8

Epilogue: A Survey of Some Techniques

This last chapter is a short epilogue to our ruminations. Having discussed many variations on the Borel conjecture and hopefully gained some appreciation for the problem, we now briefly discuss some of the very significant attempts that have been made to prove both it and its noncommutative geometric cousin, the Baum–Connes conjecture. To do an adequate job would take two or three more volumes of the length of this one[1] and clearly (indeed tautologously) that cannot be done here. Instead, we will give a breezy overview of some milestones, giving detail precisely for the parts that are easiest or that connect directly to earlier discussions.

The first four sections of the chapter will be devoted to the Borel and Farrell–Jones conjectures and we will then turn to the Baum–Connes conjecture.

8.1 Codimension-1 Methods

The first results on the Borel conjecture grew out of the study of codimension-1 submanifolds of homotopy-equivalent manifolds. We first discussed this idea in the setting of the Novikov conjecture in §4.4 (splitting theorems).

Geometrically, these ideas arose first for 3-manifolds in the classical theory of Haken manifolds, i.e. of irreducible 3-manifolds that contain an incompressible surface. A major result of that theory is that a connected irreducible[2] 3-manifold has a hierarchy iff it has an incompressible surface.

An incompressible surface is a two-sided codimension-1 submanifold that is one-to-one on π_1. A hierarchy is an inductive structure so that you start with an incompressible surface, cut open your manifold along it, and then find a new surface there, and keep on going. The key point, though, is that the process

[1] May they soon be written.
[2] That is, one in which each embedded S^2 bounds a ball.

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terminates, and you end up with a ball. You then think of the 3-manifold as obtained by the reverse process of constantly gluing 3-manifolds to themselves along pieces of their boundary.

This sequence of incompressible surfaces enables inductive proofs. Thus, Waldhausen (1978) showed that, for such 3-manifolds, the universal cover (of their interior) is $\mathbb{R}^3$ and any homotopy equivalence which is already a homeomoprhism on the boundary is homotopic, relative to the boundary, to a homeomorphism.

The proof of the splitting theorem in dimension 3 is a consequence of the basic theorems of Papakyriakopoulos, namely the Dehn lemma, loop theorem, and sphere theorem. The combination of Dehn lemma and loop theorem tells us that, if the fundamental group of the boundary of a 3-manifold M does not inject, then it is for the obvious reason that there is an embedded D^2 in M that intersects the boundary in an essential curve. The sphere theorem asserts that $\pi_2(M)$ is nontrivial (for an oriented 3-manifold) iff there is an essentially embedded S^2. This last gives a quick proof of the basic result (which so influenced our discussion in §6.4) that, for closed 3-manifolds, one is a nontrivial connected sum[3] iff the fundamental group is a nontrivial free product.

These can be found in any book on 3-manifolds, in particular, Hempel (1976) and Jaco (1980).

In higher dimensions, the Farrell fibering theorem gives an approach to the Borel conjecture for tori. Without using periodicity of structure sets, one has to argue indirectly and use periodicity of L-groups and G/Top and calculate. This was done via Wall (1968), Shaneson (1969), Hsiang and Shaneson (1970), and Farrell (1971b, 1996).[4] More generally, the splitting theorem of Cappell (1976a) can be thought of as a Mayer–Vietoris sequence for L-theory of amalgamated free products and HNN extensions (see Cappell, 1976b), except that in the non-square root closed situation one can run into UNil obstructions; or, phrased differently, there's an extra summand in one of the terms of the Mayer–Vietoris sequence.

The analogous situation in K-theory was perhaps not as immediately apparent to someone working on concrete questions, since frequently K_0 and Whitehead groups vanish.

Stallings (1965a) had shown early that $\mathrm{Wh}(G * H) = \mathrm{Wh}(G) \oplus \mathrm{Wh}(H)$. For polynomial extensions, Bass *et al.* (1964) gave the formula in the $\mathbb{Z}$ case of the K-theory Borel with coefficients. (This paper, strictly speaking, required

[3] Nontrivial excludes connected sum with homotopy spheres, but now that the Poincaré conjecture is a theorem, this does not need to be made explicit.

[4] Farrell (1971b) explains the close connection between fibering over a circle and the problem of putting a boundary on an open manifold, i.e. the problem studied in Siebenmann (1965).

coefficients to be a regular ring, which avoids Nil terms. However, Bass's 1968 book has a more complete discussion.) This formula led to Bass's definition of the negative K-groups and the desire for higher algebraic K-theory.

The twisted version of this formula was discovered by Farrell and Hsiang (1970) and was part and parcel of understanding the problem of fibering over the circle (monodromy requires allowing for twists – the problem always involves, as we've discussed, the Whitehead group). Waldhausen (1987) made a major advance when he proved the analogous statement for K-theory of amalgamated free products, using a new Nil functor to measure the lack of excision. His motivation was to understand why the homotopy equivalences between 3-manifolds discussed above were all simple! His answer was that the same structure that enables one to deform homotopy equivalences to homeomorphisms enables one to prove that the relevant Whitehead groups are trivial.

The story in algebraic K-theory is recapitulated in operator theory. The analogue of the Farrell and Hsiang (1970) formula is the Pimsner–Voiculescu sequence (Pimsner and Voiculescu, 1980) and the analogue of Cappell's theorem is Pimsner's theory of K-theory for "groups that act on trees" (Pimsner, 1986). (Interestingly, for the special problem of positive scalar curvature (see Chapter 5), the analogue of the key boundary map in the exact sequence, which above is produced via codimension-1 splitting, was constructed – at least in low dimensions – by Schoen and Yau (1979a), who showed how to use stable minimal hypersurfaces to use hierarchies to obstruct positive scalar curvature.[5])

It is surely worth observing that among the most developed methods for constructing strange groups is via amalgamated free products and HNN extensions (see e.g. Baumslag *et al.*, 1983). As a result, the Mayer–Vietoris sequences in K- and L-theory remain valuable for constructing examples.

