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On the differential equations for tide-well systems

A. Brown

The paper discusses the differential equation

$$dY/d\tau + \beta_n^{-1}|Y|^{n/2}\text{sgn}(Y) = \cos\tau \ , \quad n = 1, 2, 3 \ ,$$

from a fresh point of view, to supplement an earlier discussion by Noye. In particular, for $n = 1$ the equation can be transformed to the equation for a pendulum with viscous damping, with $\beta_1 = (1/2)^{3/2}$ corresponding to critical damping. At the end of the paper, some related equations are considered.

1. Introduction

In a recent paper [3], Noye considered the differential equation relating the height of water in a tide gauge to the sea level outside the gauge. Ideally the two levels would be the same but in practice the relationship between them is more complicated. To increase the level in the gauge, water has to flow in and the rate at which it flows in depends on the way in which the gauge is connected to the sea. Noye distinguishes three cases, where the connection is:

I via a circular orifice near the bottom of the gauge,

II via a long horizontal pipe,

III via a vertical slit of constant width.

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We shall refer to these as tide-well systems I, II and III. Noye gives the differential equation relating height inside and outside the gauge for each system and in particular discusses the case where the outside level fluctuates sinusoidally. Following Noye's notation, we write $h_0(t')$ as the sea level at time t' and $h_w(t')$ as the level inside the gauge or well. When $h_0(t') = a\sin\omega t'$, non-dimensional variables

$$\tau = \omega t' \quad \text{and} \quad Y = (1/a)(h_0 - h_w)$$

can be introduced and the differential equation becomes [3, equation (1.5)]

$$dY/d\tau + \beta_n^{-1}|Y|^{n/2}\text{sgn}(Y) = \cos\tau \,, \tag{1.1}$$

where $n = 1$ for system I, $n = 2$ for system II and $n = 3$ for system III. The coefficients β_1, β_2 and β_3 are (non-dimensional) positive constants; their precise form is given by Noye in terms of the parameters affecting the system. For the purposes of the present paper, it is sufficient to replace β_1, β_2, β_3 by a single positive constant, β. Also, we shall write $t = \tau - (\pi/2)$ and assume as initial condition that $Y = 0$ when $\tau = 0$. With these minor modifications, the equation to be discussed is

$$dY/dt + (1/\beta)|Y|^{n/2}\text{sgn}(Y) = -\sin t \,, \quad n = 1, 2, 3 \,, \tag{1.2}$$

with $Y = 0$ when $t = -\pi/2$. In Noye's discussion, he is concerned with steady state solutions of equation (1.1), so fixing the initial condition does not invalidate a comparison with his results.

2. Equation for n = 1 : relation to viscous damping problem

For $n = 1$, the equation for Y is non-linear. If we write the equation in the form

$$dY/dt = F(t, Y) \,,$$

then $\partial F/\partial Y$ is not defined for $Y = 0$ and indeed $|\partial F/\partial Y| \to \infty$ as $Y \to 0$, either from above or from below. Thus the usual test for existence and uniqueness of the solution (using the Lipschitz condition) does not apply when $Y = 0$. However, Drummond's test [1] can be used to show that the solution is unique except where $Y = 0$ and $\sin t = 0$.

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At the point $(Y = 0,\ t = -\pi/2)$, $dY/dt = 1$ and hence we can expect Y to be positive throughout an interval $-\pi/2 < t < T_1$, where T_1 is the first zero of Y to the right of $t = -\pi/2$. In this interval the equation for Y is

(2.1) $$dY/dt + (1/\beta)\sqrt{(Y)} = -\sin t\ .$$

Now introduce

(2.2) $$V = \int_{-\pi/2}^{t} \{Y(v)\}^{-1/2} dv \quad (-\pi/2 < t < T_1)\ .$$

Note that this integral converges at the lower limit because $Y \sim t + (\pi/2)$ near $t = -\pi/2$. From equation (2.2),

$$dt/dV = \sqrt{(Y)}\ ,\quad d^2t/dV^2 = (1/2)(dY/dt)\ .$$

If we use D for differentiation with respect to V , equation (2.1) becomes

(2.3) $$D^2t + (1/2\beta)Dt + (1/2)\sin t = 0\ .$$

This is the equation of motion for a pendulum with viscous damping, one of the standard examples of a non-linear differential equation [*2*, pp. 180-182; *4*, pp. 61-66]. The behaviour of the solutions is usually represented in the phase plane, working from the first order equations

