*Ergod Th & Dynam Sys* (1987), 7, 531-557
*Printed in Great Britain*

# On ergodic actions whose self-joinings are graphs

A DEL JUNCO AND D RUDOLPH

*Department of Mathematics, University of Toronto, Toronto, Canada, Department of Mathematics, University of Maryland, College Park, Maryland, USA*

(*Received* 8 *January* 1985, *revised* 4 *December* 1985 *and* 13 *October* 1986)

*Abstract* We call an ergodic measure-preserving action of a locally compact group $G$ on a probability space simple if every ergodic joining of it to itself is either product measure or is supported on a graph, and a similar condition holds for multiple self-joinings This generalizes Rudolph's notion of minimal self-joinings and Veech's property $S$

Main results The joinings of a simple action with an arbitrary ergodic action can be explicitly described A weakly mixing group extension of an action with minimal self-joinings is simple The action of a closed, normal, co-compact subgroup in a weakly-mixing simple action is again simple Some corollaries Two simple actions with no common factors are disjoint The time-one map of a weakly mixing flow with minimal self-joinings is prime Distinct positive times in a $\mathbb{Z}$-action with minimal self-joinings are disjoint

## 0 *Introduction and definitions*

The notion of minimal self-joinings for $\mathbb{Z}$-actions was introduced in [**Ru2**] as a source of counter-examples In this paper we generalize this notion to what we call simple group actions and develop some general theory for these actions This allows us to broaden the repertoire of actions displaying this sort of behaviour We deal with actions of fairly general groups because it is convenient for our purposes and not much more difficult, but the main interest lies in $\mathbb{Z}$ and $\mathbb{R}$-actions Most of our results are new even within the setting of $\mathbb{Z}$-actions

We consider a standard Borel space $(X, B)$, that is there exists a complete separable metric on $X$ such that $B = B(X)$ is the $\sigma$-algebra of Borel sets generated by the corresponding topology on $X$ (By the remarks on p 138 of [**Ma2**] one can assume that the metric on $X$ is actually compact ) Suppose that $X$ is equipped with a Borel probability measure $\mu$ and that $G$ is a locally compact group By a (left) action of $G$ on $X$ we mean a Borel map $G \times X \to X$ denoted $(g, x) \mapsto gx$ such that

$$(hg)x = h(gx) \qquad \forall h, g \in G, x \in X,$$

and

$$ex = x \qquad \forall x \in X,$$

where $e$ denotes the identity element of $G$ and $x \mapsto gx$ is a measure-preserving map for each $g \in G$ We then say that $\mathscr{X} = (X, B(X), \mu, G)$ is a $G$-action or a $G$-space We will often shorten this to $\mathscr{X} = (X, \mu, G)$ or $(X, \mu)$ For convenience, throughout this paper $\mathscr{X}$ always represents $(X, B(X), \mu, G)$ and $\mathscr{Y}$ represents $(Y, B(Y), \nu, G)$ We require that all our actions be ergodic, that is all (everywhere) invariant Borel sets have measure 0 or 1 Equivalently, $gA = A$ a e implies $\mu(A) = 0$ or 1, (see Theorem 3 of [**Ma1**])

If $\mathscr{X}_i = (X_i, B(X_i), \mu_i, G)$, $i = 1, \ldots, k$ are $G$-actions, by a joining of $\mathscr{X}_1, \ldots, \mathscr{X}_k$ we mean a Borel measure $\lambda$ on $X_1 \times \cdots \times X_k$ which is invariant under the natural diagonal $G$-action $g(x_1, \ldots, x_k) = (gx_1, \ldots, gx_k)$ and whose marginal (projection) on each $X_i$ is $\mu_i$ Thus $(X_1 \times \cdots \times X_n, \lambda, G)$ is an action and we will frequently identify the joining with the corresponding action When we need to emphasize the role of $G$ we will speak of a $G$-joining By a $k$-joining of the single $G$-action $\mathscr{X}$ we mean a joining of $k$ copies of $\mathscr{X}$ We denote by $J(\mathscr{X}_1, \ldots, \mathscr{X}_n)$ the space of joinings of $\mathscr{X}_1, \ldots, \mathscr{X}_n$

We denote by $C(\mathscr{X})$ the centralizer of the action $\mathscr{X}$, that is the semi-group of (equivalence classes of) measure-preserving maps commuting a e with the action of each $g \in G$ For $S \in C(\mathscr{X})$ we denote by $\mu_S$ the Borel measure on $X \times X$ which is the image under the map $x \mapsto (x, Sx)$ of the measure $\mu$ Thus

$$\mu_S(A \times B) = \mu(A \cap S^{-1}B),$$

which makes it clear that $\mu_S$ does not depend on the choice of representative of $S$ $\mu$ may also be defined as $(\mathrm{id} \times S)\mu_\Delta$ where $\mu_\Delta = \mu_{\mathrm{id}}$ is the diagonal measure on $X \times X$