Codimension-1 methods also are critical to the proofs that go through the controlled world. The basic results about controlled topology over finite-dimensional ANRs or bounded control over cones of polyhedra, etc., are all proved via appropriate codimension-1 splitting theorems (since, after all, the hard part in proving that something is a homology theory is almost always checking excision, also known as Mayer–Vietoris); see, for example, Quinn (1979, 1982b,c, 1986), Pedersen and Weibel (1989), and Ferry and Pederson (1995).

[5] In a recent preprint (Schoen and Yau, 2017), they remove the low-dimensional condition.

8.2 Induction and Control

Our next goal is to explain the ideas of Farrell and Hsiang (1970) that, for example, prove the Borel conjecture for flat manifolds. These are "merely" finite torsion-free extensions of free abelian groups, yet they are hard to understand directly. The arguments are a beautiful mix of algebra (induction theory) and controlled methods (one of the first applications of these to computing something) and have been very influential.

Actually the result of Farrell and Hsiang (1970) is more general: it gives a topological characterization of almost-flat manifolds in the sense of Gromov. A manifold M is almost flat if it has a sequence of Riemannian metrics with $|K| < 1$ and $\operatorname{diam}(M) \to 0$ (or equivalently with bounded diameter and $K \to 0$). Such manifolds are exactly (see Buser and Karcher, 1981) infranilmanifolds.

Theorem 8.1 *A closed topological manifold has an almost-flat structure iff it is aspherical and its fundamental group is virtually nilpotent.*

Sketch The result boils down to showing that there is a unique aspherical manifold with the given fundamental group. We will ignore the low-dimensional cases, because they hold for the usual reasons: Perelman (geometricization) does dimension 3, and Freedman's work applies in dimension 4, because the relevant fundamental groups are all "small" (see Freedman and Quinn, 1990).

The nilpotent case is easy. Nilpotent groups are poly-$\mathbb{Z}$: there is a natural induction on the cohomological dimension, and such a group Γ always surjects to $\mathbb{Z}$ with a smaller such group as its kernel. Ultimately, one deduces the result from many applications of Farrell fibering and the Farrell and Hsiang (1970) variant of the Bass–Heller–Swan formula.

The remaining part of this argument is inspired by representation theory (see Serre, 1977, for everything we say). For finite groups, representations are determined by their characters, i.e. their restrictions to cyclic subgroups. However, not every representation is a sum of representations induced from their cyclic subgroups. (It is, if one allows rational coefficients, according to Artin's theorem.) To get these, one needs a larger class of groups (this is the content of Brauer induction).

These are proved by making use of $R(H)$ for all H in G. There are operations $\operatorname{ind}\colon R(H) \to R(G)$ and $\operatorname{res}\colon R(G) \to R(H)$ (and, of course, G can be replaced by any subgroup of G that contains H). It suffices to prove a formula in the algebra of operations of the form $1 = \sum a_H \operatorname{ind}_H^G(\operatorname{res}_H)$, where the a are some coefficients. □

Suitable reciprocity would also give that, for any module over this algebra (i.e., suitable functors of groups that have appropriate behavior with respect to induction and restriction), restrictions to the family H will detect elements.

To make a long story short, the work of Dress (1975) on equivariant Witt rings can be used in a similar way to prove induction theorems for structure sets of manifolds, whenever one has a map $\pi_1 M \to G$ for a finite group G. Farrell and Hsiang (1983) use this for L-groups; Nicas (1982) gives a version for structure groups that is a bit more natural for our purposes:[6]

$$S(M) \to \bigoplus S(M_H)$$

is injective localized at a prime p, if H ranges over the p-hyperelementary subgroups (i.e. the groups containing normal cyclic subgroups with index a power of p), where M_H is the cover of M corresponding to the subgroup H (and the map is "transfer" to this cover). Rationally, it is injective when H ranges over the cyclic subgroups (just like in character theory!).

We now can sketch some of the ideas that go into the proof of the Farrell–Hsiang rigidity theorem. The actual proof for the general case involves more complex fibering rather than just over the circle (but from Chapter 6, this causes us no fear!). In the flat case where the holonomy is odd order, the proof is much simpler (Farrell and Hsiang, 1978b). In that case, they show that the following algebraic fact holds:

Proposition 8.2 *Assume Γ is the fundamental group of a flat manifold with holonomy G of odd order, so that there is an exact sequence*

$$1 \to A \to \Gamma \to G \to 1,$$

where A is (free) abelian. Then either there is a surjection $\Gamma \to \mathbb{Z}$, or for infinitely many $s = 1 \bmod \#(G)$, one has Γ/sA having the property that any hyperelementary subgroup that maps onto G, maps isomorphically onto G.

The conclusion of the proposition enables an induction on the cohomological dimension combined with the holonomy, except for the awkward case where one has an element that is transferred to the cover corresponding to a hyperelementary subgroup that is isomorphic to the manifold itself. Naively it looks like we're stuck with an impossible circular argument.

But we are not at all: the method of Example 4.17 in §4.6 now kicks in. When we identify the cover with the original M, one has used an expanding

[6] Of course, the upshot of modern surgery is that structure sets are essentially L-groups of a suitable object. As a result, precisely the same structure that gives induction for L-groups of groups gives it for structures. Induction arguments in smooth surgery are sometimes possible (see, for example, Madsen *et al.*, 1976).

map, so the point inverses are smaller. Ultimately, we are in the circumstances of Ferry's theorem and one gets that all the transfers are 0.

Consequently, by Dress induction, every element of the structure set vanishes,[7] and therefore the structure set does (and the assembly map is an isomorphism).

In §8.3, I will explain another example where transfers actually suffice for proving vanishing of structure sets. (Actually, I will focus on the easier case of Whitehead groups, for a reason that will be clearer in §8.3.)

Remark 8.3 Although the control ideas in the above proof are the most important piece for the Borel and Farrell–Jones conjectures,[8] the use of induction in L-theory and structure sets is fundamental throughout equivariant topology.

