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DEATH-RATES IN GREAT BRITAIN AND SWEDEN: EXPRESSION OF SPECIFIC MORTALITY RATES AS PRODUCTS OF TWO FACTORS, AND SOME CONSEQUENCES THEREOF

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(With 6 Figures in the Text)

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INTRODUCTION

IN a preliminary paper[1] the specific death-rates of England and Wales, of Scotland, and of Sweden of the various age groups for different years have been analysed, and the chief result which emerged may be stated in the following terms. If $v_{t,\theta}d\theta$ denote the number of persons at a time t, between the ages θ and $\theta+d\theta$, then[2]

$$-\frac{1}{v_{t,\theta}}\left(\frac{\partial v_{t,\theta}}{\partial t}+\frac{\partial v_{t,\theta}}{\partial \theta}\right)=f(t,\theta) \qquad \ldots\ldots(1)$$

is the specific death-rate for the age θ at the time t. It appears from our previous paper that $f(t,\theta)$ may, to a close approximation, be represented as the product of two factors, of which one, β_θ, is a function of the age alone, whilst the other, $\alpha(t-\theta)$, is a function of the date of birth $(t-\theta)$. Thus

$$f(t,\theta)=\alpha(t-\theta)\,\beta_\theta \qquad \ldots\ldots(1a).$$

Clearly both α and β are arbitrary to the extent of a multiplying constant. In the case of the statistics of England and Wales, and of Scotland, the deviations from this form appear to be of the nature of random irregularities,

[1] Kermack, McKendrick and McKinlay (1934). *Lancet*, i, 698.

[2] McKendrick (1925–26). *Proc. Edinb. Mathemat. Soc.* **44**, 98.

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with the exception of those relating to the period which includes the war and the pandemic of influenza. The English figures here show a definite irregularity for ages up to 40, but this does not appear in the Scottish figures as they refer to three-yearly periods centred at the census years, and so exclude the years of the war. In the case of Sweden, for which statistics are available from 1751, the same general statement is true provided that the age groups (centred at 10, 20 and 30) be excluded from the year 1850 onwards. Some disturbance affecting those age groups would appear to have manifested itself between 1840 and 1850.

In the preliminary communication the calculations were in some respects only approximate, and the methods used, although they gave substantially correct results, were not necessarily those which gave the best fit. The simple, though somewhat crude, methods which were used had the advantage of being more direct and convincing than the more complicated and refined processes which it is necessary to employ to obtain the theoretically best numerical values. Furthermore no attempt was made to calculate the probable errors of the statistics extracted from the data, and a number of the more technical points which arose were either omitted from the discussion or only briefly referred to.

The present paper is therefore essentially a supplement to the previous one. An attempt is made to obtain the most satisfactory numerical values, and generally to fill in the lacunae left in the previous analysis, and in addition to work out some of the consequences which follow if the general conclusions suggested are substantially correct. To this end we shall assume the truth of the general result as stated, viz. that $f(t, \theta) = \alpha(t-\theta)\beta_\theta$, and we shall attempt to determine what, on this assumption, are the best sets of values of $\alpha(t-\theta)$ and β_θ which may be deduced from the data.

Data employed

The fundamental data employed are given in Tables I–III. Table I, relating to males, females, and both sexes for England and Wales from 1845 to 1925, is taken from the Registrar-General's statistical review[1]. Table II relates to males and females in Scotland from 1861, taken from McKinlay's report[2], and to both sexes in Scotland from 1861, separately calculated by McKinlay. The figures for both sexes in Table III, relating to Sweden, from 1775, are taken from the *Statistisk Årsbok*[3]; those for the separate sexes have been kindly supplied to us by the Director of the Bureau central de statistique at Stockholm.

In all cases the age groups, 5–14, 15–24, 25–34, 35–44 years, etc., extend over 10 years and for brevity are denoted by their approximately central

[1] *Registrar-General's Statistical Review of England and Wales for* 1931, Tables, Part 1, Medical.

[2] *Seventy-eighth Annual Report of the Registrar General for Scotland*, 1932, p. xl.

[3] *Statistisk Årsbok fŏr Sverige*, 1931.

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Table I. *Specific mortality rates per thousand for males, females, and for both sexes, England and Wales*, 1845–1925.

	1845		1855		1865		1875		1885		1895		1905		1915		1925	
10	7·2 7·2	7·2	6·7 6·8	6·8	6·4 6·4	6·3	5·2 5·0	5·1	4·2 4·2	4·2	3·4 3·5	3·4	2·7 2·9	2·9	2·9 3·0	3·0	2·1 2·0	2·0
20	8·3 8·5	8·4	7·8 8·0	7·9	7·4 7·3	7·3	6·4 6·2	6·3	5·0 5·0	5·0	4·5 4·1	4·2	3·7 3·2	3·4	[4·2] 3·1	3·8	3·0 2·8	2·9
30	9·9 10·6	10·3	9·6 9·9	9·8	9·9 9·7	9·8	9·3 8·6	9·0	7·8 7·4	7·6	6·8 6·1	6·4	5·6 4·7	5·1	[6·5] 4·8	5·5	3·9 3·5	3·7
40	12·9 12·9	12·9	12·5 12·1	12·3	13·5 12·1	12·7	13·8 11·6	12·7	12·4 10·6	11·5	11·5 9·6	10·5	9·2 7·5	8·3	8·6 6·4	7·4	6·4 4·8	5·5
50	18·2 16·0	17·0	18·0 15·2	16·5	19·3 15·6	17·4	20·1 15·6	17·8	19·4 15·1	17·1	18·9 14·7	16·8	16·2 12·5	14·3	14·5 10·9	12·6	11·6 8·6	10·0
60	31·8 28·4	29·9	30·9 27·0	28·9	33·1 27·9	30·4	34·9 28·7	31·6	34·7 28·5	31·4	34·9 28·4	31·5	31·8 24·8	28·1	29·0 21·5	25·0	24·4 18·1	21·1
70	67·5 61·0	63·6	65·3 58·7	61·7	67·1 59·1	62·8	69·7 61·0	65·0	70·5 60·4	65·0	70·4 60·7	65·0	64·8 53·9	58·8	63·5 49·6	55·8	59·2 46·0	51·9
80	148·3 135·9	141·5	146·7 134·5	139·9	147·2 134·9	140·4	150·8 135·4	142·2	146·6 130·6	137·6	146·1 130·6	137·2	137·7 119·8	127·2	139·3 116·3	125·5	136·9 114·0	123·1

Table II. *Specific mortality rates per thousand for males, females, and for both sexes, Scotland*, 1861–1931.

	1861		1871		1881		1891		1901		1911		1921		1931	
10	7·2 7·0	7·1	7·6 7·5	7·6	6·1 6·3	6·0	4·6 4·9	4·8	3·6 3·8	3·8	3·1 3·2	3·1	2·5 2·5	2·5	2·2 2·0	2·1
20	8·7 7·6	8·2	9·2 8·5	8·7	7·1 7·0	7·1	6·5 6·2	6·4	5·4 5·1	5·3	4·0 3·9	3·9	3·5 3·4	3·5	3·0 2·8	2·9
30	10·3 9·6	9·9	11·2 10·4	10·8	8·9 9·2	9·1	8·3 8·8	8·6	8·5 7·3	7·9	5·6 5·5	5·6	4·9 5·0	5·0	3·9 3·9	3·9
40	12·3 11·5	11·9	14·0 12·4	13·1	12·4 11·0	11·7	12·2 11·6	11·9	11·7 10·0	10·8	8·5 8·0	8·2	7·2 6·7	7·0	6·6 5·5	6·0
50	17·2 14·4	15·7	20·0 16·1	17·9	19·4 14·8	16·9	20·8 15·8	18·1	19·5 15·6	17·4	15·6 12·5	14·0	12·7 10·4	11·5	11·6 9·4	10·4
60	29·5 24·9	27·0	32·5 27·2	30·0	33·4 26·6	29·6	37·2 31·6	33·4	37·9 30·5	33·8	31·3 25·2	28·1	27·8 21·9	24·7	24·6 19·7	22·1
70	64·6 56·6	60·1	64·1 54·6	58·7	63·6 53·3	57·7	72·1 61·8	66·1	71·6 60·2	65·0	65·3 52·8	58·1	61·4 48·5	54·2	60·5 47·1	53·1

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Table III. *Specific mortality rates per thousand for males, females, and for both sexes, Sweden*, 1755–1925.

	1755		1765		1775		1785		1795		1805		1815		1825		1835	
10	9·79 9·19	9·49	10·20 9·56	9·88	12·92 11·45	12·21	11·31 10·37	10·85	7·98 7·51	7·75	10·09 9·22	9·66	7·89 7·39	7·64	6·21 5·87	6·04	6·31 5·86	6·09
20	7·98 6·50	7·19	8·10 6·59	7·31	9·44 8·22	8·80	9·34 7·42	8·35	6·70 5·84	6·25	9·29 7·77	8·50	7·63 6·87	7·23	6·54 5·67	6·10	6·59 5·73	6·14
30	10·89 10·18	10·51	11·34 10·32	10·79	12·65 11·66	12·13	13·30 10·39	11·78	9·22 8·96	9·08	11·80 10·33	11·03	10·69 9·23	9·92	10·69 8·26	9·45	11·16 8·64	9·87
40	14·81 12·91	13·82	14·86 13·18	13·98	15·76 15·04	15·38	16·28 13·39	14·78	12·91 12·17	12·53	15·69 14·26	14·93	15·66 13·23	14·37	15·87 11·81	13·72	16·55 12·33	14·35
50	22·61 16·60	19·35	23·42 17·45	20·20	23·89 19·85	21·72	23·35 17·56	20·28	21·07 16·44	18·64	25·64 20·37	22·85	24·56 18·23	21·15	24·69 16·59	20·30	24·94 17·80	21·05
60	36·61 30·13	32·89	38·97 30·17	33·17	38·51 34·35	36·24	38·58 31·82	34·88	37·47 31·36	34·13	45·34 38·66	41·70	42·71 34·63	38·23	41·23 31·68	35·88	41·30 32·27	36·22
70	68·50 64·83	66·30	74·85 68·47	71·08	85·41 79·16	81·79	84·25 77·78	80·58	81·76 75·31	78·12	89·76 80·14	84·21	86·83 77·68	81·57	80·72 69·55	74·70	83·66 73·21	77·51