(2.4) $$Dt = u\ ,\quad Du = -(1/2\beta)(u+\beta\sin t)\ .$$

If we forget about the restriction on V and allow V to increase indefinitely, the solutions of (2.4) which start at $u = 0$, $t = -\pi/2$, will spiral in toward the origin when β is large (light damping), whereas for β small enough the origin is a node and the solutions are tangential to a straight line as they approach the origin. (A diagram showing solution curves for a pendulum with light damping is given by Stoker [*4*, p. 63].) More precisely, it can be shown that the origin is a focus for $\beta > \beta_0$, where $\beta_0 = (1/2)^{3/2}$, a result obtained by linearising equations (2.4) for t small. For $\beta \le \beta_0$, the origin is a node and the approach to the origin (as $V \to \infty$) is along the line $u = -mt$, where $m = (1/4\beta)\left\{1-\left(1-8\beta^2\right)^{1/2}\right\}$. For β small, $m = \beta + O\left(\beta^3\right)$.

3. Behaviour of solution of equation (1.2), for n = 1

For zero damping $(\beta = \infty)$, equation (1.2) gives $Y = \cos t$ as the appropriate solution and the N-th zero (for $t > -\pi/2$) occurs when $t = (2N-1)(\pi/2)$. As β decreases, we can expect the N-th zero to occur earlier and the maximum value of $|Y|$ to become smaller. (We shall assume later that $|Y| < 1$ for $0 < \beta < \infty$.)

For $n = 1$ and β small, we have seen that the first zero of Y occurs at $t = 0$. For $t > 0$, we can use the asymptotic expansion given by Noye [3, §4]. With t used instead of τ , this expansion is

$$(3.1) \qquad Y = -\beta^2\sin^2 t\left[1-4\beta^2\cos t-4\beta^4(2-7\cos^2 t)+O(\beta^6)\right] ,$$

for $0 \le t \le \pi$, with

$$(3.2) \qquad Y = (-1)^m Y(t-m\pi) , \text{ for } m\pi \le t \le (m+1)\pi .$$

With this expansion, we see that the N-th zero occurs at $t = (N-1)\pi$ for β small.

For $n = 1$ and $\beta > \beta_0$, we shall assume that the N-th zero occurs at $t = T_N$, where $(N-1)\pi < T_N < (2N-1)\pi/2$. This means that Y is negative for $T_{2N-1} < t < T_{2N}$ and positive for $T_{2N} < t < T_{2N+1}$. Hence if we write

$$(3.3) \qquad Y_N = (-1)^N Y , \quad t_N = t - N\pi ,$$

then in the interval $\left(T_N, T_{N+1}\right)$ we have $Y_N > 0$ and $-\pi < t_N < \pi/2$. Also, equation (1.2) gives

$$(3.4) \qquad dY_N/dt_N + (1/\beta)\surd\left(Y_N\right) = -\sin t_N ,$$

with $Y_N = 0$ when $t_N = -\left(\pi-T_N^*\right)$, where

$$(3.5) \qquad T_N^* = T_N - (N-1)\pi .$$

Equation (3.4) has the same form as (2.1) and we can again make use of equations (2.4) and their phase plane solution. To do this, let D_N denote differentiation with respect to V_N , where

(3.6) $$V_N = \int_{T_N^*-\pi}^{t_N} \{Y_N(v)\}^{-1/2} dv \ .$$

Then the same argument as before gives

(3.7) $$D_N t_N = u_N \ , \quad D_N u_N = -\ (1/2\beta)\left(u_N + \beta \sin t_N\right) \ ,$$

with $u_N = 0$ when $t_N = T_N^* - \pi$. If we follow the phase plane solution of these equations as V_N increases, we can continue the trajectory until u_N again becomes zero. This occurs when $t_N = T_{N+1} - N\pi = T_{N+1}^*$ and this fixes the starting point for the solution for u_{N+1} and t_{N+1} .