$\mu_S$ is a 2-joining of $\mathscr{X}$ its marginals are $\mu$ because $S$ is measure-preserving and it is $G$-invariant because $S$ commutes with the $G$-action The corresponding action is isomorphic to $\mathscr{X}$ via the map $x \mapsto (x, Sx)$ so $\mu_S$ is ergodic because of our standing assumption of ergodicity We will call joinings of the form $\mu_S$ off-diagonal $\mu \times \mu$ is also a 2-joining which is ergodic precisely if $\mathscr{X}$ is weak-mixing (We may take this as the definition of weak-mixing ) We shall say $\mathscr{X}$ is 2-fold simple if every ergodic 2-joining is either product measure $\mu \times \mu$ or an off-diagonal (This does not mean that $\mathscr{X}$ is weak-mixing!) For the case of $\mathbb{Z}$-actions this notion is due to Veech who called it property $S$ If in addition each $S \in C(\mathscr{X})$ agrees a e with the action of some $g \in G$ then we say $\mathscr{X}$ has 2-fold minimal self-joinings (MSJ)

If $T \in C(\mathscr{X})$ denote by $\tilde{\mu}_T$ the image of $\mu$ under $x \mapsto (Tx, x)$ Then $\tilde{\mu}_T$ is an ergodic 2-joining, so if $\mathscr{X}$ is 2-fold simple $\tilde{\mu}_T = \mu_S$ for some $S \in C(\mathscr{X})$ Evaluating this equation on the rectangle $S^{-1}B \times B$ we obtain

$$\mu(T^{-1}S^{-1}B \cap B) = \mu(S^{-1}B \cap S^{-1}B) = \mu(B),$$

so $T^{-1}S^{-1}B = B$ a e for $B \in B(X)$ As is well known, in a standard Borel space this implies that $ST = \mathrm{id}$ a e Similarly $TS = \mathrm{id}$ a e Thus 2-simplicity forces $C(\mathscr{X})$ to be a group This removes the evident asymmetry in the definition of 2-simplicity an equivalent definition is that $C(\mathscr{X})$ is a group and every ergodic 2-joining of $\mathscr{X}$ is either product measure or a measure of the form $(S_1 \times S_2)\mu_\Delta$, $S_1, S_2 \in C(\mathscr{X})$

We now want to make a definition which restricts in a similar way the $k$-joinings of $\mathscr{X}$ to the obvious ones What are the obvious ones? If $S_1, \ldots, S_k \in C(\mathscr{X})$ then we call the image of $\mu$ under the map $x \mapsto (S_1 x, \quad, S_k x)$ an off-diagonal measure An off-diagonal measure is evidently an ergodic $k$-joining By a product of off-diagonals (POOD) on $X^k$ we mean that the index set $(1, \quad, k)$ has been split into subsets $k_1, \quad, k_r$, on each $X^{k_i}$ we put an off-diagonal measure and then take the product of these off-diagonal measures A POOD is evidently a self-joining of $\mathscr{X}$ Note that product measure is itself a POOD – an off-diagonal may sit on a single factor of $X^k$ We say that $\mathscr{X}$ is simple if $C(\mathscr{X})$ is a group and for every $k$ each ergodic $k$-joining of $\mathscr{X}$ is a POOD If in addition each $S \in C(\mathscr{X})$ agrees a e with the action of some $g \in G$ then we say $\mathscr{X}$ has MSJ

Some comments about our terminology are in order In [**Ju1**] (and, following [**Ju1**], in [**Ve**]) the term simple was (unhappily, we now feel) used to mean 2-fold minimal self-joinings for $\mathbb{Z}$-actions For $\mathbb{Z}$-actions generated by a map $T$ our definition of minimal self-joinings restricts only the joinings of $T$ with itself whereas [**Ru1**] also restricts joinings of unequal powers of $T$ We feel that the present terminology is apter the term self-joining should refer only to joinings of $T$ with itself Moreover, as we shall see later (§ 6) the present definition is almost equivalent to the stronger one Finally, simplicity generalizes Veech's property S ([**Ve**]) which is 2-fold simplicity (Veech works only with $\mathbb{Z}$-actions)