	1845		1855		1865		1875		1885		1895		1905		1915		1925	
10	6·38 6·12	6·11	8·59 7·84	8·22	6·93 6·54	6·74	6·35 6·33	6·34	5·87 5·89	5·89	4·68 4·89	4·78	3·49 3·74	3·61	3·16 3·28	3·22	1·86 1·84	1·83
20	5·93 5·23	5·56	6·74 5·54	6·12	5·92 4·92	5·40	5·82 4·98	5·37	5·54 4·92	5·21	5·69 5·24	5·47	5·53 5·15	5·34	6·61 5·60	6·11	4·13 3·80	3·96
30	9·02 7·19	8·04	9·25 7·66	8·40	7·80 6·66	7·17	7·87 6·96	7·36	6·79 6·36	6·55	6·67 6·32	6·49	6·13 6·09	6·10	7·43 6·67	7·04	4·40 4·17	4·29
40	14·15 10·55	12·22	13·31 10·86	12·00	10·99 9·27	10·06	10·06 8·64	9·29	8·60 7·84	8·19	8·20 7·57	7·87	7·36 6·99	7·16	7·36 7·00	7·17	5·17 4·99	5·08
50	22·12 14·96	18·21	21·23 15·27	17·99	17·88 13·11	15·30	15·16 11·35	13·12	13·14 10·16	11·54	12·23 9·76	10·91	11·39 9·19	10·22	10·70 8·97	9·78	8·68 7·83	8·24
60	37·08 28·45	32·12	36·50 29·53	32·56	33·33 25·45	29·18	27·11 21·65	24·12	23·87 19·18	21·34	22·37 17·79	19·90	20·70 16·58	18·48	20·11 16·32	18·06	18·09 15·30	16·59
70	80·27 68·53	73·13	76·03 67·50	70·84	71·24 59·75	64·45	61·22 51·80	55·80	53·19 46·33	49·33	50·79 44·27	47·20	46·70 40·27	43·15	46·58 40·62	43·30	43·09 38·44	40·44

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years, 10, 20, 30, 40, etc., respectively. The values are correct up to the significant figures given, except that the 10 group in all three countries has been calculated as the simple average of the 5–9 and the 10–14 groups, and in the case of Sweden the 20, 30, 40, etc. groups have been similarly obtained, each as the average of two 5-yearly age periods. It is not likely, however, that the errors involved by this procedure are important. In the case of England and Wales and of Sweden the calendar years are grouped in 10-yearly periods, in the former 1841–50, 1851–60, etc., and in the latter 1751–60, 1761–70, etc. It is convenient to consider these periods as centred at 1845, 1855, etc., and at 1755, 1765, etc. The figures for Sweden for 1925 were obtained by taking the average of those for the three years 1924, 1925 and 1926, as those for the whole decennium were not readily available. In the case of Scotland each set of figures refers to the average of 3 years centred round the census years 1861, 1871, ..., etc. Thus the 1861 figures are really the average of 1860, 1861, and 1862.

CALCULATION OF CHARACTERISTICS OF MORTALITIES

The method of calculation employed in the previous paper consisted in assuming that the early columns of the tables represented the population in a steady state, so that the death-rate $f(t, \theta) = \alpha(t-\theta)\beta_\theta$ is equal to $\alpha_0\beta_\theta$, since the assumption that a steady state exists implies that at this time $\alpha(t-\theta)$ is a constant. We chose arbitrarily the first decennial period given in the tables, which is in fact the first 10-yearly period for which reliable figures are available, for our set of β_θ values. From the values of β_θ so obtained (the arbitrary constant can be taken at will) a set of relative mortality rates was obtained by dividing the observed specific death-rates by the value of β_θ corresponding to the particular age in question. These represent estimates of $\alpha(t-\theta)$, and would be constant along any diagonal for which $t = \theta + \text{constant}$, if the fundamental law $f(t, \theta) = \alpha(t-\theta)\beta_\theta$ were strictly fulfilled. It was actually found that the value of α along any diagonal varied somewhat, but in an apparently random manner, hence the mean value along the diagonal was calculated and chosen as representing the value of $\alpha(t-\theta)$ characteristic of the diagonal, that is, of the year of birth $(t-\theta)$. It is clear that this process could not be properly applied unless the first column of the table in question referred to a period such that for a considerable time previously the population had been in a steady state. It is, however, not difficult to see that in the absence of a steady state a set of β_θ values could be chosen by using a diagonal and not a column as a standard, provided we assume that the law $f(t, \theta) = \alpha(t-\theta)\beta_\theta$ is correct. This follows because, along a diagonal, $\alpha(t-\theta)$ is constant. Furthermore, any diagonal can be taken. We can thus make use of practically all the data in calculating a set of β_θ values, and we are not limited to one or at most a few columns at the beginning of the table. Once the best set of β_θ values has been found it may be employed to calculate the $\alpha(t-\theta)$ figures as previously.

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The β_θ figures involve an arbitrary constant, that is to say, only the ratios are significant. We therefore attempt to find the best values of $\frac{\beta_{20}}{\beta_{10}}, \frac{\beta_{30}}{\beta_{20}}, \ldots$, etc. We notice that assuming (1*a*), $\frac{f(t, \theta)}{f(t-10, \theta-10)} = \frac{\beta_\theta}{\beta_{\theta-10}}$; thus, for example, the calculated value of $\frac{f(1875, 20)}{f(1865, 10)}$ gives an estimate of $\frac{\beta_{20}}{\beta_{10}}$. By taking different values of t and the same value of θ, that is, going along a row, we obtain a set of estimates of $\frac{\beta_\theta}{\beta_{\theta-10}}$, and the average of those is taken as a suitable value of the ratio. When this process is applied to all the rows of the table in successive pairs, the whole set of values of β_θ is obtained, and β_θ is completely defined apart from an arbitrary multiplier. In the case of the English figures the data referring to the 10, 20, and 30 age groups for the decade centred at 1915 were omitted as they showed an obvious abnormality presumably due, directly or indirectly, to the war. It is to be remembered that the great pandemic of influenza occurred during this period. This method seems to be free from the arbitrariness of that previously employed, but it was not considered advisable to use it in the previous paper, as the employment of the diagonals for the purpose of calculating the values of β_θ might lead to the suspicion that the final result, namely, the constancy of α along the diagonals, was nothing more than an arithmetical artefact, dependent upon the use of the diagonals in the calculation. As, however, the essential correctness of the diagonal law has been demonstrated by the previous method there would seem to be no objection to the employment of this more refined process in working out the consequences of the law.

The values of β_θ so obtained are given in Table IV, in which β_{10} is given the arbitrary value of unity.

Table IV. *Values of β_θ ($\beta_{10}=1$) for males, females, and for both sexes, in England and Wales, in Scotland and in Sweden.*

	England and Wales			Scotland			Sweden		
	M.	F.	Both sexes	M.	F.	Both sexes	M.	F.	Both sexes
10	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00
20	1·05	1·00	1·03	1·13	1·05	1·05	0·87	0·80	0·83
30	1·31	1·17	1·26	1·31	1·26	1·28	1·23	1·14	1·18
40	1·82	1·45	1·66	1·66	1·46	1·55	1·71	1·53	1·60
50	2·66	1·92	2·31	2·55	1·94	2·22	2·51	1·98	2·21
60	4·66	3·39	4·06	4·58	3·58	4·01	4·19	3·55	3·79
70	9·51	7·08	8·34	9·22	7·26	8·06	8·66	8·03	8·18
80	20·35	15·51	18·00	—	—	—	—	—	—

When the set of β's has been obtained the calculation of the values of α is quite straightforward. The method is in fact exactly the same as that employed in the preliminary paper except that the more correct values of β_θ are used. Each specific mortality rate is divided by the corresponding β value, and in this way a series of numbers which are approximately constant along any

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one diagonal array is obtained. The mean of the figures along any particular diagonal array is taken as the value of α $(t-\theta)$ where $(t-\theta)$ is the year of birth of the generation which is characteristic of the diagonal array.

In Table V the $\alpha \times 10^3$ values for England and Wales and for Scotland are given for males and females separately. In the case of England the data referring to the war years are excluded as before.

The value of the specific mortality for age θ at time t may be calculated from Tables IV and V. For example, in the case of English males

$$f(1915, 60) = \alpha(1855) \times \beta(60) = 6{\cdot}2 \times 4{\cdot}66 \text{ per thousand} = 28{\cdot}9.$$

In Table I the actual value is 29·0 per thousand: similarly

$$f(1925, 40) = \alpha(1885) \times \beta(40) = 3{\cdot}5 \times 1{\cdot}82 = 6{\cdot}37,$$

and by Table I it is 6·4, and .

$$f(1875, 30) = \alpha(1845) \times \beta(30) = 6{\cdot}9 \times 1{\cdot}31 = 9{\cdot}04,$$

and by Table I it is 9·3.

These examples, which are selected at random, are typical.

ERRORS OF α AND β VALUES

It is desirable to get some measure of the probable errors of the α and β values calculated by the above process. As it is only the ratios which are of significance, we shall first of all attempt to determine the probable errors of the ratios of the β values.

We shall first consider the following system containing n rows and l columns, each column being characterised by a particular α, and each row by a particular β, so that the expected number in the kth row and the sth column is ${}_k\beta_s . \alpha_s$. The observed values, however, deviate from the calculated.

	1	2	3	4 … s … l
1				
2				
3				
⋮				
k				
⋮				
n				

We shall, in the first instance, assume that the α values are constant, and that the fluctuations along any row are due to variations in the β's. As a reasonable approximation we shall assume that

$\frac{\sigma_{{}_k\beta_s}}{{}_k\bar{\beta}} = \pi$ for all values of k (where ${}_k\bar{\beta}$ is the mean of the β's in the kth row).

It follows that $\frac{\sigma_{{}_k\bar{\beta}}}{{}_k\bar{\beta}} = \frac{\pi}{\sqrt{l}}$. We cannot find ${}_k\beta_s$ itself but we can find $\frac{{}_{k+1}\beta_s}{{}_k\beta_s} = {}_k\gamma_s$, and thus we can calculate the mean value ${}_k\bar{\gamma}$ $(k=1, 2, \ldots, n)$.