In effect, we keep solving the same pair of equations in the phase plane, following the solution as long as u is positive and using the terminal point of one solution to fix the starting point of the next. If we begin with $u = 0$ for $t = -\pi/2$ and use equations (2.4), this fixes T_1 and we note that

$$0 < T_1 = T_1^* < \pi/2 \ .$$

Hence the solution for u_1, t_1 starts at $u_1 = 0$, $t_1 = T_1^* - \pi$, that is to the left of the (u, t) solution, since $-\pi < T_1^* - \pi < -\pi/2$. The (u_1, t_1) solution cannot intersect the (u, t) solution because the only singular points are where u (or u_1) and $\sin t$ (or $\sin t_1$) are simultaneously zero. It follows that $T_1^* < T_2^*$. Also, we have $T_2^* < \pi/2$, because $T_2^* + \pi = T_2 < 3\pi/2$. Combining the inequalities gives

$$0 < T_1^* < T_2^* < \pi/2 \ .$$

The solution for u_2, t_2 starts at $u_2 = 0$, $t_2 = T_2^* - \pi$, where

$$-\pi < T_1^* - \pi < T_2^* - \pi < -\ \pi/2 \ .$$

It follows that the (u_2, t_2) solution lies between the two previous solution **curves** and terminates at $t_2 = T_3^*$, where $T_1^* < T_3^* < T_2^*$. Continuing in this way, it can be seen that

(3.8) $$0 < T_1^* < T_3^* < \ldots < T_{2N+1}^* < T_{2N}^* < \ldots < T_4^* < T_2^* < \pi/2 \ ,$$

for $N = 1, 2, 3, \ldots$. Hence T_N^* approaches a limiting value T^* , as $N \to \infty$, and this implies that the $\left(u_N, t_N\right)$ solution curve approaches a

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limiting position. The graph of Y against t must also settle down to a limiting form.

The above discussion applies for $\beta > \beta_0$, that is in the case where the origin is a focus for the phase plane solution. For $\beta \le \beta_0$, we know from the phase plane solution that $T_1 = 0$. From the asymptotic solution, we can say that $T_N = (N-1)\pi$ for β small and it seems likely that this will apply whenever $\beta \le \beta_0$. This would mean that the graph of Y against t has its limiting form for $t \ge 0$, that is for $\tau \ge \pi/2$.

4. Behaviour of solution of equation (1.2) for n = 2 and n = 3

As noted by Noye, equation (1.2) is linear for $n = 2$ and an explicit solution can be written down. When $Y = 0$ at $t = -\pi/2$, the solution is

$$Y = \{\beta/(1+\beta^2)\}\left\{(\beta\cos t-\sin t) - \exp\left[-\{t+(\pi/2)\}/\beta\right]\right\} . \tag{4.1}$$

For large values of t, Y behaves like the sinusoidal function $-\{\beta/(1+\beta^2)^{1/2}\}\sin(t-\phi)$, where $0 < \phi < \pi/2$ and

$$\tan\phi = \beta . \tag{4.2}$$

More precisely, if S_N is the N-th positive zero of Y and if

$$S_N^* = S_N - (N-1)\pi ,$$

then it can be seen that

$$0 < S_1^* < S_3^* < \ldots < S_{2N+1}^* < \phi < S_{2N}^* < \ldots < S_4^* < S_2^* < \pi/2 , \tag{4.3}$$

for $N = 1, 2, 3, \ldots$. This is the analogue of equation (3.8) and we can deduce that $S_N^* \to \phi$ as $N \to \infty$.