8.3 Dynamics and Foliated Control

To say that the subject was revolutionized by the work of Farrell and Jones through their long series of brilliant[9] and beautiful papers (Farrell and Jones, 1986, 1987, 1988, 1989, 1991a,b, 1993a,b, 1998a,b, 2003)[10] is an understatement.[11]

Here I will explain just one of the ideas, as an introduction to this body of work. Among their achievements, they proved the Borel conjecture for closed non-positively curved manifolds and for many manifolds with infinite volume, again under a curvature condition. But, also, they formulated (and proved many cases of) the "Farrell–Jones isomorphism conjecture" as a natural byproduct of what the method leads to. This conjecture is proved for lattices in Lie groups in Farrell and Jones (1993a) for K-theory and in Bartels *et al.* (2014a,b) for L-theory. We can see this already in their earliest result in this series,

Theorem 8.4 (Farrell and Jones, 1986, 1987) *If* Γ *is the fundamental group of a closed negatively curved manifold, then* $\mathrm{Wh}(\Gamma) = 0$.

To get a feeling for this achievement, realize that until this point all previous groups that were understood had a clear algebraic nature, a place from which to

7 Please see the cartoon in Figure 1.1 at the end of Chapter 1.

8 However, see Bartels and Lück (2012b) for a use of their general inductive scheme.

9 Difficult.

10 It is also worth mentioning that there was follow-up work by Farrell and Jones and by Ontaneda that develops methods for proving results about negatively curved manifolds and spaces of negatively curved metrics building on these topological rigidity ideas.

11 To say that this is an understatement, is an understatement; and so on, for another few iterations.

grab an inductive hold. Here we are impelled to use geometry, which seemed almost unheard-of in high dimensions.[12]

Here's the general idea. One starts with M the closed hyperbolic manifold with fundamental group Γ, which we assume is of dimension greater than 4. An element of the Whitehead group $\mathrm{Wh}(\Gamma)$ is represented by an h-cobordism W, with $\partial W = M \cup M'$. The idea is to find a bundle X over W, so that this bundle has enough geometry that one can geometrically "flow" this new h-cobordism to "control it." Controlled h-cobordisms will be products, and if there is no information lost in the transfer map $\mathrm{Wh}(M) \to \mathrm{Wh}(X)$, then one will have proved that the original h-cobordism is trivial, and thus the theorem.

Of course, at the level of algebra there is a map $\mathrm{Wh}(M) \to \mathrm{Wh}(X)$, and the composite is just multiplying by $\chi(\mathrm{F})$, the Euler characteristic of the fiber. (In L-theory, the monodromy plays a larger role, but it usually is multiplying by sign (F). This fairly straightforward topological step is a major obstacle in the Baum–Connes conjecture.[13])

None of these steps are straightforward.

For simplicity we will work on M rather than W. This means that we should work in a setting of geometric modules. The Whitehead group $\mathrm{Wh}(M)$ is made out of free $Z\Gamma$ modules with automorphisms (or acyclic chain complexes of free based modules). Now imagine that each basis element is given a location on M, and that morphisms include paths that link generator to generator (see Quinn, 1982a, 1985a, 1987b). This enables defining controlled Whitehead group elements, and the "fundamental theorem of controlled topology" (i.e. the main theorem in Quinn, 1979, 1982b,c, 1986) asserts that controlled Whitehead groups are essentially the homology of the control space with coefficients in the Whitehead spectrum (so, for $\mathbb{Z}$, they vanish). We will transfer up our geometric uncontrolled chain complex from M to X and flow it there. Hopefully, the curves that arise in the module will become smaller, and thus become controlled, and trivialize.[14]

12 Of course, for most of a decade, Thurston had already been preaching the importance of geometry for low-dimensional topology. In the work of Thurston, the geometry is there because the manifold ends up being geometrizable. In the work of Farrell and Jones, it's because we can transport our problems to live over a geometric object, and study them there.

13 Usually the signature of a fiber bundle is the product of the signature of base and fiber – at least if the monodromy is simple enough – but for other operators this is rarely the case. Part of the fascination with the elliptic genus is based on the magic that there *are* operators for which this is true for connected compact structural groups (see, for example Bott and Taubes, 1989) in smooth settings.

14 This cannot actually happen – because of the Nil term in the Bass–Heller–Swan formula. Nothing in the geometry knows that the coefficients of the paths lie in $\mathbb{Z}$, so we could do all this algebra with an arbitrary coefficient ring R; this proof, if it worked, would imply that $\mathrm{Nil}(R) = 0$ for all R. However, as we will see, the kind of control that is gained is less than the control in Chapman and Ferry, or Quinn, and the Nil term will naturally come up in the end.

The first try for the bundle is the unit sphere bundle of M. This won't have the right transfer properties, so it will ultimately be modified. (It would suffice for M to be odd-dimensional to prove the vanishing of $\mathrm{Wh}(M) \otimes \mathbb{Z}[1/2]$.)

The unit sphere bundle of a Riemannian manifold has a natural flow on it, the geodesic flow: A point is a pair consisting of (m, v), with m a point of M and v a unit tangent vector at m. One then considers the geodesic going through m in the direction v for t seconds (transporting v to the other end).

The geodesic flow on a negatively curved manifold is Anosov. What this means is this. The tangent space to X breaks up into three pieces $F_s + F_u + \mathbb{R}$. In the F_s direction, the flow contracts (exponentially fast). In the F_u direction, the flow expands things (in negative time, it contracts). The $\mathbb{R}$ is the direction of the geodesic, where the flow leaves things alone.

A key element, then, is the asymptotic transfer, which lifts curves on M to ones on X.[15] It is like a connection, telling one to not just use the abstract bundle properties to lift homotopies, but how to use the geometry. The choice Farrell and Jones use is the "asymptotic transfer," defined to make curves shrink with respect to the geodesic flow, to at least end up close to the $\mathbb{R}$ directions, i.e. have no F_u parts.