We now briefly describe our results § 1 reviews some background on $G$-actions, group extensions and joinings It also includes a characterisation of group extensions (Theorem 1 8 2) essentially due to Veech [**Ve**], which is essential for our main result (Theorem 4 1) In § 2 we show that when a locally compact group acts ergodically on a compact group by left translations then the action is simple and the centralizer consists of all the right translations This is in some sense the trivial case and we show that every non-weak-mixing simple action must be of this type The main interest lies in the weakly mixing case

Veech [**Ve**] has shown that a simple $\mathbb{Z}$-action is a group extension of any non-trivial factor In § 3 we reprove this in the general setting We go on to characterize joinings of factors of a given simple action and determine when a factor of a simple action is again simple

§ 4 contains our main result, a characterization of joinings of a simple $G$-action $\mathscr{X}$ with an arbitrary ergodic $G$-action $\mathscr{Y}$ Just as a simple $G$-action has only the obvious joinings with itself it turns out that it has only the 'obvious' joinings with other actions For $\mathbb{Z}$-actions with MSJ S Glasner [**Gl**] has given a different proof of this result Glasner does not actually describe all the joinings but rather characterizes those $\mathscr{Y}$ which are not disjoint from $\mathscr{X}$ A corollary of our result is that two simple actions with no common factors are disjoint

In § 5 we show that a weakly mixing group extension of an action with MSJ is simple We present an example due to Glasner which shows that a weakly mixing group extension of a simple action need not be simple For the proof of the result on group extensions we introduce the auxiliary notion of a pairwise determined action – one for which any self-joining which is pairwise independent must be independent – a notion which we think is of independent interest

In § 6 we show that in a weakly mixing simple action of a group $G$ any closed, normal cocompact subgroup $H$ acts simply and that its centralizer is the centralizer of the full action A corollary of this is that in a weakly-mixing flow with MSJ the time one map is prime, i e it has only the trivial invariant $\sigma$-algebras Moreover if $\mathscr{X}$ and $\mathscr{Y}$ are weakly mixing simple $G$-actions such that any ergodic joining of $\mathscr{X}$ and $\mathscr{Y}$ is weakly mixing then any $H$-joining of $\mathscr{X}$ and $\mathscr{Y}$ is a $G$-joining We make some further applications to flows and show that our definition of minimal self-joinings in the case of $\mathbb{Z}$-actions is almost as strong as the original one in **[Ru1]** We conclude with some open problems

As we have already stated our main interest lies in weakly mixing, simple or MSJ, $\mathbb{Z}$ or $\mathbb{R}$ actions Examples of such actions with MSJ are already available, see **[Ru1]**, **[JRS]**, **[Ju1]** for $\mathbb{Z}$-examples, **[Ra]**, **[J, P]** for $\mathbb{R}$-examples Theorems 5 4 and 6 1 show how to obtain simple actions from actions with MSJ The construction in **[Ru2]** can probably be modified to obtain a simple map with a Bernoulli shift in its centralizer All these examples of simplicity depend on a very explicit knowledge of the centralizer Elsewhere we will construct a completely different sort of example a weakly mixing simple prime $\mathbb{Z}$-action which is also rigid, that is there exists a sequence of powers of the map which converges weakly to the identity As is well-known this forces the centralizer to be uncountable

We owe a large debt of gratitude to S Glasner He was the first to formulate a theorem like our theorem 4 1, in the case of a $\mathbb{Z}$-action with MSJ, which in addition has a strong condition on generic points (This condition was established for the Chacón example in **[J, K]** ) We then realized that we could prove theorem 4 1 for any $\mathbb{Z}$-action with MSJ and Glasner independently found a different proof **[Gl]** Then we extended the result to arbitrary groups and simple actions Several of our examples and proofs have also been simplified by suggestions of Glasner's

We also wish to thank the referee for providing us with the short elegant proof of theorem 4 1, which replaces our original long ugly one