For the interval $-\pi/2 < t < S_1$, Y is positive and we can introduce V as before (see equation (2.2)). The equation for Y is transformed to

$$D^2t + (1/2\beta)(Dt)^2 + (1/2)\sin t = 0 , \tag{4.4}$$

which is the equation of motion for a pendulum with damping proportional to the square of the angular velocity [4, pp. 59-61]. In this case the origin is a focus for all positive values of β and there is not a critical value for β, as there was for $n = 1$. (This agrees with the

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conclusions drawn from equation (4.1).) In terms of u and t, equation (4.4) can be replaced by

$$(4.5) \qquad Dt = u \ , \quad Du = -(1/2\beta)(u^2+\beta\sin t) \ ,$$

and the initial condition becomes $u = 0$, $t = -\pi/2$ for $V = 0$.

At any point (u, t) in the region $0 < u < 1$ of the phase plane we have $0 < u^2 < u < 1$ and hence, for a given value of β,

$$(4.6) \qquad \left(\frac{du}{dt}\right)_{n=1} = -\frac{u+\beta\sin t}{2\beta u} < -\frac{u^2+\beta\sin t}{2\beta u} = \left(\frac{du}{dt}\right)_{n=2} .$$

(As mentioned in Section 3, we assume $|Y| < 1$ for $\beta > 0$, so the restriction to the region $0 < u < 1$ is appropriate.) Thus if we have the same initial point $(u = 0,\ t = -\pi/2)$ and the same value of β, the solution curve of equations (4.5) will rise above that of equations (2.4) and will return to zero for a larger value of t, that is $S_1 > T_1$.

A similar argument shows that $\phi > T^*$, for a given value of β. The solution of equations (4.5) which starts at $u = 0,\ t = T^* - \pi$ will come back to zero to the right of the corresponding solution of equations (2.4) and thus has a "span" greater than π. To get a smaller span, we would have to move the initial point to the right. In particular, the initial point $u = 0,\ t = \phi - \pi$, which gives a solution curve with span π for equations (4.5), must be to the right of $u = 0,\ t = T^* - \pi$.

For $n = 3$, equation (1.2) is again non-linear and an explicit solution is not available. However, we can discuss the equation in terms of a corresponding equation in the (u, t) phase plane in much the same way as before. If we again take $Y = 0,\ t = -\pi/2$ as initial point and if W_N is the N-th zero of Y for $t > -\pi/2$, then it can be shown that

$$(4.7) \qquad 0 < W_1^* < W_3^* < \ldots < W_{2N+1}^* < W_{2N}^* < \ldots < W_4^* < W_2^* < \pi/2 \ ,$$

for $N = 1, 2, 3, \ldots$, where

$$W_N^* = W_N - (N-1)\pi \ .$$

Hence W_N^* tends to a limit, W^*, as $N \to \infty$. Also, for a given value of β, $W_1 > S_1 > 0$ and $W^* > \phi > T^*$.

5. Numerical results

The most interesting result that emerges from the transformation to the phase plane is the discovery of a precise limiting value of β for critical damping in the case of the tide-well with an orifice $(n = 1)$. The value obtained, that is $\beta = (1/2)^{3/2} = 0.354$, agrees well with Shipley's results for phase lag, as shown in Figure 3 of Noye's paper. Some numerical calculations using equations (2.4) also confirmed this. The table below shows the values of T_1 obtained for $\beta = 0.4, 0.5, \ldots, 1.0$.

β	0.4	0.5	0.6	0.7	0.8	0.9	1.0
T_1	0.0025	0.0489	0.1230	0.2015	0.2754	0.3450	0.4065 .

For $\beta < \beta_0$, the solution curve in the phase plane should approach the origin as $V \to \infty$, so the solution breaks down in this case before T_1 is attained. A solution with $\beta = 0.3$ gave:

$$u = 0.000414 \quad \text{for} \quad t = -0.001055 ,$$
$$u = 0.000101 \quad \text{for} \quad t = -0.000257 ,$$
$$u = 0.000025 \quad \text{for} \quad t = -0.000063 .$$

Theoretically, the solution should approach the origin along the line $u = -0.3924t$ and the numerical results are in good agreement.