The picture in Figure 8.1 describes the construction very clearly. A point on the universal cover of M and a vector determines a well-defined point at infinity.

[15] In transferring a torsion there is the base direction that we are in the midst of discussing in the body of the text, and also the fiber direction. The fiber direction, however, is the chain complex of the fiber, which is as controlled as one wishes (in this, and all the other known applications of this method), and we shall not discuss it.

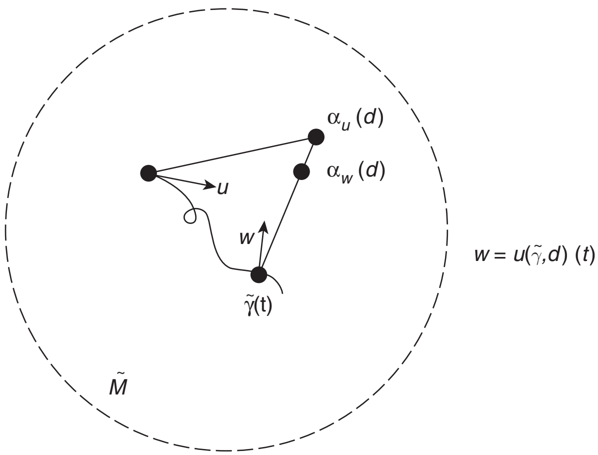

Figure 8.1 A picture of the asymptotic transfer. (Reproduced with thanks from Farrell's 2002 Trieste notes.)

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Then, for any other point, there is a unique vector that points to that same point at infinity. Along a curve, one translates the vectors by this common asymptotic rule. (This rule is equivariant with respect to the action of the covering group, and is thus well defined on M as a way of transporting on curves.)

Note that if one transfers up a geodesic segment, it lifts to a geodesic segment (with its parallel translation of the initial vector) and nothing shrinks during the flow. But, at least nothing gets larger, and all of the other directions are exponentially shrinking. So after a while one has an h-cobordism (or acyclic geometric chain complex) that is "foliated-controlled over X," i.e. sizes can be made arbitrarily small in directions orthogonal to the leaves of the foliation.

More precisely, X (the unit sphere bundle) is foliated by the orbits of geodesic flow (i.e. the geodesics on M with their unit tangent vectors), i.e. it has a one-dimensional foliation. Most of the leaves are isomorphic to $\mathbb{R}$, but there is a countable number of exceptions – the closed geodesics[16] of M. After flowing, all the morphisms in the geometric module (i.e. the tracks of the homotopies in the h-cobordism[17]) end up lying as close as we want to leaves.

Essentially what happens is this. The $\mathbb{R}$ leaves contribute nothing – the neighborhoods of these can be rescaled shrinking the $\mathbb{R}$ direction, to be fully controlled. However, the S^1 are more serious. They can't be rescaled away, but they cause no trouble because $\mathrm{Wh}(\mathbb{Z}) = 0$.

It is here that, if we used another ring, the Nil terms in the Bass–Heller–Swan formula would enter. We get one for each closed geodesic (as had been mentioned in §5.5).

Now we have to deal with the issue that the transfer to the unit sphere bundle is not injective. What Farrell and Jones did was associate to an n-dimensional negatively curved manifold M, a negatively curved metric on $M \times \mathbb{R}$, and on this manifold there is an invariant upper and lower hemispherical tangent bundle. These are disk bundles. One uses h-cobordisms with compact supports and flows on this space, following the above pattern. Topologically, we know that nothing differs in the Whitehead theory of M from that of $M \times \mathbb{R}$ with compact supports, but metrically we have replaced spheres by disks.

In Bartels *et al.* (2008) a Rips complex, with Mineyev's version of geodesic flow on a hyperbolic group (Mineyev, 2005), replaces ordinary geodesic flow, and enables the proof of the Farrell–Jones conjecture in K-theory for hyperbolic groups.

Remark 8.5 The foliated control theory is critical to the Farrell–Jones pro-

[16] Remember that M is assumed negatively curved.

[17] For any homotopy equivalence, one considers the homotopy H from fg to the identity, and for each point p one has the tracks $H(t, p)$ for $0 < t < 1$.

gram (although it does not play as large an explicit role in some of the later work of Bartels, Lück, Reich, and others).

First of all, the fibered case of foliated control is essentially the same thing as controlling with respect to the quotient space. In many situations in foliated geometry, one wants to analyze algebraic topological invariants of the quotient space that does not exist in the conventional sense – see e.g. Connes (1985, 1994) and our discussion in Chapter 5. In all cases, the basic idea is to deal with the Hausdorff object that exists and never really take the quotient.

Remark 8.6 (Digression) Foliations also occur very naturally in the study of the asymptotics of topological phenomena if there is a bound on the local geometry. One can, for example, think about the Borel conjecture as a statement about vanishing of certain "periodic structure sets" – by passing to the universal cover, and then begin inquiring about aperiodic analogues – in ways that ape the theory of quasicrystals (see, for example, Bellissard, 1995). This would lead to a foliated Borel conjecture. Needless to say, Baum and Connes, in formulating their conjecture, also considered a foliated version.

Such foliations and their homology also naturally arise in topological data analysis (see, for example, Weinberger, 2014) because of their connection to "testable properties" or statistically "sampleable" invariants of manifolds (see, for example, Bergeron and Gaboriau, 2004; Elek, 2010; Abert *et al.*, 2017). Essentially one asks for invariants of manifolds that can be approximated by knowing the balls around a number of randomly chosen points. For this to be true, the invariant needs to be continuous in a suitable topology (a modification of the usual Gromov–Hausdorff metric that takes measure into account; see Benjamini and Schramm, 2001, for the case of graphs – also Lovasz, 2012. and Gromov, 1999). Limits of sequences of compact manifolds in this topology are actually foliated spaces (with a transverse measure) – where the leaves have the same dimension as the approximating manifold.