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Table V. *Values of $\alpha \times 10^3$ for males, females, and for both sexes, for England and Wales, Scotland, and Sweden, along with their percentage errors.*

	Males		Females		Both sexes	
	$\alpha \times 10^3$	Percentage error	$\alpha \times 10^3$	Percentage error	$\alpha \times 10^3$	Percentage error
			England and Wales			
1765	7·30	2·8	8·8	2·9	7·9	2·0
1775	7·05	2·0	8·7	2·1	7·7	1·5
1785	6·97	1·6	8·5	1·7	7·5	1·2
1795	6·97	1·4	8·4	1·5	7·5	1·1
1805	7·1	1·3	8·4	1·3	7·6	1·0
1815	7·3	1·2	8·5	1·2	7·8	0·9
1825	7·4	1·1	8·3	1·1	7·7	0·8
1835	7·3	1·0	7·9	1·0	7·5	0·8
1845	6·9	1·0	7·3	1·0	7·0	0·8
1855	6·2	1·1	6·4	1·1	6·2	0·8
1865	5·2	1·2	5·3	1·2	5·1	0·9
1875	4·4	1·3	4·2	1·3	4·2	1·0
1885	3·5	1·4	3·3	1·5	3·3	1·2
1895	2·9	1·6	3·0	1·7	2·9	1·5
1905	2·9	2·0	2·9	2·1	2·8	2·0
1915	2·1	2·8	2·0	2·9	2·0	2·0
			Scotland			
1791	7·0	3·8	7·8	3·5	7·5	3·9
1801	6·7	2·7	7·3	2·5	7·0	2·8
1811	6·9	2·2	7·4	2·0	7·2	2·3
1821	7·6	1·9	8·0	1·8	7·9	2·0
1831	8·0	1·7	8·2	1·6	8·1	1·8
1841	7·9	1·6	7·8	1·4	7·9	1·6
1851	7·2	1·5	7·4	1·2	7·1	1·4
1861	6·6	1·5	6·7	1·2	6·7	1·4
1871	5·7	1·6	5·8	1·4	5·7	1·6
1881	4·5	1·7	4·7	1·6	4·7	1·8
1891	3·7	1·9	3·8	1·8	3·9	2·0
1901	3·1	2·2	3·2	2·0	3·4	2·3
1911	2·6	2·7	2·6	2·5	2·7	2·8
1921	2·2	3·8	2·0	3·5	2·1	3·9
			Sweden			
1685	7·9	4·9	8·1	4·8	8·1	4·5
1695	8·7	3·6	8·5	3·6	8·7	3·3
1705	9·4	2·9	8·9	2·9	9·2	2·7
1715	9·2	2·5	9·2	2·5	9·3	2·4
1725	9·2	2·3	9·2	2·2	9·3	2·1
1735	9·4	2·1	9·1	2·1	9·3	1·9
1745	9·7	1·8	9·3	1·8	9·6	1·7
1755	9·9	1·8	9·4	1·8	9·7	1·7
1765	10·0	1·8	9·3	1·8	9·7	1·7
1775	9·5	1·8	8·8	1·8	9·2	1·7
1785	9·2	1·8	8·4	1·8	8·8	1·7
1795	9·0	1·8	8·1	1·8	8·6	1·7
1805	8·0	1·8	7·2	1·8	7·7	1·7
1815	6·9	1·8	6·4	1·8	6·7	1·7
1825	6·2	2·1	5·9	2·1	6·1	1·9
1835	5·7	2·3	5·4	2·2	5·5	2·1
1845	5·1	2·5	5·0	2·5	5·1	2·4
1855	4·8	2·5	4·8	2·5	4·8	2·4
1865	4·3	2·9	4·5	2·9	4·4	2·7
1875	3·9	3·6	4·3	3·6	4·1	3·3
1885	3·0	4·9	3·3	4·8	3·2	4·5

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From this set of values of $\bar{\gamma}$, we can then, if we fix on some arbitrary constant β_1, determine a set of numbers ${}_k\bar{\beta}=\beta_1 \cdot {}_1\bar{\gamma}\cdot{}_2\bar{\gamma}\cdot\ldots\cdot{}_{k-1}\bar{\gamma}$, which give a suitable estimate of the values of the β's. It is to be noted that the ${}_k\beta_s$'s, and the ${}_k\bar{\beta}$'s have only a theoretical significance, whereas the ${}_k\bar{\beta}$'s can be calculated from the data, apart from the arbitrary multiplier β_1. We require an estimate of the probable errors of the n $\bar{\beta}$'s in terms of the standard deviations of the ${}_k\beta_s$'s, that is to say, of π.

Now
$$\log {}_k\gamma_s=\log {}_{k+1}\beta_s-\log {}_k\beta_s,$$

so that
$$\frac{\Delta_k\gamma_s}{{}_k\bar{\gamma}}=\frac{\Delta_{k+1}\beta_s}{{}_{k+1}\bar{\beta}}-\frac{\Delta_k\beta_s}{{}_k\bar{\beta}},$$

whence
$$\sum_{s=1}^{l}\frac{\left(\frac{\Delta_k\gamma_s}{{}_k\bar{\gamma}}\right)^2}{l(l-1)}=\frac{1}{l(l-1)}\sum_{s=1}^{l}\left\{\left(\frac{\Delta_{k+1}\beta_s}{{}_{k+1}\bar{\beta}}\right)^2+\left(\frac{\Delta_k\beta_s}{{}_k\bar{\beta}}\right)^2-2\frac{\Delta_{k+1}\beta_s}{{}_{k+1}\bar{\beta}}\frac{\Delta_k\beta_s}{{}_k\bar{\beta}}\right\},$$

hence
$$\frac{\sigma^2{}_{k\bar{\gamma}}}{{}_k\bar{\gamma}^2}=\frac{1}{l}\left(\frac{\sigma^2{}_{k+1}\beta_s}{{}_{k+1}\bar{\beta}^2}+\frac{\sigma^2{}_k\beta_s}{{}_k\bar{\beta}^2}\right)$$
$$=\frac{2\pi^2}{l} \qquad \ldots\ldots(2).$$

Further
$$\log {}_{k+1}\gamma_s=\log {}_{k+2}\beta_s-\log {}_{k+1}\beta_s,$$

whence
$$\sum_{s=1}^{l}\frac{\frac{\Delta_k\gamma_s}{{}_k\bar{\gamma}}\cdot\frac{\Delta_{k+1}\gamma_s}{{}_{k+1}\bar{\gamma}}}{l(l-1)}=-\sum_{1}^{l}\frac{\frac{\Delta^2{}_{k+1}\beta_s}{{}_{k+1}\bar{\beta}^2}}{l(l-1)},$$

the other terms being omitted as, where summed, they are equal to zero provided that the β's are uncorrelated, hence

$$r_{{}_{k+1}\gamma,{}_k\gamma}\frac{\sigma_{k\bar{\gamma}}}{{}_k\bar{\gamma}}\frac{\sigma_{k+1\bar{\gamma}}}{{}_{k+1}\bar{\gamma}}=-\left(\frac{\sigma_{k+1\bar{\beta}}}{{}_{k+1}\bar{\beta}}\right)^2$$
$$=-\frac{\pi^2}{l}.$$

Thus from (2)
$$r=-\tfrac{1}{2} \qquad \ldots\ldots(3).$$

By definition
$$\frac{{}_{k+1}\bar{\beta}}{{}_k\bar{\beta}}={}_k\bar{\gamma},$$

hence
$$\frac{\sigma\left(\frac{{}_{k+1}\bar{\beta}}{{}_k\bar{\beta}}\right)}{\frac{{}_{k+1}\bar{\beta}}{{}_k\bar{\beta}}}=\frac{\sigma_{k\bar{\gamma}}}{{}_k\bar{\gamma}}=\pi\sqrt{\frac{2}{l}}\text{ (by (2) above)} \qquad \ldots\ldots(4).$$

Again
$$\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}=\frac{{}_{k+2}\bar{\beta}}{{}_{k+1}\bar{\beta}}\cdot\frac{{}_{k+1}\bar{\beta}}{{}_k\bar{\beta}}$$
$$={}_{k+1}\bar{\gamma}\,{}_k\bar{\gamma},$$

thus
$$\log\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}=\log {}_{k+1}\bar{\gamma}+\log {}_k\bar{\gamma},$$

whence
$$\frac{\sigma^2\left(\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}\right)}{\left(\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}\right)^2}=\frac{\sigma^2_{k+1\bar{\gamma}}}{{}_{k+1}\bar{\gamma}^2}+\frac{\sigma^2_{k\bar{\gamma}}}{{}_k\bar{\gamma}^2}+2r_{k+1\bar{\gamma},k\bar{\gamma}}\frac{\sigma_{k+1\bar{\gamma}}}{{}_{k+1}\bar{\gamma}}\frac{\sigma_{k\bar{\gamma}}}{{}_k\bar{\gamma}}$$
$$=\frac{4\pi^2}{l}\left(1+r_{k+1\bar{\gamma},\,k\bar{\gamma}}\right).$$

But it may be shown that
$$r_{k+1\bar{\gamma},\,k\bar{\gamma}}=r_{k+1\gamma,\,k\gamma},$$
therefore
$$r_{k+1\bar{\gamma},\,k\bar{\gamma}}=-\tfrac{1}{2} \qquad \text{......(5)},$$
and finally
$$\frac{\sigma^2\left(\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}\right)}{\left(\frac{{}_{k+2}\bar{\beta}}{{}_k\bar{\beta}}\right)^2}=\frac{2\pi^2}{l}.$$

It may readily be shown that
$$\frac{\sigma^2\left(\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}\right)}{\left(\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}\right)^2}=\frac{2\pi^2}{l},$$
or
$$\frac{\sigma\left(\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}\right)}{\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}}=d \qquad \text{......(6)},$$
where $d=\pi\sqrt{\frac{2}{l}}$.

d is therefore a measure of the uncertainty of the ratio of any two of the $\bar{\beta}$'s, and, as only the ratios of the $\bar{\beta}$'s are of any significance, d is an appropriate measure of the uncertainty of the $\bar{\beta}$'s.

Normalisation of β and α values

The problem, however, arises of comparing two sets of $\bar{\beta}$'s, in order to ascertain whether they differ significantly from each other. This might be done by comparing the two sets of $\bar{\gamma}$'s, that is, of the $\frac{{}_{k+1}\bar{\beta}}{{}_k\bar{\beta}}$'s.