*Section* 1

1 1 *Boolean G-spaces* Suppose that $\mathscr{X}$ is a $G$-space The action of $G$ may also be viewed as an action on sets, that is, as a Boolean $G$-action as defined by Mackey **[Ma1]** We denote by $B(\mu)$ the $\sigma$-Boolean algebra of Borel subsets of $X$, two subsets being identified when they differ by a null set The measure $\mu$ is well-defined on $B(\mu)$ and again denoted $\mu$ $B(\mu)$ is a complete metric space under the metric $\mu(E\Delta F)$ We say $B(\mu)$ is a Boolean $G$-space if $G$ acts on $B(\mu)$ by measure-preserving $\sigma$-Boolean algebra automorphisms and for $E\in B(\mu)$ the map $g\mapsto gE$ is Borel (We have followed the definition in **[Ram]** which is easily seen to be equivalent to Mackey's Note also that we do not consider abstract Boolean $G$-spaces ) If $\mathscr{X}$ is a $G$-space then $B(\mu)$ becomes a Boolean $G$-space under the natural $G$-action as is shown in Lemma 1 of **[MA1]** Boolean $G$-spaces arise in another natural way Let $G(\mu)$ denote the group of all measure-preserving invertible Borel maps of $X$, two maps being identified when they agree a e $G(\mu)$ is a complete separable metric group under the weak topology ($S_n\to S\Leftrightarrow S_n(A)\to S(A)$ in $B(\mu)\forall A\in B(\mu)$) If $G$ is a locally compact subgroup of $G(\mu)$ then the natural $G$-action on $B(\mu)$ is a Boolean $G$-action since $g\mapsto gA$ is continuous by definition of the weak topology

Two Boolean $G$-spaces $B(\mu)$ and $B(\nu)$ are said to be isomorphic if there is a measure-preserving Boolean algebra isomorphism $B(\mu)\to B(\nu)$ which is $G$-equivariant $B(\nu)$ is said to be a factor of $B(\mu)$ if there is a $G$-equivariant measure-preserving Boolean algebra homomorphism $\psi$ $B(\nu)\to B(\mu)$

1 2 *Isomorphism of G-actions* If $\mathscr{X}$ and $\mathscr{Y}$ are $G$-actions, by an isomorphism $\phi$ $\mathscr{X}\to\mathscr{Y}$ we mean a measure-preserving Borel isomorphism $\phi$ between $G$-invariant Borel co-null subsets $X^*$ and $Y^*$ of $X$ and $Y$ which is also $G$-equivariant Suppose that $\phi'$ is a measure-preserving Borel isomorphism between Borel co-null subsets of $X$ and $Y$ such that for each $g\in G$, $\phi'(gx)=g(\phi'x)$ for a a $x$ (the null set may depend on $g$!) Then there is an isomorphism $\phi$ $\mathscr{X}\to\mathscr{Y}$ which agrees with $\phi'$ a e $\phi'$ induces an isomorphism of the Boolean $G$-spaces associated with $X$ and $Y$ and by theorem 2 of [**Ma1**] this Boolean isomorphism is induced by an isomorphism of $\mathscr{X}$ and $\mathscr{Y}$ Since $\phi$ and $\phi'$ induce the same Boolean map they must be equal a e

Suppose that $\phi_i$ $\mathscr{X}_i\to\mathscr{Y}_i$, $i=1,\quad,k$, are $G$-space isomorphisms defined between $G$-invariant Borel co-null subsets $X_i^*$ and $Y_i^*$ Then $\phi=\phi_1\times\quad\times\phi_k$ is a Borel isomorphism between $X_1^*\times\quad\times X_k^*=U$ and $Y_1^*\times\quad\times Y_k^*=V$ If $\lambda\in J(\mathscr{X}_1,\quad,\mathscr{X}_n)$ then $\lambda$ is supported on $U$ so $\phi(\lambda|_U)$ is a Borel probability measure on $V$ This measure may be regarded as a Borel measure on $Y_1\times\quad\times Y_n$ which is evidently a joining Thus the notion of joining is preserved under isomorphism and in particular simplicity and MSJ are preserved under isomorphism

1 3 *Factor maps* By a factor map $\phi$ $\mathscr{X}\to\mathscr{Y}$ we mean a measure-preserving Borel map from a $G$-invariant Borel co-null subset $X^*$ of $X$ to $Y$ which is also $G$-equivariant If $\phi'$ is a measure-preserving Borel map from a Borel co-null subset $X^*$ of $X$ to $Y$ such that $\phi'(gx)=g(\phi'x)$ a e then there exists a factor map $\phi$ $\mathscr{X}\to\mathscr{Y}$ which agrees with $\phi'$ a e This is proved in the course of the proof of Proposition 2 1 of [**Zi1**] The same argument shows that if $\psi$ $B(\nu)\to B(\mu)$ is a $G$-equivariant injective measure-preserving Boolean algebra homomorphism then there is a factor map $\phi$ $\mathscr{X}\to\mathscr{Y}$ such that $\phi^{-1}=\psi$