8.4 Tensor Square Trick

The results on L-theory and for the Borel package require other transfers.

Crossing with a sphere (of dimension greater than 1) is trivial in L-theory as is crossing with a disk, so it is necessary to find a new fibration (and transfer) $X \to M$. In Farrell and Jones (1989) the fiber is a modification of $F = \mathcal{S}^{m-1} \times \mathcal{S}^{m-1}/\mathbb{Z}_2$, the set of unordered pairs (s, s') on the sphere at ∞. When $s \neq s'$ there is a unique geodesic in the universal cover asymptoting to that pair. When $s = s'$ there is a unique geodesic going through a given a given point in $\tilde{M}$ at time 0 and asymptoting to (s, s). Thus, one considers the union of F with a $\mathcal{D}^m$.

It turns out that this stratified space has the property that crossing a manifold surgery problem with it does not lose any surgery obstruction. A similar approach to Siebenmann periodicity $S(M) \to S(M \times \mathcal{D}^4)$ via an "exotic product" with $\mathbb{CP}^2 \cup \mathcal{D}^3$ is given in Weinberger and Yan (2001).

In both cases the key feature is that the "main part" of the space is a homology manifold and so has a signature and that signature is 1. (Indeed, the space of Farrell and Jones is modified to give rise to an equivariant version of Siebenmann periodicity in Weinberger and Yan (2005) for compact group actions.[18])

As the program developed, more and more complicated transfers were constructed. A major problem for the situation of hyperbolic groups comes because their boundaries are almost always not even ANRs, let alone manifolds! Bestvina and Mess (1991), though, do show that the compactification of Eπ is an ANR.

The solution to this was a breakthrough in Bartels and Lück (2012a) and relies on a tensor square trick. The first point is that there is no reason that one has to "cross with a space" (perhaps in a twisted way) to induce a transfer. One can cross with a symmetric Poincaré complex, which should be geometric over a control space – so one can gain control to good effect – but it need not be the controlled symmetric signature of the control space (or some fancy variant thereof).

This is akin to the use of elliptic operator to set up an (equivariant) Thom isomorphism for complex bundles in Atiyah (1974). One does not need to write down a bundle – just a construction that leads to a suitable family of operators. If one thinks of K-theory as being related to, for example, normal invariants, then one sees an isomorphism that is associated to a nontopological construction on the fibers – as the Dirac operator is not topological and the signature operator causes difficulties at the prime 2.[19]

The basic point of the construction is that, if P is a projective module, then $P \otimes P^*$ naturally supports a symmetric bilinear form. More generally, if P is a chain complex, then $P \otimes P_{-*}$ supports a symmetric Poincaré structure, $(P \otimes P_{-*})^* \cong (P_{-*} \otimes P) \cong P \otimes P_{-*}$ interchanging factors. If P has Euler characteristic 1, then this tensor square has signature 1, and one has a formal

[18] The earlier paper only succeeding in doing this for abelian groups.

[19] And this made the periodicity theorems of Weinberger and Yan (2001) more difficult still. As we discussed in Chapter 7 regarding equivariant products, the equivariant signature operator for even-order groups does not give an orientation even rationally, because the localized contribution near 0 is a zero divisor in $R(G)$. Thus one is *forced* to do non-topologically invariant constructions.

process of turning the kind of transfer used in K-theory into one suitable for L-theory. This is a construction that is perfectly well controlled, as verified in Bartels and Lück (2012a), when one changes the control Z space of P to $Z \times Z/\mathbb{Z}_2$ (this being necessary because of the interchange of factors in the above).

Remark 8.7 Bartels and Lück (2012a) introduce another important technical innovation in that paper (necessary for their CAT(0) results) – namely the use of homotopy actions rather than actions.

This paper ushered in a sequence of important new advances on this problem (see, for example, Rüping, 2013; Bartels, 2014; Bartels *et al.*, 2014a,b; Bartels and Bestvina, 2019). It is too soon to be sure where the new "natural boundary" of the current technique is. One can hope that all linear groups over some field, and groups with some "non-positive curvature," will ultimately follow to extensions of these methods.

8.5 The Baum–Connes Conjecture

The serious reader should turn to the excellent survey of Higson and Guentner (2004)[20] for a very useful and insightful treatment.[21] While there have been developments since that paper was written, notably Lafforgue's work clarifying the obstacle of Property (T),[22] it remains, to my mind, the best single survey.

What follows is intended for the frivolous reader.

Remember playing the Novikov game (way back in Chapter 5)? The setting for the game involved improving the index of elliptic operators to lie in the K-theory of some appropriate C^*-algebra. We have focused on the analogy between the normal invariants of degree-1 normal maps, living in $L(e)$-homology theory of a group, and $K_i(\mathrm{B}\pi)$ and correspondingly between $L(\pi)$ and $K(C^*\pi)$.

More precisely, associated to a group π, there are C^*-algebras $C_r^*\pi$ and $C_{\max}^*\pi$ that are completions of $\mathbb{C}\pi$ thought of as an algebra of unitary operators on appropriate Hilbert spaces. Of the two, $C_{\max}^*\pi$ is perhaps the more naive choice – it is the completion with respect to all unitary representations. It has the advantage of being a functorial construction on the category of groups. The

[20] This seems like a good place to express my deep gratitude to Erik [Guentner] for spending a couple of very intensive weeks at Jerusalem cafes explaining this all to me (including assigning and critiqueing homework). And thanks to Nigel [Higson] for sending Erik.

[21] Other recommended surveys are Valette (2002) and Gomez Aparicio *et al*. (2019) and, of course, Connes's 1994 book *Noncommutative Geometry*.

[22] The two Bourbaki expositions by Skandalis (1999) and Puschnigg (2012) on Lafforgue's work are excellent next steps.

other, $C^*_r\pi$, is the completion with respect to the regular representation. It is not, in general, functorial, although it is functorial with respect to injections. We will soon return to the issue of functoriality.