As, however, we often find it convenient to plot the set of values of $\bar{\beta}$, there is some advantage in introducing a multiplying factor designed to make the $\bar{\beta}$'s of any two sets directly comparable. For this purpose the most convenient factor appears to be
$${}_1\beta'=({}_1\bar{\beta}\cdot{}_2\bar{\beta}\cdot{}_3\bar{\beta}\cdot\ldots\cdot{}_n\bar{\beta})^{-1/n} \qquad \text{......(7)}.$$

We can then devise the set of *normalised* $\bar{\beta}$'s, namely ${}_1\beta'$, ${}_2\beta'$, etc., up to ${}_n\beta'$, where ${}_k\beta'={}_1\beta'\cdot{}_k\bar{\beta}$. This process may be briefly justified from the following considerations. The assumption made above that $\frac{\sigma_{k\beta_s}}{{}_k\beta}=\pi$, implies

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that $\sigma(\log {}_k\beta_s)$ is equal to a constant, so that on the logarithmic scale the probable errors of the β's and therefore of the $\tilde{\beta}$'s are of equal magnitude. The introduction of the normalising factor is equivalent on the logarithmic scale to altering the origin in such a manner that the mean value is equal to zero. It is therefore the solution obtained by the method of least squares applied to the logarithms, assuming equal weights at all points of the logarithmic scale.

It is to be observed that this normalising factor cannot have any absolute significance attached to it, as it has been derived from a limited part of the β curve, namely, from 5 to 75 years of age. It would doubtless be improved, if instead of the simple mean of the logarithms being taken, a weighted mean were substituted, the weights being related to the average size of the population in different age periods. As in practice, however, the relative age distributions of the populations are found to have altered very considerably during the years covered by the tables, at best some arbitrary mean age distribution would require to be assumed. It seems doubtful, however, whether the alteration made in the normalising factor in this way would be significantly different from that obtained by taking the simple mean.

We shall now calculate the probable errors of the normalised β values.

$$\frac{1}{{}_k\beta'}=\left(\frac{{}_1\tilde{\beta}}{{}_k\tilde{\beta}}\cdot\frac{{}_2\tilde{\beta}}{{}_k\tilde{\beta}}\cdot\ldots\cdot\frac{{}_{k-1}\tilde{\beta}}{{}_k\tilde{\beta}}\cdot 1\cdot\frac{{}_{k+1}\tilde{\beta}}{{}_k\tilde{\beta}}\cdot\ldots\cdot\frac{{}_n\tilde{\beta}}{{}_k\tilde{\beta}}\right)^{1/n}$$

$$=\left\{\frac{1}{{}_1\bar{\gamma}\cdot{}_2\bar{\gamma}\cdot\ldots\cdot{}_{k-1}\bar{\gamma}}\times\frac{1}{{}_2\bar{\gamma}\cdot{}_3\bar{\gamma}\cdot\ldots\cdot{}_{k-1}\bar{\gamma}}\times\ldots\frac{1}{{}_{k-1}\bar{\gamma}}\times 1\times{}_k\bar{\gamma}\times{}_k\bar{\gamma}\cdot{}_{k+1}\bar{\gamma}\times\ldots\right.$$
$$\left.\times\left({}_k\bar{\gamma}\cdot{}_{k+1}\bar{\gamma}\cdot\ldots\cdot{}_{n-1}\bar{\gamma}\right)\right\}^{1/n},$$

therefore

$$-\log{}_k\beta'=\frac{1}{n}\{(\log{}_{n-1}\bar{\gamma}+2\log{}_{n-2}\bar{\gamma}+3\log{}_{n-3}\bar{\gamma}+\ldots(n-k)\log{}_k\bar{\gamma})$$
$$-(\log{}_1\bar{\gamma}+2\log{}_2\bar{\gamma}+3\log{}_3\bar{\gamma}+\ldots(k-1)\log{}_{k-1}\bar{\gamma})\}.$$

Thus

$$-\frac{\Delta_k\beta'}{{}_k\beta'}=\frac{1}{n}\left(\frac{\Delta_{n-1}\bar{\gamma}}{{}_{n-1}\bar{\gamma}}+2\frac{\Delta_{n-2}\bar{\gamma}}{{}_{n-2}\bar{\gamma}}+\ldots(n-k)\frac{\Delta_k\bar{\gamma}}{{}_k\bar{\gamma}}-\frac{\Delta_1\bar{\gamma}}{{}_1\bar{\gamma}}-2\frac{\Delta_2\bar{\gamma}}{{}_2\bar{\gamma}}-\ldots(k-1)\frac{\Delta_{k-1}\bar{\gamma}}{{}_{k-1}\bar{\gamma}}\right),$$

whence $$\frac{\sigma^2{}_{k\beta'}}{{}_k\beta'}=\frac{1}{n^2}[\{1^2+2^2+3^2+\ldots(n-k)^2+1^2+2^2+3^2+\ldots(k-1)^2\}d^2$$
$$-\{1.2+2.3+3.4+\ldots(n-k-1)(n-k)+1.2+2.3+\ldots(k-2)(k-1)$$
$$+(k-1)(n-k)\}d^2]$$

(since $r_{s\bar{\gamma},t\bar{\gamma}}=-\frac{1}{2}$ when $|s-t|=1$, and $=0$ when $|s-t|>1$)

$$=\frac{d^2}{n^2}[\{1+2+3+\ldots(n-k)\}+\{1+2+3+\ldots(k-1)\}+(k-1)(n-k)]$$

$$=\frac{d^2}{n^2}\left\{\frac{(n-k)(n-k+1)}{2}+\frac{k(k-1)}{2}+n(k-1)-k(k-1)\right\}$$

$$=\left(1-\frac{1}{n}\right)\frac{d^2}{2}.$$

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But $d=\frac{2\pi^2}{l}$, hence $\frac{\sigma_{k\beta'}}{{}_k\beta'}=\pi\sqrt{\frac{(1-1/n)}{l}}$.

In practice we calculate $$\frac{\sigma_{k\bar{\gamma}}}{{}_k\bar{\gamma}}=\sqrt{\frac{2}{l}}\,\pi=d,$$

so that $$\frac{\sigma\left(\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}\right)}{\frac{{}_s\bar{\beta}}{{}_t\bar{\beta}}}=d \quad (t\neq s)$$

and $$\frac{\sigma_{k\beta'}}{{}_k\beta'}=d\sqrt{\frac{(1-1/n)}{2}} \qquad \text{......(8).}$$

In the particular problem under consideration two modifications of the above theory are necessary. (1) The rows are sometimes incomplete and therefore l is not constant for all rows. (2) Two members at least of each row are usually omitted in virtue of the overlap arising from the diagonal arrangement. The effect of this second perturbation is rather difficult to calculate, but it seems clear that unless the number of terms in the row is very small, its effect can be neglected, provided that in place of l we take λ equal to the number of ratios actually employed in calculating $\bar{\gamma}$ from the two rows. As the probable errors are themselves subject to considerable random fluctuation, it seems justifiable to neglect the influences of these two disturbing conditions. Further, if a particular row contains λ instead of l γ's, where $l-\lambda$ is small, then the value of $\frac{\sigma_{k\beta'}}{{}_k\beta'}$ to be used for that row would be $\pi\sqrt{\frac{(1-1/n)}{\lambda}}$.

In this case the best plan would be to calculate the value of $\frac{\sigma_{k\gamma_s}}{{}_k\bar{\gamma}}$ for each row (equal to ${}_k y$, say), then if $\bar{y}$ be the average of the ${}_k y$'s,

$$\frac{\sigma_{k\beta'}}{{}_k\beta'}=\bar{y}\sqrt{\frac{(1-1/n)}{2\lambda}} \qquad \text{......(9).}$$

This formula is not necessarily absolutely accurate, but it is probably not much in error.

In Table VI are given the normalised β' values along with their probable errors in the cases of England and Wales, and of Scotland, calculated as above. Comparison of the differences of the corresponding β''s, along with their probable errors, shows that the values for the females in England and Wales on the one hand, and in Scotland on the other, show no definitely significant differences—with the possible exception of the 20 and 40 age groups, the same is true for the males of these countries—whereas there is a significant difference between the values for the two sexes in either country.

In the case of the Swedish figures inspection of the values of $\frac{\sigma_\gamma}{\bar{\gamma}}$ indicates that this factor decreases steadily with increasing age, so that it would seem that the fundamental assumption in sections 2 and 3, namely that $\frac{\sigma_\beta}{\bar{\beta}}$ is

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independent of the age, is not fulfilled. This may be associated with the existence of the obvious disturbance in the lower age groups. The situation is further complicated by the fact that the number of ratios available in the lower age groups is much smaller than in the higher ones. For these reasons we have omitted the probable errors for the Swedish figures in Table VI, but have given the values of $\frac{\sigma_\gamma}{\bar{\gamma}}$ and also of $100 \times 0{\cdot}6745 \frac{\sigma_{\bar{\gamma}}}{\bar{\gamma}}$ in Table VII. The normalised values are charted for males in Fig. 1 and for females in Fig. 2.

