have an element g, such that all $v_i = 1$, then $g \in C\langle u_1, \dots, u_s, H'\rangle \cap L$.

If $g \notin Z_2(H)$, there is h in H, such that $[g,h] \notin Z(H)$ and $[[g,h],z] \neq 1$ for some z in H. In particular, there is some $u_i, i \leqslant w$, such that $[[g,h],u_i] \neq 1$, since $[g,h]$ is contained in L.

But, $1 = [[g,h],u_i][[h,u_i],g][[u_i,g],h]$ and $[[h,u_i],g] = [u_i,g] = 1$, a contradiction. Now $g \in Z_2(H) = H'$ and

$$C\langle u_1, \dots, u_s, H'\rangle \cap L \subseteq H' \cap L.$$

By combining the last two steps of argument, we have

$$|L/L \cap H'| \leqslant p^{k-w+w+\binom{w}{2}}.$$

Now $|H/H'| = |H/H'L|\,|LH'/H'| = |H/H'L|\,|L/L \cap H'| \leqslant p^q$, where

$$\begin{aligned} q &= 2(k-w) + \binom{k-w}{2} + k - w + w + \binom{w}{2} \\ &= 3k - 2w + \binom{k}{2} - w(k-w) \\ &= 2k + \binom{k}{2} - ((w-1)(k-w+1)+1) \\ &< 2k + \binom{k}{2}, \end{aligned}$$

since $k > w \geqslant 1$.

The Theorem is shown for $p \geqslant 5$.

For $p = 3$, the statement "If x is an element of H, such that $[x, H] \subseteq L$, then $x \in LZ(H)$" need not be true, since x^3 may be different from 1 and not contained in L. In this case we construct a basis of H which contains x and use the modification, indicated before, again to obtain a new group H^* and a generator x^* of order 3. Now, for all u in the new normal subgroup L, $(ux^*)^3$ is contained in L and here the statement can be proved by successive modification of H. The remainder is unchanged. $\square$

4.

DEFINITION 1: The group G is a central product of A and B, if $G = AB$, where A and B are normal subgroups of G and $A \subseteq C(B)$.

PROPOSITION 1. *Assume that G is a central product of the two p-groups A and B and that $A' \cap B' \neq 1$. Then G is not capable.*

PROOF: Choose an element $x \neq 1$ contained in $A' \cap B'$. Since G is a central product of A and B, this element is contained in $Z(G)$. If G is capable, there is a group H, such that $G = H/Z(H)$. Let x^* be a pre-image of x with respect to the epimorphism from G to H. There is an element y of H, such that $[x^*, y] \neq 1$, since x^* is contained in $Z_2(H)$ but not in $Z(H)$. This element y can be chosen to be contained in the pre-images of either A, or B; assume that it is contained in A^*. Since x was contained in B', we can describe x^* as a product w of commutators of elements in B', multiplied by an element c of $Z(H)$. So,

$$w = \prod_i [u_i, v_i] \text{ with all } u_i, v_i \text{ in } B^*$$

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and

$$1 \neq [x^*, y] = [w, y] = \left[\left(\prod_i [u_i, v_i]\right), y\right].$$

Using formula b of [**3**, Hilfssatz 1.2, p.253] we find that $[x^*, y]$ is a product of conjugates of commutators $[[u_i, v_i], y]$, and applying the Witt-Hall identity for u_i, v_i^{-1} and y (see [**3**, Satz 1.4, p.254] we obtain:

$$1 = [[u_i, v_i], y]^{v_i^{-1}} [[v_i^{-1}, y^{-1}], u_i]^y [[y, u_i^{-1}], v_i^{-1}]^{u_i} = 1.$$

Since $[B^*, y] \subseteq Z(H)$ by construction, we find that the second and third commutators are trivial and so also $[[u_i, v_i], y] = 1$ and finally, $[x^*, y] = 1$ regardless of the special choice of H. This is a contradiction, and the Proposition is shown. $\square$

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