What is important for us here is that, associated to any elliptic operator D on M^n with fundamental group π, there is an index $\mathrm{ind}(D) \in K_n(C^*\pi)$ for either of these algebras. The symbol of D lies in $K_n(M)$ (as observed by Atiyah, 1975; see also Higson and Roe, 2000).

There is a natural[23] group homomorphism

$$K_n(M) \to K_n(C^*\pi)$$

that takes an elliptic operator to its index. Indeed, this factors through

$$K_n(\mathrm{B}\pi) \to K_n(C^*\pi),$$

and, ultimately (using proper equivariant elliptic operators)

$$K_n^\pi(\mathrm{E}\pi) \to K_n(C^*\pi),$$

analogous to our story about the (equivariant) controlled symmetric signature of manifolds, and their algebraic uncontrolled versions (or equivalently the surgery obstruction map in surgery).

Moreover, one can take "twisted coefficients," as we had done in K- and L-theory to accommodate problems of (block or approximate) fibration (and stratified spaces[24]). This leads to the following statement:

$$K_\top(G, D) \to K(C^*_r(G, D)).$$

When D is just $\mathbb{C}$, G acting trivially, then the left-hand side is the equivariant K-homology group $K_G(\mathrm{EG})$, as above.

As mentioned in Chapter 5, knowing the injectivity of such a map is an analytic variant of the Novikov conjecture (and it's sometimes called the strong Novikov conjecture in the literature). It implies the usual Novikov conjecture when applied to the signature operator. When applied to an equivariant signature operator, it implies the pseudo-equivalence invariance statement discussed in Chapter 7. (As hinted at in §4.5, the topological invariance of the equivariant signature operator[25] can be proved – along the lines of Pedersen *et al.* (1995)

[23] For maps that preserve the fundamental group, using the pushforward of pseudodifferential operators that is introduced for the K-theoretic proof of the index theorem in Atiyah and Singer (1968a).

[24] Although there is much more that needs to be done to understand the natural elliptic operators on stratified spaces than arises for the signature operator (or in topology).

[25] Which implies, for example, that for odd-order groups nonlinear conjugacy of linear representations only occurs for linearly equivariant representations, the theorem of Hsiang–Pardon and Madsen–Rothenberg discussed in Chapter 6.

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for the ordinary signature operator – using a metric space version of this kind of statement, which is indeed true for cones of G-ANRs by, for example, the reasoning in Roe, 1996).)

And applied to other operators it has further implications, e.g. for positive scalar curvature, and to higher Riemann–Roch kinds of theorems, etc. The first nontrivial case is when $G = \mathbb{Z}$, when this conjecture is verified by the Pimsner–Voiculescu exact sequence (Pimsner and Voiculescu, 1980). It is the result of applying Mayer–Vietoris to computing the left-hand side and combining it with the isomorphism above. It is the analogue of the Bass–Heller–Swan(–Farrell–Hsiang) theorem in algebraic K-theory. It is simpler in that there is no Nil. Indeed, there is no need for the Nil and UNils that arise in K- and L-theory Farrell–Jones conjectures. The analogue of the work of Waldhausen and Cappell described in the first section of this chapter is Pimsner (1986).

As in the previous paragraph, this implies all the cases of the Novikov conjecture discussed in §8.1.

Much of the immediately subsequent development took somewhat parallel turns in topology and operator theory. It is important to mention the work of Kasparov (1988) on non-positively curved complete manifolds as a highpoint (which inspired the work of Ferry and Weinberger (1991) that paralleled it, although it looked quite different at the time because of difference of emphasis[26]). For this work, Kasparov developed KK-theory, a bivariant version of K-theory that accepts a pair of C^*-algebras – which, together with variants such as E-theory, tend to be key technical tools in the area. The serious reader should study Blackadar (1998) and Higson (2000) to learn this tool.

One result on the operator algebra side that has no known analogue in topology is the theorem of Higson and Kasparov on the (idiosyncratically named[27]) a-T-menable groups.

Theorem 8.8 (Higson and Kasparov, 2001) *If G acts metrically properly and isometrically on a Hilbert space (i.e. is a-*T*-menable), then the Baum–Connes map (with coefficients) is an isomorphism (with either completion)*

$$K^G(\underline{EG}, D) \to K(C^*(G, D)).$$

The condition of a-T-menability is somehow an opposite of Property (T): Property (T) groups always have fixed points for continuous isometric actions on Hilbert space.

[26] It was only later that Higson, Roe, and others elucidated the close parallels between these theories.

[27] By Gromov (1993).

A couple of hundred pages ago, amenability was also described as an opposite to Property (T). Amenable groups are, indeed, examples of a-T-menable groups.

This is not at all obvious. (Indeed, Gromov had asked the question in 1993, expecting the positive solution.) This was soon shown by Bekka *et al.* (1995). We will return to some of the relevant concepts in §8.6.

The groups $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ are also a-T-menable, as had been showed by Vershik, Gel'fand, and Graev almost 40 years ago (Vershik *et al.*, 1974; see also Cherix, 2001).

The Higson–Kasparov theorem has one aspect that cannot be improved upon – their ability to accommodate $C^*_{\max}$. At the opposite extreme, Property (T) groups, every finite-dimensional representation is isolated in the Fell topology;[28] these give rise to elements of infinite order in $K_0(C^*_{\max}\pi)$.

For instance, if π is, say, a lattice in a higher-rank Lie group or even $\mathrm{Sp}(n, 1)$,[29] it has many finite-dimensional irreducible representations, so the right-hand side $K_0(C^*_{\max}\pi)$ is infinitely generated (while the domain of the assembly map is finitely generated, e.g. by Borel–Serre).

This is one of the difficulties with the conjecture. From its outset, one realized that, because of the general functoriality of the domain, one would want to use $C^*_{\max}$, but that Property (T) is an obstacle. In some sense the Higson–Kasparov theorem carves out the natural place to look where this difficulty will not arise.