Table VI. *"Normalised" values of β with percentage errors.*

	England and Wales	Scotland	Sweden
		Males	
10	0·444 (0·83)	0·451 (1·56)	0·482
20	0·466 (0·90)	0·510 (1·56)	0·419
30	0·582 (0·83)	0·589 (1·56)	0·592
40	0·808 (0·78)	0·749 (1·56)	0·822
50	1·180 (0·78)	1·150 (1·56)	1·209
60	2·067 (0·78)	2·065 (1·56)	2·018
70	4·200 (0·78)	4·158 (1·56)	4·172
		Females	
10	0·539 (0·86)	0·520 (1·72)	0·536
20	0·539 (0·86)	0·546 (1·72)	0·429
30	0·630 (0·86)	0·655 (1·72)	0·609
40	0·781 (0·81)	0·759 (1·72)	0·819
50	1·034 (0·81)	1·009 (1·72)	1·063
60	1·772 (0·81)	1·861 (1·72)	1·903
70	3·814 (0·81)	3·774 (1·72)	4·306
		Both sexes	
10	0·481 (0·82)	0·499 (1·27)	0·513
20	0·496 (0·82)	0·514 (1·27)	0·426
30	0·607 (0·76)	0·626 (1·27)	0·604
40	0·798 (0·71)	0·758 (1·27)	0·822
50	1·109 (0·71)	1·086 (1·27)	1·134
60	1·951 (0·71)	1·961 (1·27)	1·940
70	4·010 (0·71)	3·942 (1·27)	4·191

Table VII. *Sweden, values of $\bar{\gamma}$ and percentage probable errors.*

	Males			Females			Both sexes		
	$\bar{\gamma}$	$\sigma_\gamma/\bar{\gamma}$	$67{\cdot}45\ \sigma_{\bar{\gamma}}/\bar{\gamma}$	$\bar{\gamma}$	$\sigma_\gamma/\bar{\gamma}$	$67{\cdot}45\ \sigma_{\bar{\gamma}}/\bar{\gamma}$	$\bar{\gamma}$	$\sigma_\gamma/\bar{\gamma}$	$67{\cdot}45\ \sigma_{\bar{\gamma}}/\bar{\gamma}$
10	0·870	0·200	4·5	0·801	0·191	4·3	0·833	0·164	4·4
20	1·420	0·173	3·9	1·418	0·173	3·9	1·420	0·231	3·7
30	1·382	0·141	3·0	1·344	0·116	2·5	1·361	0·168	2·6
40	1·469	0·124	2·0	1·299	0·115	1·9	1·381	0·147	1·7
50	1·670	0·085	1·4	1·790	0·103	1·7	1·713	0·157	1·5
60	2·067	0·069	1·1	2·263	0·083	1·4	2·162	0·169	1·3

The calculation of the probable errors of the α's is quite straightforward. Each $\bar{\alpha}$ is obtained as the mean of a set of α's, found by dividing $f(t, \theta)$ by ${}_\theta\beta'$. σ_{α_s} (where s is written for $t-\theta$) may then be calculated in the usual way. It is probably best to calculate the values of $\frac{\sigma_{\alpha_s}}{\bar{\alpha}_s}$, and then to take the mean (weighted if necessary) $= w$. Then $\frac{\sigma_{\bar{\alpha}_s}}{\bar{\alpha}_s} = \frac{w}{\sqrt{m}}$, where m is the number of values

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from which $\bar{\alpha}$ was calculated. If the $\tilde{\beta}$'s are multiplied by the normalising factor so as to get the normalised β's, it is necessary to divide the α's by the same factor, so that the product of α and β remains unchanged. This, however, does not modify the value $\frac{\sigma_{\bar{\alpha}_s}}{\bar{\alpha}_s}$. The values of $67{\cdot}45\,\frac{\sigma_{\bar{\alpha}_s}}{\bar{\alpha}_s}$ are given in Table V.

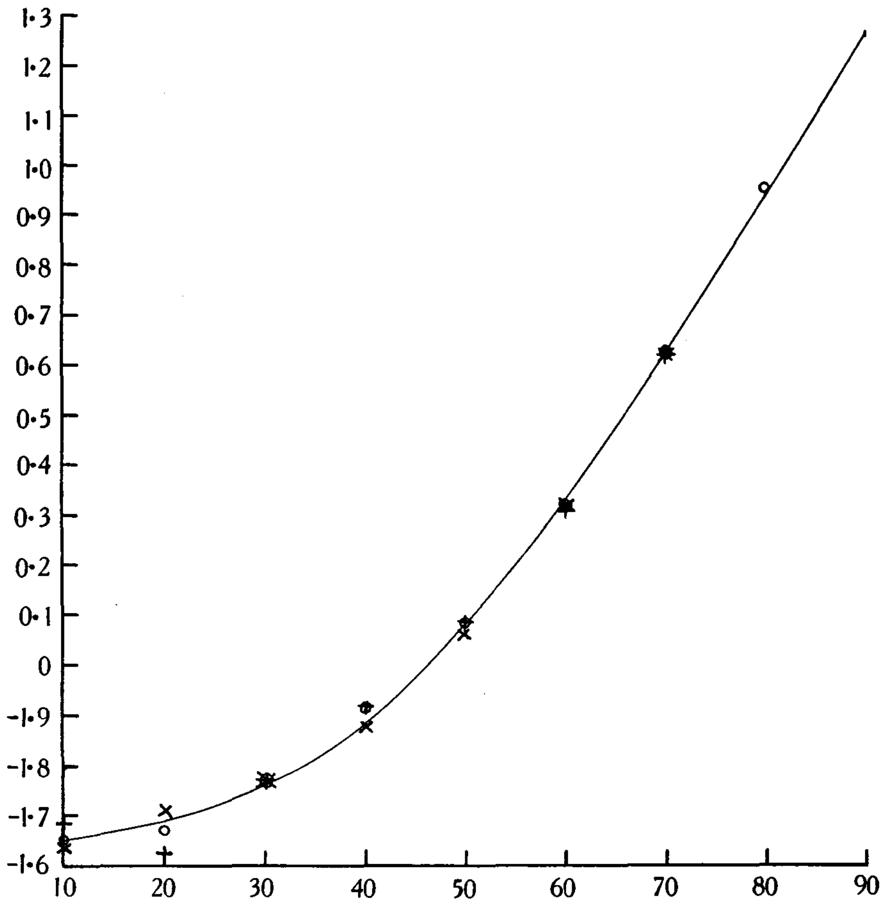

Fig. 1. Logarithms of normalised β values for males of England and Wales, Scotland, and Sweden. The full line is calculated by the Makeham-Gompertz formula.

○ = β value for England and Wales extracted from the data,
× = ,, ,, Scotland extracted from the data,
+ = ,, ,, Sweden extracted from the data.

The values of α—after normalisation—are charted in Fig. 3 for males and in Fig. 4 for females.

Some consequences of the above hypothesis

Although a knowledge of the α and β values gives a complete specification of the progress of any particular group of persons born in a certain year, the complete meaning of the figures may not be intuitively obvious. It is therefore

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of some utility to consider what happens if we begin with a population containing a large number n of children and observe their progress when they are under the influence of a series of specific death-rates such as we have been considering. We may then keep the β_θ values constant and allow the α factor to vary and observe how the course of affairs is altered when this takes place.

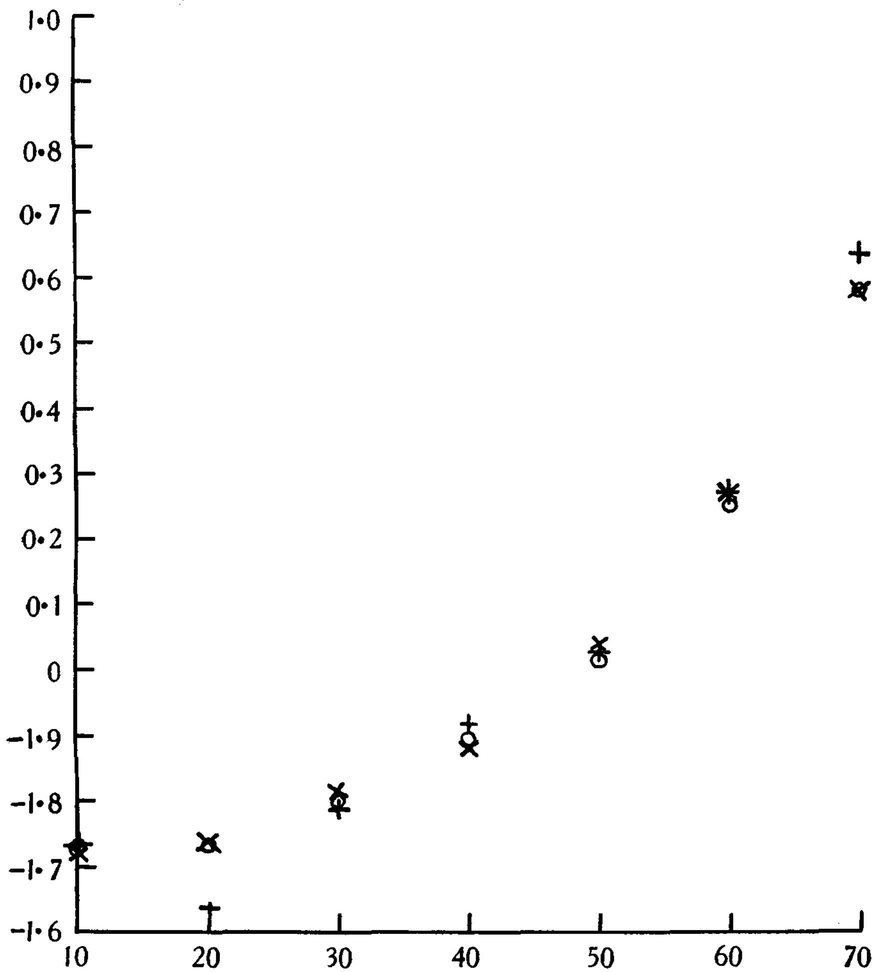

Fig. 2. Logarithms of normalised β values for females of England and Wales, Scotland, and Sweden.

○ = β value for England and Wales extracted from the data,
× = „ „ Scotland extracted from the data,
+ = „ „ Sweden extracted from the data.