For $\beta = 0.8$, the values for $T_1^*, T_2^*, T_3^*, T_4^*$ and T_5^* were calculated and came out as

$$0.275408, 0.334081, 0.333883, 0.333884, 0.333884 ,$$

respectively, with the solution repeating itself after that. Thus for this intermediate value of β, the solution settled down to a steady state oscillation quickly and the limiting value of T_N^* was obtained with reasonable accuracy after three of four iterations. In the steady state oscillation, the maximum value of Y was 0.4364 and the time interval to rise from zero to the maximum was 1.77635, compared with a time interval of 1.36525 for the decrease from the maximum to zero. This asymmetry indicates the difficulty of trying to represent Y by a sine curve solution.

6. Related equations

The transformation employed to change from equation (2.1) to equation (2.3) can be used for some other first order equations also. An immediate generalization is to replace $-\sin t$ in equation (2.1) by another function of t, say $G'(t)$; then $-\sin t$ is replaced by $G'(t)$ in equation (2.3) also. If we start from this more general form, that is,

$$dY/dt + (1/\beta)\surd(Y) = G'(t) \,, \tag{6.1}$$

other forms can be obtained essentially by change of variable. For example, if we assume $Y > 0$ and replace Y by v^p, the equation becomes

$$dv/dt = (1/p\beta)\{\beta G'(t)v^{1-p} - v^{(2-p)/2}\} \,, \tag{6.2}$$

with $p = 2$ and $p = -2$ giving the most interesting special cases.

For $p = -2$, equation (6.2) becomes

$$dv/dt = (1/2\beta)v^2 - (1/2)v^3G'(t) \,. \tag{6.3}$$

This a special case of the more general form

$$dv/dt = va(t) + v^2b(t) + v^3c(t) \,, \tag{6.4}$$

but we can show that, with suitable assumptions, equation (6.4) can be simplified to the form (6.3). To do this, let

$$A(t) = \exp\left\{\int_{t_0}^{t} a(u)du\right\} \,, \quad v = A(t)w \,.$$

Then for $A(t) \neq 0$, equation (6.4) becomes

$$dw/dt = w^2b(t)A(t) + w^3c(t)\{A(t)\}^2 \,. \tag{6.5}$$

For $b(t) \neq 0$, let $s(t) = \int_{t_0}^{t} 2\beta b(u)A(u)du$; then

$$dw/ds = (1/2\beta)w^2 + H(s)w^3 \,, \tag{6.6}$$

where $H(s) = c(t)\{A(t)\}/\{2\beta b(t)\}$. Equation (6.6) is now of the same form as equation (6.3), thus linking (6.4) with (6.1).

For $p = 2$, equation (6.2) becomes

$$dv/dt = (1/2v)G'(t) - (1/2\beta) \,. \tag{6.7}$$

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For $G'(t) \neq 0$, let $z = G(t)$ and $W = v + (t/2\beta)$. Then

$$\frac{dz}{dt} = 2v\left\{\frac{dv}{dt} + \frac{1}{2\beta}\right\} = \left(2W - \frac{t}{\beta}\right)\frac{dW}{dt} ,$$

and

(6.8) $$dz/dW + M(z) = 2W ,$$

where

$$M(z) = t/\beta = (1/\beta)G^{-1}(z) .$$

If we now set $q = dW/dz$ and differentiate with respect to z, equation (6.8) becomes

(6.9) $$dq/dz = q^2M'(z) - 2q^3 ,$$

an equation which can be used to link equations (6.4) and (6.1) via an alternative assumption. Instead of assuming $b(t) \neq 0$, we can assume $c(t) \neq 0$ and set

$$s_1(t) = -(1/2)\int_{t_0}^{t} c(u)\{A(u)\}^2du .$$

This allows equation (6.5) to be replaced by

(6.10) $$dw/ds_1 = w^2E(s_1) - 2w^3 ,$$

where $E(s_1) = -2b(t)/\{c(t)A(t)\}$, and we now have an equation of the same form as (6.9). Thus equation (6.4) can be transformed to the form (6.1) and hence to the corresponding phase plane problem in many cases.

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Australian National University,
Canberra, ACT.