We shall also see that that theorem has some extraordinary implications.

However, it underscores the extent to which Property (T) is an obstacle. It is thus very remarkable that Lafforgue (2002) was able to overcome this obstacle in some cases by including uniform lattices in $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$, and hyperbolic groups (Lafforgue, 2012) – even with coefficients. These are based on making a variant of *KK*-theory for Banach algebras, which allow more deformations of representations that allow one to "pass through" the gaps that prevent deformations in *KK*-theory. However, Lafforgue (2008) describes a strengthening of Property (T) that obstructs all known techniques, and shows that $\mathrm{SL}_3(\mathbb{Q}_p)$ has this property, so that it (and its lattices) definitely lie outside current technology.

In §8.6 we will discuss that, for general discrete groups, the Baum–Connes conjecture with coefficients fails – and the reason for this is because of expanders, a class of graphs that we have met in Chapter 3 as one of the first applications of Property (T).

28 The reader might want to review some ideas from Chapter 3 to unravel this discussion.

29 Since $\mathrm{Sp}(n, 1)$ is a rank-1 group, its lattices are fundamental groups of negatively curved manifolds. As a result, the Higson–Kasparov theorem does not even extend to the situation of negative curvature.

8.6 A-T-menability, Uniform Embeddability, and Expanders

The hypothesis of the Higson–Kasparov theorem, a-T-menability, was first introduced by Haagerup in an equivalent form. A useful source on a-T-menability is Cherix (2001). The equivalent forms of a-T-menability have parallels among equivalent definitions of Property (T). These equivalences are generally useful for making constructions and in different applications.

(1) There is a proper function $\psi\colon \pi \to \mathbb{R}^+$ that is conditionally negative-definite (i.e. $\psi(g) = \psi(g^{-1})$) and, for any n-tuple of elements of π, the matrix $\psi(g_i g_j^{-1})$ is conditionally negative-definite (i.e. negative-definite on tuples $(a_1, \ldots, a_n)$ so that the sum of the a_i equals zero.)
(2) There is a sequence φ_n of continuous positive-definite functions on π with $\varphi_n(1) = 1$ that vanish at infinity but converge to 1 uniformly on compact subsets of π.
(3) π acts isometrically and metrically properly on a Hilbert space.

For Property (T) groups, every conditionally negative-definite function is bounded (i.e. not proper if π is noncompact). If a sequence of normalized positive-definite functions converges to 1 uniformly on compact sets, then it converges to 1 uniformly, so they can't vanish at infinity. And, finally, as noted before, every isometric action has a fixed point.

Many groups are a-T-menable (and, of course, many are not). As we mentioned, $SO(n,1)$ and $SU(n,1)$ and products of such, and amenable groups are. Also groups that act on CAT(0) cubical complexes or more generally "spaces with walls" are included in this class (Niblo and Reeves, 1997). So (as proved by Farley in his thesis – see Farley, 2003) the morally amenable (but not yet known to be (non-)amenable) Thompson group satisfies the Baum–Connes conjecture.

The Higson–Kasparov theorem is proved by an analogue of Atiyah's proof of Bott periodicity (Atiyah, 1968).

Rather than make any attempt at explaining the proof of the theorem, let's instead (for example, following the line of thought of §§4.8 and 4.9) think about the metric-space analogue of the theorem and then see what that buys us in terms of the Novikov conjecture. Of course, the metric-space version is of interest in its own right: there are many more bounded geometry metric spaces[30] than there are finitely generated groups!

[30] A metric space has bounded geometry if it is a path metric (i.e. distances are generated by a path geometry) and there are only "finitely many types" (or a compact space of types) of balls of a fixed radius. So all Cayley graphs of finitely generated groups have bounded geometry. As

Theorem 8.9 (Yu, 2000; Skandalis *et al.*, 2002) *If* Γ *is a (discrete) metric space of bounded geometry which uniformly embeds in a Hilbert space, then the bounded Baum–Connes assembly map is an isomorphism:*

$$K_n^{\mathrm{lf}}(|\Gamma|) = \lim K_n^{\mathrm{lf}}(N_k|\Gamma|) \to K_n^{\mathrm{lf}}(C^*|\Gamma|)\,.$$

The left-hand side is the limit of K-homology of the nerve of covers of Γ by k-balls, as $k \to \infty$. (The discreteness of Γ is a convenience to make this cover locally finite: otherwise, one can replace a metric space of bounded geometry by a coarsely dense discrete subset.) The range is the K-theory of the Roe algebra: it is the closure of the bounded propagation speed operators on Γ. If one thinks of operators as being described by kernels (like matrices in a geometric module), then one is taking the limit of the operators where $k(x, y) = 0$ if $d(x, y) > R$ as $R \to \infty$. So an operator in this algebra can be well approximated by operators with finite propagation speed. We shall later see the implications of this approximation – which have no analogue in the purely topological world.

Note that if Γ acts properly and isometrically on a Hilbert space, then it uniformly embeds – indeed, the map $\Gamma \to H$ given by $\gamma \to \gamma(h)$ for any fixed h is a uniform embedding. Propriety means that one only returns finitely many times to any fixed neighborhood of h. The isometry condition then translates this into a condition for any group element (uniformly), i.e. that there is a proper non-negative increasing function f, so that

$$\|\gamma(h) - \gamma'(h)\| > f\big(d(\gamma, \gamma')\big).$$

Moreover, the map is Lipschitz with Lipschitz constant supremum sup $\|\gamma(h) - h\|$ as γ runs over generators of Γ.

The arguments given in §4.9 (i.e. the principle of descent) now enable one to show that, if in addition $B\Gamma$ is a finite complex, then the analytic Novikov conjecture with coefficient is true, i.e. the Baum–Connes assembly map with coefficients is split injective. However, things are better yet. Higson (2000) has shown that one can dispense with this finiteness and still get the result.