It is then possible to see at what age an alteration in α produces the greatest effect. To do this conveniently, however, it is desirable to express the set of β's in terms of some convenient formula containing only a few constants. Otherwise the arithmetical labour involved becomes very great. We have therefore examined the various sets of β's obtained above to find out whether

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any of them fitted a Gompertz or other formula. We have found that the Scottish and English males which, as previously remarked, do not differ significantly from one another, can both be fitted to the Makeham-Gompertz formula,

$$\beta'_\theta = A + B_{10} e^{c(\theta-10)} \qquad \text{......(10),}$$

where $A=0{\cdot}93$, $B=0{\cdot}07305$, and $c=0{\cdot}07907$. The fitting was carried out in part empirically, and, although not necessarily absolutely the best fit, it is

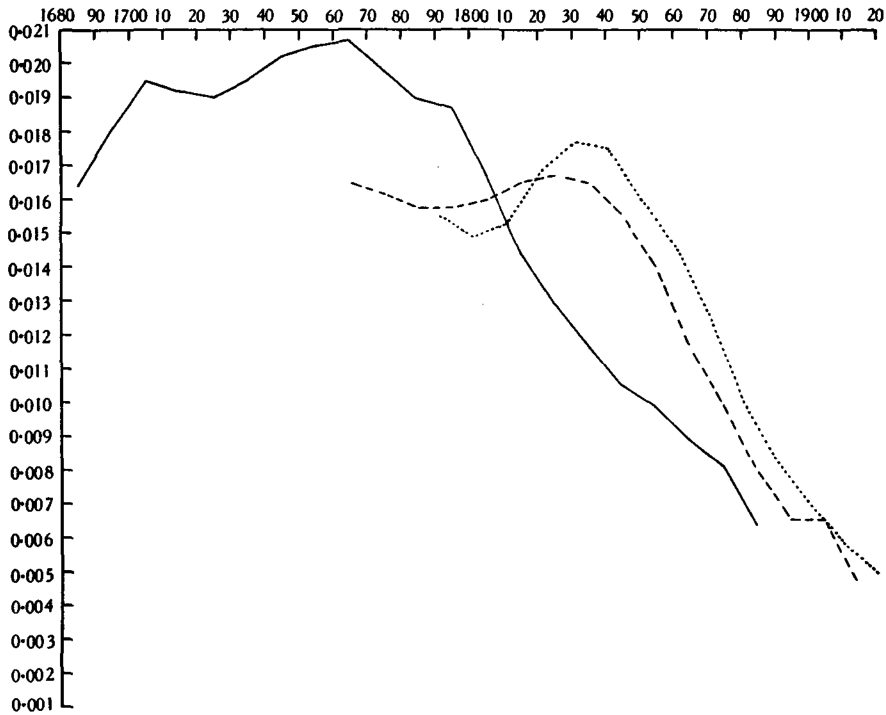

Fig. 3. Curves of normalised α values for males of England and Wales, Scotland, and Sweden.
- - - - - - - - = England and Wales,
. . . . = Scotland,
———— = Sweden.

sufficiently good for the present purpose, as shown by Table VIII, in which to facilitate comparison the normalised figures are given.

If we assume the above value for β'_θ then $-\dfrac{1}{v_\theta}\dfrac{dv_\theta}{d\theta} = \alpha(A + B_{10}e^{c(\theta-10)})$, where v_θ is the number of survivors of age θ, whence

$$v_\theta = v_0 e^{-\alpha\left\{A(\theta-10) + \frac{B_{10}}{C}\left(e^{c(\theta-10)}-1\right)\right\}} \qquad \text{......(11),}$$

and $$z\,d\theta = -\frac{dv}{d\theta}\,d\theta = \alpha\,(A + B_{10}e^{c(\theta-10)})\,v_\theta\,d\theta \qquad \text{......(12)}$$

gives the number who die aged between θ and $\theta + d\theta$.

We take $v_0 = 1$, and assume that the population begins at age 10, since the

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β formula does not accommodate the infantile death-rate. v_θ and z are then expressed as fractions of the total population at age 10. Fig. 5 shows v_θ

Table VIII. *Comparison of "normalised" values of β calculated by the Makeham-Gompertz formula with those of Table VI (males) (see also Fig.* 3).

	Calculated by M.-G.	Males: England and Wales	Scotland	Sweden
10	0·447	0·444	0·451	0·482
20	0·487	0·466	0·510	0·419
30	0·577	0·582	0·589	0·592
40	0·764	0·808	0·749	0·822
50	1·184	1·180	1·150	1·109
60	2·118	2·067	2·065	2·018
70	4·161	4·220	4·158	4·171
80	8·710	9·029	—	—
90	18·640	—	—	—

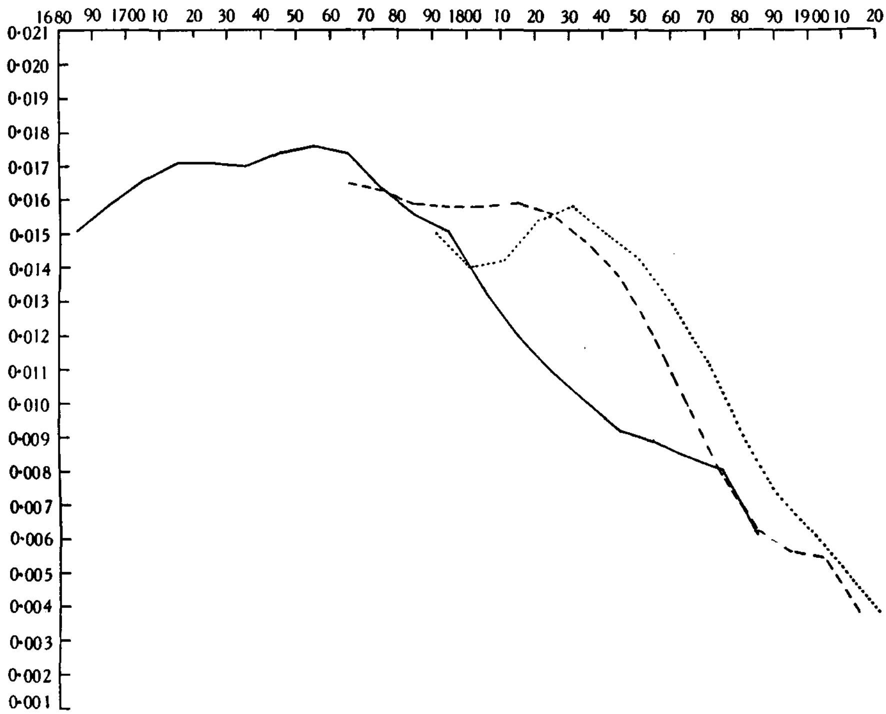

Fig. 4. Curves of normalised α values for females of England and Wales, Scotland, and Sweden.

- - - - - - - - = England and Wales,
. . . . = Scotland,
———— = Sweden.

calculated for $\alpha = 0{\cdot}001$, $0{\cdot}002$ ... $0{\cdot}008$, $0{\cdot}016$, $0{\cdot}0213$, $0{\cdot}032$, whilst in Fig. 6, z_θ has been calculated over a similar range of values. Fig. 6 is of special interest as it shows that when $\alpha = 0{\cdot}007$ or $0{\cdot}008$ (a situation which existed in England and Scotland for those born in the earlier parts of last century), the

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incidence of death was maximum at an age of about 70, whereas for $\alpha = 0{\cdot}002$—the level apparently reached by those born in 1915—the maximum incidence occurs at 89 years. Further, the actual height of the maximum gradually rises with decreasing values of α. It may be shown that the locus of the maxima and minima for various values of α is given by

$$y = \frac{B_{10}\, c\, e^{c(\theta-10)}}{A + B_{10}\, e^{c(\theta-10)}}\, e^{-\frac{B_{10}\, c\, e^{c(\theta-10)}}{(A+B_{10}\, e^{c(\theta-10)})^2}\left\{A(\theta-10)+\frac{B_{10}}{c}(e^{c(\theta-10)}-1)\right\}} \qquad \ldots(13).$$

This is represented by the dotted line in Fig. 6.

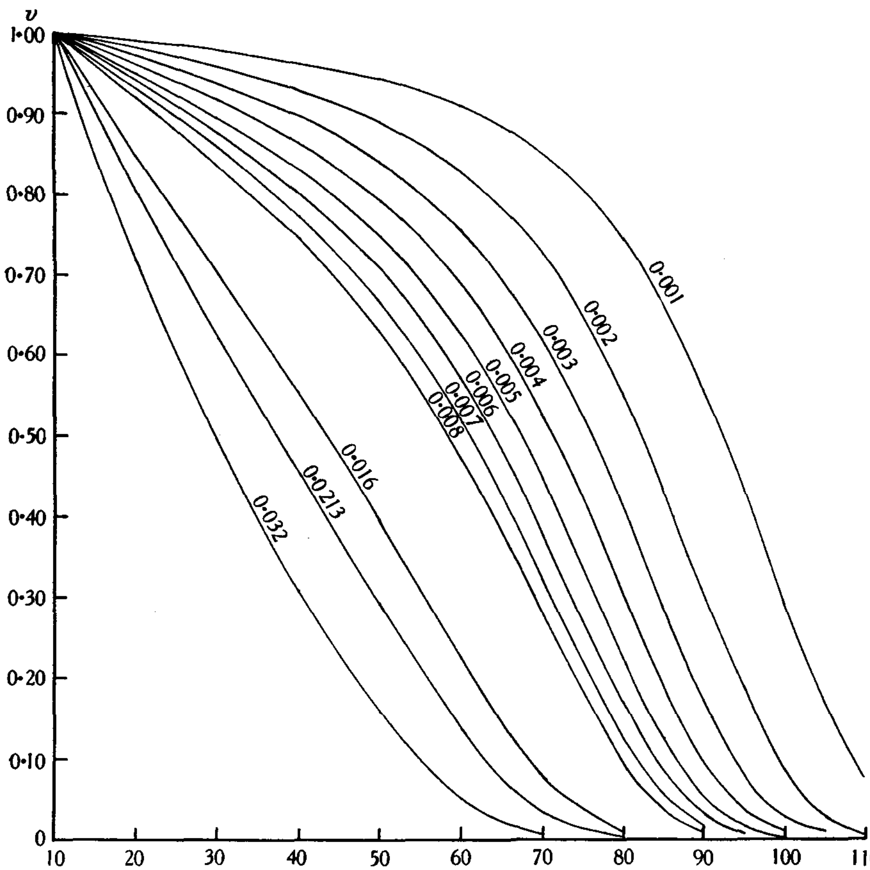

Fig. 5. v_θ curves for various values of α. v_θ gives the number surviving at age θ, as a fraction of those surviving at age 10.