Theorem 8.10 (The Novikov conjecture for groups that uniformly embed; Skandalis *et al.*, 2002) *If* Γ *is a countable group which uniformly embeds in*

mentioned in §4.8 and explained in §5.3, without bounded geometry there are older counterexamples to the bounded Borel and Baum–Connes conjectures based on different principles than the examples we are about to explain regarding the Baum–Connes conjecture. I am not aware of any counterexamples to the bounded Borel conjecture.

Hilbert space,[31] *then for all coefficients D,*

$$K^{\Gamma}(\underline{E}\Gamma, D) \to K\big(C^*(\Gamma, D)\big)$$

is split injective.

Later work by Kasparov and Yu (2006) has weakened the hypothesis on which Banach space one needs to embed in for this result.

This theorem has the following corollary:

Corollary 8.11 (Guentner *et al.*, 2005) *The Novikov conjecture holds for any countable* Γ *in* $\mathrm{GL}_n(\mathbb{F})$ *for any field* $\mathbb{F}$*.*

Of course, this supplements the cases of non-positive curvature, amenable (a-T-menable) groups, hyperbolic groups, etc. that have been discussed before!

The proof uses a variant of condition (1) at the start of this section that describes a condition sufficient for uniformly embedding a discrete metric space into Hilbert space. Instead of a function from Γ to $\mathbb{R}$, one uses a function $\Gamma \times \Gamma \to \mathbb{R}$ that is a negative-type kernel, defined exactly the same as in condition (1). We now need the kernel to behave well with respect to the metric on Γ, i.e. that $\psi(g, h)$ can be bounded above and below in terms of $d(g, h)$.

If Γ were discretely embedded in $\mathrm{GL}_n(\mathbb{C})$, one could use explicitly the geometry of $\mathrm{GL}_n(\mathbb{C})/U(n)$ to construct the desired embedding in Hilbert space. ($\mathrm{GL}_n(\mathbb{C})/U(n)$ is isomorphic to the parabolic group of upper triangular matrices, which is amenable, indeed solvable.) In general, the idea is to find enough valuations, so that Γ is discretely embedded in a product of GL_n of valuated rings and that each of these is embedded in a way appropriate to the geometry of building for that valuation.[32] Thus, the workhorse lemma is this:

Lemma 8.12 *For any finitely generated field, there is a countable number of valuations* d_i *(both archimedean and discrete) such that, for any finitely generated subring, R, and for any positive numbers* N_i*, the* $\{r \in R \mid d_i(r) < N_i\}$ *is finite.*

For $\mathbb{Q}$ one uses the usual valuations. The finitely generated subrings are of the form $\mathbb{Z}[1/N]$ for some N. Then one only needs finitely many valuations, namely the archimedean one and the ones corresponding to primes in N.

Remark 8.13 There is now a rather different approach to this corollary that works in the topological setting, at least with the hypothesis of finiteness of

[31] Assume that all of its finitely generated subgroups do, in order to avoid any questions about metrics.

[32] Needless to say, some care needs to be taken in combining a perhaps infinite number of embeddings to guarantee convergence and that one remains, for example, discrete.

$B\Gamma$. This is due to Guentner *et al.* (2012) and is based on clever limiting arguments and takes its start from the Novikov conjecture for groups of finite asymptotic dimension (Yu, 1998; Carlsson and Goldfarb, 2004; Chang *et al.*, 2008; Dranishnikov *et al.*, 2008; Bartels, 2014).

We close with a brief discussion of failure. First of all, not all discrete metric spaces of bounded geometry uniformly embed in Hilbert space. Although not the first examples, an example can be built from expanders, as observed by Gromov.

Proposition 8.14 *If X_i is a sequence of d-regular expander graphs, then their disjoint union cannot be uniformly embedded.*

Without loss of generality we can assume that X_i is embedded via f_i so that its mean value (in H) is trivial. We will now use the Laplacian characterization of expansion. Let's assume that neighbors in X_i are moved a distance at most 1. In that case

$$|(\Delta f_i, f_i)| = 1/d(\text{sum over neighbors } (v, w), \|f_i(v) - f_i(w)\|^2) < \|X_i\|.$$

But this gives an upper bound on $\|f_i\|$ by the expander property, which means that the average distance of f_i from the origin is uniformly bounded (in terms of d and the expansion constant), contradicting uniformity of the embedding (i.e. that far vertices are mapped far apart).

In fact, consider $e^{-t\Delta}$. It is a bounded propagation speed operator that converges to the projection to the locally constant functions on this disjoint union. Thus that projection is in $C^*|X|$. This is a bounded propagation speed operator whose definition requires expansion: one might expect that it does not lie in the image of the coarse Baum–Connes map. This is true. It is analogous to the fact that G-indices for free actions – i.e. ones that come from $K(X/G)$ – are multiples of the regular representation.

Gromov then showed that an expander family of large girth expanders (e.g. the ones that come from the Selberg theorem; see the appendix to §3.5) can be coarsely embedded in "random quotients" of hyperbolic groups (Gromov, 2003; Silberman, 2003),[33] which therefore do not uniformly embed in Hilbert space. Higson *et al.* (2002) converted such groups into counterexamples to Baum–Connes with coefficients.

Of course, this raises a number of questions. Do expanders obstruct the untwisted Baum–Connes conjecture? Can they be used to disprove the Novikov conjecture (in any form)? Or to disprove the Borel conjecture?

[33] See Sapir (2014) for a method of embedding these groups into geometrically finite groups, as the Gromov examples are only finitely generated.

However, now that we have seen that the original versions of these analytic analogues of the Borel conjecture fail, it seems, in the spirit of what we have argued throughout this book, that understanding what is true remains an important problem.

For example – it seems to me that understanding bounded propagation speed algebras could be useful in scientific situations far removed from manifold theory, and the hypotheses that the underlying metric space – network – is uniformly embeddable seems shockingly naive. Indeed, besides the issues caused by expanders, one frequently would want to dispense even with bounded geometry, bringing on many new issues.