When α tends to zero, y tends to $0{\cdot}02909$, so that the locus curve is asymptotic to this value. The value of α for which a maximum or a minimum occurs at a given age θ is given by

$$\frac{d^2 v}{d\theta^2} = 0, \quad \text{or} \quad \alpha = \frac{B_{10}\, c\, e^{c(\theta-10)}}{\{A + B_{10}\, e^{c(\theta-10)}\}^2} \qquad \ldots\ldots(14),$$

whence

$$e^{c(\theta-10)} = -\frac{A}{B_{10}} + \frac{c}{2\alpha B_{10}}\left(1 \pm \sqrt{1 - 4\frac{\alpha A}{c}}\right) \qquad \ldots\ldots(15)$$

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The two values of θ_m coincide for $\alpha = 0{\cdot}0213$, and then $\theta_m = 42$. For values of α less than 0·0213 the two curves are of the same type as those drawn for $\alpha = 0{\cdot}001$ to $\alpha = 0{\cdot}008$, except that as α approaches 0·006 it begins to develop a minimum which in fact corresponds to the other root of the quadratic. The two roots are positive only when $e^{c(\theta-10)}$ has two values each of which is greater than 1, this gives $\alpha > \frac{cB_{10}}{(A+B_{10})^2}$, *i.e.* $> 0{\cdot}0057$. When $\alpha > 0{\cdot}0057$, the z curve does not rise steadily but first falls to a minimum then rises to a

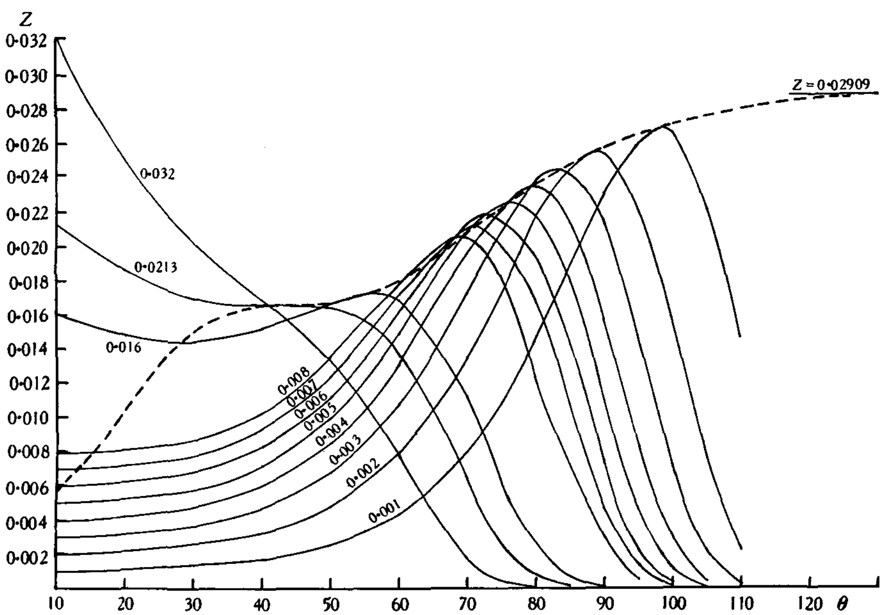

Fig. 6. z_θ **curves for various values of** α. z_θ **gives the number dying at age** θ **as a fraction of those surviving at age 10. The broken line is the locus of the maxima and minima.**

maximum before finally falling. This is shown, for example, by the curve for $\alpha = 0{\cdot}016$. The minima which exist for $\alpha > 0{\cdot}0057$ and $\alpha < 0{\cdot}0213$ lie on that part of the y curve between $\theta = 10$ and $\theta = 42$, which is shown by the dotted line in Fig. 6. The y curve has a point of inflection with a horizontal tangent at $\theta = 42$. This corresponds to $\alpha = 0{\cdot}0213$. The curve for $\alpha = 0{\cdot}0213$ is shown in the figure. It falls steadily from $\theta = 10$ to a point of inflection with a horizontal tangent at $\theta = 42$, and then falls steadily. For $\alpha > 0{\cdot}0213$, the curves fall steadily from their original value for $\theta = 10$, the latter value increasing indefinitely with increase in α. In fact for all values of α, $z_{10} = \alpha(A+B_{10})\, v_{10}$. It will be seen that over the various values of α operative during the last 50 years in Great Britain the most marked feature is a change in the maximum from $\theta = 69$ to $\theta = 89$. Furthermore the curve is skew, the slope being less for values of θ below the maximum. Thus we may say that the chief effect is to change the age period when the largest number of deaths will occur from

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65–70, up to 85–90. It thus appears that it is between these age periods that the greatest effect of the alteration in the value of α will be felt. The same point is brought out by a comparison of the curves in Fig. 5. For example, if we consider the v_θ curves for $\alpha=0{\cdot}002$ and $\alpha=0{\cdot}008$ the greatest vertical distance between them is in the neighbourhood of $\theta=80$. The point is brought out in greater detail by the following calculation. The change of v with α is $\frac{\partial v}{\partial \alpha}$. This will be maximum at a value θ_α given by $\frac{\partial}{\partial \theta}\frac{\partial v}{\partial \alpha}=0$. This gives the following equation for θ_α:

$$\frac{B_{10}}{c}e^{c(\theta_\alpha-10)}=\frac{1}{\alpha}+\frac{B_{10}}{c}-A\,(\theta_\alpha-10) \qquad \ldots\ldots(16).$$

Table IX gives the values of θ_α corresponding to different values of α.

Table IX

$\alpha=0{\cdot}001$	$\theta_\alpha=97{\cdot}5$
$=0{\cdot}002$	$=87{\cdot}7$
$=0{\cdot}003$	$=81{\cdot}7$
$=0{\cdot}004$	$=76{\cdot}7$
$=0{\cdot}005$	$=73{\cdot}7$
$=0{\cdot}006$	$=70{\cdot}4$
$=0{\cdot}007$	$=67{\cdot}9$
$=0{\cdot}008$	$=65{\cdot}4$

This value θ_α is the age which is most markedly affected by an alteration in α, and it will be seen that it changes from 65·4 at $\alpha=0{\cdot}008$ to 87·7 at $\alpha=0{\cdot}002$.

For low values of α, θ_m and θ_α both tend to be given by the equation $e^{c(\theta-10)}=\frac{c}{\alpha B_{10}}$, whence $c\,(\theta-10)=\log\frac{c}{B_{10}}-\log\alpha$ and $\Delta\theta=-\frac{1}{c}\frac{\Delta\alpha}{\alpha}$. Thus for low values of α geometrical changes in α correspond to arithmetical increases in θ_m and θ_α. For example, a 10 per cent. change in α would cause a change of about 1·25 years in θ_m and θ_α, provided that α is sufficiently small.

The same question may be approached from the point of view of the expectation of life or the average age at death.

It can readily be shown that E_r, the expectation of life of people aged r, is given by

$$E_r=\int_r^\infty \frac{v_\theta}{v_r}\,d\theta=\int_r^\infty e^{-\alpha\int_r^\theta \beta_\xi\,d\xi}\,d\theta \qquad \ldots\ldots(17).$$

By substituting $\theta=\theta'+r$, and $\xi=\xi'+r$ we find

$$E_r=\int_0^\infty e^{-\alpha\int_0^{\theta'}\beta_{\xi'+r}d\xi'}\,d\theta'$$

$$=\int_0^\infty e^{-\alpha\int_0^{\theta'}\beta'_{\xi'}d\xi'}\,d\theta',$$

where $\beta'_{\xi'}$ is written for $\beta_{\xi'+r}$.

If
$$\beta_\theta=A+Be^{c\theta},$$
$$\beta'_\theta=A+Be^{cr}e^{c\theta}$$
$$=A+B_re^{c\theta}\quad\text{where } B_r=Be^{cr}.$$

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Thus
$$E_r=\int_0^\infty e^{-\alpha\int_0^{\theta'}(A+B_r e^{c\xi})\,d\xi}\,d\theta'$$
$$=e^{\alpha\frac{B_r}{c}}\int_0^\infty e^{-\alpha A\theta'-\alpha\frac{B_r}{c}e^{c\theta'}}\,d\theta'$$
$$=\frac{e^{\alpha\frac{B_r}{c}}}{c}\left(\frac{c}{\alpha B_r}\right)^{-\alpha\frac{A}{c}}\int_{\alpha\frac{B_r}{c}}^\infty e^{-u}\,u^{-\left(\alpha\frac{A}{c}+1\right)}du \qquad \text{......(18)},$$

where $u=\alpha\frac{B_r}{c}e^{c\theta'}$.

Thus E_r is obtained in terms of an incomplete gamma function, and approximations to its value may be obtained as follows.

Let $\bar{B}_r=\alpha\frac{B_r}{c}$ and $\bar{A}=\alpha\frac{A}{c}$. It will be noted that $\bar{A}$ is a small quantity compared with unity.

The required expression is
$$E_r=\frac{e^{\bar{B}_r}\bar{B}_r^{\bar{A}}}{c}\int_{\bar{B}_r}^\infty e^{-u}u^{-(\bar{A}+1)}\,du \qquad \text{......(19)},$$

and it is not difficult to show by repeated integration by parts that its value is given by the series
$$E_r=\frac{1}{c\bar{A}}\left\{1+\frac{\bar{B}_r}{1-\bar{A}}+\frac{\bar{B}_r^2}{(1-\bar{A})(2-\bar{A})}+\dots\frac{\bar{B}_r^{\rho-1}}{(1-\bar{A})(2-\bar{A})\dots(\rho-1-\bar{A})}+\text{etc.}\right\}$$
$$-\frac{e^{\bar{B}_r}\bar{B}_r^{\bar{A}}}{c\bar{A}}\Gamma(1-\bar{A}) \qquad \text{......(20)},$$

and that this series is always convergent. For values of $\bar{B}_r>1$, however, many terms are required in order to obtain an approximately correct result. For large values of $\bar{B}_r$ and small values of $\bar{A}$, the following expression is approximately correct:
$$E_r=\frac{1}{c(\bar{A}+\bar{B}_r+1)} \qquad \text{......(21)}.$$

Table X gives values of E_r for various values of r and α.

Table X

	$r=10$	40	65	80	90
$\alpha=0{\cdot}002$	68·2	41·7	21·8	11·6	6·2
0·004	56·5	32·6	15·2	8·0	4·1
0·006	49·6	28·0	12·0	5·8	3·0
0·008	44·7	24·7	10·0	4·3	2·4

The figures in Table XI give an approximate idea of how the average age at death of all people who survive to the age r is affected by changes in the value of α.

It is obvious that as r increases the proportionate effect of changes in α on

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the expectation of life becomes greater and greater, but the effect on the average age at death becomes progressively smaller. Thus for $r=10$, the change in the value of α from 0·008 to 0·002 results in increasing E_r from 44·7 to 68·2, so that the average age at death increases from 54·7 to 78·2, but for $r=90$ a similar change in α results in rising from 2·4 to 6·2, that is, a rise of over 200 per cent., whilst the average age at death increases only from 92·4 to 96·2. It is clear from these figures that with the values of α under consideration, a large effect upon the expectation of life, when it is produced, affects chiefly the ages from 60 to 80. This result is in harmony with that obtained above from a consideration of the v and z curves. It is interesting to note that $\theta=E_{10}+10$ gives the position of the mean of the z curve for any particular value of α, whilst $\theta=E_r+r$ gives the mean of the tail of the same curve truncated at the point $\theta=r$.

Table XI

	$r=10$	40	65	80	90
$\alpha=0{\cdot}002$	78·2	81·7	86·8	91·6	96·2
0·004	66·5	72·6	80·2	88·0	94·1
0·006	59·6	68·0	77·0	85·8	93·0
0·008	54·7	64·7	75·0	84·3	92·4

On the hypothesis made in this communication, persons of ages 60 and upwards are still following a course characterised by relatively high values of α. As they have been born in 1875 or earlier, α will not be less than 0·005 or 0·006, so that it is to be expected that the next few decades will show an increase in old persons of ages from 70 to 80, and that gradually a similar increase will begin to appear in the 80–90 group.

If it be asked what the actual age constitution of the population will be, assuming that α remains 0·002, it is not possible to give a definite answer unless some assumption be made as to the birth-rate. The equation to be solved is (1) above, and a general solution (2) is given by

$$v_{t,\theta}=v_{t-\theta,0}\,e^{-\int_0^\theta f(t-\theta+\xi,\,\xi)\,d\xi} \qquad \ldots\ldots(22).$$

This, however, contains the completely arbitrary function of $(t-\theta)$, namely $v_{t-\theta,0}$, and in any actual population $v_{t-\theta,0}=Nb_{t-\theta}$, where $b_{t-\theta}$ is the birth-rate at the time $t-\theta$, and N is the total population. (Of course $v_{t-\theta,0}$ can also be expressed in terms of the specific birth-rates for different ages.) If we assume that a steady state has been reached, and that the birth-rate is constant, then $v_{t-\theta,0}=\text{constant}=v_0$, and the equation becomes

$$v_{t,\theta}=v_0 e^{-\int_0^\theta f(t-\theta+\xi,\,\xi)\,d\xi}=v_0 e^{-\alpha(t-\theta)\int_0^\theta \beta_\xi d\xi}=v_\theta \qquad \ldots\ldots(23),$$

in Fig. 5, provided that we take $\alpha\,(t-\theta)$ as a constant.

Thus under these conditions v_θ represents the age distribution of the population in the various age groups.

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Similarly, if we consider the equation

$$v_{t,\theta}=v_{10}e^{-\int_{10}^{\theta}f(t-\theta+\xi,\,\xi)\,d\xi}=v_{10}e^{-\alpha(t-\theta)\int_{10}^{\theta}\beta^{\xi}\,d\xi} \qquad \ldots\ldots(24),$$

we see that the curves in Fig. 5 represent the age distributions from 10 years upwards in a community in a steady state, where v_{10} represents the rate at which children reach their tenth birthday.

In conformity with the above observations, it is clear that the biggest gap between the curves for $\alpha=0{\cdot}002$ and $\alpha=0{\cdot}005$ occurs in the neighbourhood of θ between 70 and 80, showing that the largest effect on the age distribution of the population will probably occur at that age period. In practice the above assumptions will not be realised. It is quite probable that α will continue to fall, and that the birth-rate may also tend to decrease. Both these effects, however, will only accentuate the tendency shown by the curves, namely an increase in number of the older sections of the community. It would seem that the chief increase will be between the ages of 65 and 85, and that it would require a very large fall in α to cause an appreciable proportion of the population to be over 90. It is not proposed to discuss here the bearing of these conclusions on questions of social policy, but it may perhaps be emphasised that they are of importance in considering many vital questions such as employment, pensions, etc.

In the above discussion of the effects of a fall in the value of α, the β curve relating to the males of England and Wales and of Scotland has been used as a basis, because, as explained above, this happens to be represented by a Makeham-Gompertz formula. In the case of the females of these countries a satisfactory representation by a Makeham-Gompertz formula does not seem to be possible. The curve rises rather too slowly between the ages of 35 and 50. Similarly with the Swedish figures, neither the β values for males nor those for females follow the Makeham-Gompertz law, the most noticeable deviation in this case being the minimum in the curves for both sexes in the age group centred at 20. In all cases, however, the general trends of the curves, especially in the higher ages, agree closely, and it is quite obvious that, though in these other cases the results may slightly differ quantitatively, the general qualitative effects will be similar, and the result of alterations in the value of α will be of the same order of magnitude as in the examples discussed above.

CRITICAL CONSIDERATIONS

It seems desirable to emphasise that the results obtained in this communication are largely based upon a definite hypothesis, namely that the tendencies observed during the last 50 to 100 years will continue to exert themselves, and that no serious deviation will occur. Extrapolation of any type is attended by some degree of uncertainty, and the more distant the extrapolation is extended the greater the uncertainty necessarily becomes. The extrapolation employed in this paper is of the simplest possible type, in that it depends on the continuance of straight lines, but on the other hand it is obvious that it is in the nature of the case that a wide deviation from the

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expected course of events might at any time occur. In Sweden, for instance, as has been shown, such a deviation affecting a particular fraction of the community seems to have occurred about 1850, and is only now tending to disappear. Examination of the figures shows that a similar deviation occurred in Great Britain during the war years, especially 1918. It is interesting to note that in both cases the age groups centred at 10, 20, and 30, were those chiefly affected. It is of course impossible to exclude the chance of similar deviations occurring in the future, nor is it possible to predict them.

Further, it is conceivable that the generalisation might break down in another way. There might for instance be some progressive alteration in the diagram as time went on, and higher age groups became involved. For example, the lines down the diagonals, approximately straight up to 70 or 80 years of age, might gradually curve round and flatten out horizontally at still higher ages. Many may on general grounds consider that this is a phenomenon likely to be realised. It would correspond to a prevailing impression that the onset of old age cannot ultimately be arrested, and that curative and preventive medicine and improvement of social conditions cannot be expected to have any appreciable effect on the senile. We can only say at present that the statistics, as far as they go, give no indication of any flattening out of the curves. On the other hand, we have demonstrated above that even if the curves continue to run straight, that is, even if the hypothesis made is completely fulfilled, the numbers of the very aged (above 100, say) will not, at least for many years to come, increase to such an extent as to become a substantial proportion of the population, although they will become relatively more numerous.

Another point which must be emphasised as a possible ground for criticism of the general results of the thesis of this communication arises from the fact that the abnormal effects shown in the Swedish figures, and in the English figures during the war years, suggest that any abnormally adverse conditions tend to influence most markedly the younger age groups. Persons of over 45 years of age are scarcely affected in either case. In the case of the war years in England males are affected more than females, but this may be due partly to the withdrawal of many healthy male lives from the civil population to which the statistics refer. The increase in specific mortality is quite definite amongst the females, and in this case the direct effects of the war can scarcely be the cause. It might be argued that if adverse conditions so markedly affect the younger age groups, then it is only to be expected that improving conditions will show themselves first in those same younger age groups. The gradual improvement in conditions over many years might then show itself in gradual and progressive improvement which would affect the older age groups only when the conditions had been operating for a long time. Mathematically this would be equivalent to splitting the function $f(t, \theta)$ into a product of two functions $\phi(t)$ and $\psi(\theta)$, $\psi(\theta)$ being a measure of the susceptibility of the age group θ to changes in the conditions as measured by $\phi(t)$. This would be consistent with the existence of straight diagonal contours only

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if $\phi(t)$ were an exponential function. If for example $\phi(t)=re^{-\rho t}$ then $f(t, \theta)=re^{-\rho(t-\theta)}e^{-\rho\theta}\psi(\theta)$. From this form, similar to (1*a*) above, it follows that the observed lines would be realised. It is to be noted, however, that the figures do not bear out the hypothesis that $\phi(t)$ is exponential in character. An examination of the figures suggests that the improvement was slow to begin with and gradually accelerated up to a point. This is in harmony with the finding that it has not, in fact, been possible to split $f(t, \theta)$ into two factors $\phi(t)$ and $\psi(\theta)$. It is of course possible that the effect here suggested has played some part in the complex system of causes, of which the statistical result is relatively simple, and takes the form of the linear relationships which we have been treating. From a practical point of view the abnormalities which we have been discussing emphasise the importance of environment for the younger age groups, and seem to confirm rather than to damage the conclusions which we have arrived at as to the importance of environment during the early age periods in determining national health.

SUMMARY

1. The specific mortality rates for males, females and the total population for England and Wales, for Scotland and for Sweden, have been fitted to a formula $f(t, \theta)=\alpha(t-\theta)\beta_\theta$, where $f(t, \theta)$ is the specific mortality rate at a time t for age θ, β_θ is a function depending solely on the age θ, and $\alpha(t-\theta)$ depends only on the time of birth $(t-\theta)$. The results are in substantial agreement with those obtained by less refined methods in the previous paper. The probable errors of the values found for α and for β have been calculated.

2. It is shown that the β_θ curves for the Scottish and the English males are approximately represented by the Makeham-Gompertz formula $A+Bc^\theta$, where A, B and c have suitable values. The other β_θ curves do not appear to conform exactly to a formula of this type.

3. With the help of the representation of β_θ by the Makeham-Gompertz expression the effect of variation of α on the survival curves, the death curves, and the expectation of life has been determined. It is shown that with the range of values of α experienced in Britain during the last 50 years, the most marked effect is most likely to be experienced in the future between the ages of 65 and 85, a very considerable increase of people of these ages being likely provided that the relationship exhibited by the statistics up to the present date is maintained in the future.

Though the Makeham-Gompertz formula does not hold in the case of the English and Scottish females, nor for the Swedish statistics, these approximate sufficiently closely to the values for the English and Scottish males, to allow of the conclusion deduced in the latter case being extended to the former.

4. It is strongly emphasised that the validity of all the predictions depends upon a hypothesis of extrapolation which, however attractive in the light of the figures so far available, might not be fulfilled under certain contingencies.

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