If $\phi$ $\mathscr{X}\to\mathscr{Y}$ is a factor map $\phi^{-1}(B(Y))$ is a $G$-invariant sub-$\sigma$-algebra of $B(X^*)$ which can be extended in a natural way to the $G$-invariant sub-$\sigma$-algebra $\mathscr{G}$ of $B(X)$ consisting of all Borel sets agreeing a e with some set in $\phi^{-1}B(Y)$ Evidently $\mathscr{G}$ is unchanged if $\phi$ is replaced by a factor map $\phi'=\phi$ a e We write $\mathscr{G}=\phi^{-1}(\mathscr{Y})$ and call it the factor algebra generated by $\phi$ In general we call any $G$-invariant sub-$\sigma$-algebra of $B(X)$ which contains all the null sets a factor algebra of $\mathscr{X}$ Proposition 2 1 of [**Zi1**] guarantees that every factor algebra of $\mathscr{X}$ is generated by a factor map Note further that if $\phi$ $\mathscr{X}\to\mathscr{Y}$ and $\phi$ $\mathscr{X}\to\mathscr{Y}'$ are factor maps generating the same factor algebra then $\mathscr{Y}$ and $\mathscr{Y}'$ are isomorphic, since the associated Boolean $G$-actions are isomorphic Thus a factor algebra of $\mathscr{X}$ gives rise to a factor $\mathscr{Y}$ which is unique up to isomorphism

If $\phi_i$ $\mathscr{H}_i\to\mathscr{Y}_i$, $i=1,\quad,n$, are factor maps generating factor algebras $\mathscr{G}_i$ there is a natural correspondence between $J(\mathscr{Y}_1,\quad,\mathscr{Y}_n)$ and $G$-invariant measures on $\mathscr{G}_1\times\quad\times\mathscr{G}_n$ with marginals $\mu$ On the one hand any such measure $\lambda$ on $\mathscr{G}_1\times\quad\times\mathscr{G}_n$ clearly projects under $\phi=\phi_1\times\quad\times\phi_n$ to a joining $\bar\lambda$ of $\mathscr{Y}_1,\quad,\mathscr{Y}_n$ just as we

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

discussed in the case of isomorphisms, if we note in addition that the domain of $\phi$ is a product of co-null subsets, hence belongs to $\mathcal{G}_1 \times \quad \times \mathcal{G}_n$ On the other hand if $\bar{\lambda} \in J(\mathcal{Y}_1, \quad, \mathcal{Y}_n)$ it lifts under $\phi$ to a measure $\lambda$ on the $\sigma$-algebra $\phi^{-1}(B(Y_1 \times \quad \times Y_n)) = \phi_1^{-1}B(Y_1) \times \quad \times \phi_n^{-1}B(Y_n)$ in $X_1^* \times \quad \times X_n^*$ Because $\lambda$ has marginals $\mu_i|_{X_i^*}$, it may be extended to a measure on $\mathcal{G}_1 \times \quad \times \mathcal{G}_n$

1 4 *Integration of measures* If $(X, B(X))$ is a standard Borel space we denote by $M(X)$ the space of Borel probability measures on $X$ We give $M(X)$ the Borel structure generated by all the functions $\mu \mapsto \mu(f) = \int_X f\, d\mu$ for bounded Borel $f$ Since such an $f$ is an increasing pointwise limit of simple functions $f_n$ and $\mu(f_n) \to \mu(f)$ for each $\mu$, this Borel structure is also generated by the functions $\mu \mapsto \mu(1_A)$, $A$ Borel It is also not hard to see that this Borel structure is generated by the weak-$*$ topology on $M(X)$ when $M(X)$ is viewed as a set of linear functionals on $C(X)$, $X$ being given any compact metric topology generating its Borel structure

If $(Y, G)$ is a measurable space a measurable map $Y \to M(X)$, denoted $y \mapsto \mu_y$ will be called a measurable field of measures We can integrate $\mu_y$ to obtain a measure $\mu = \int \mu_y \, d\nu(y)$ defined by

$$\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$$

Approximating by simple functions one sees that

$$\mu(f) = \int_Y \mu_y(f)\, d\nu(y)$$

for each bounded measurable $f$

If $Y$ is complete metric and $\mathcal{G} = B(Y)$ then $y \mapsto \delta_y$ is a continuous map into $M(Y)$ Since $(\sigma, \tau) \mapsto \sigma \times \tau$ is a continuous map from $M(Y) \times M(X) \mapsto M(Y \times X)$ we conclude that $y \mapsto \delta_y \times \mu_y$ is a measurable field Thus whenever $Y$ is a standard Borel space we may define the direct integral measure $\lambda = \int^{\oplus} \mu_y \, d\nu(y)$ on $Y \times X$ by

$$\lambda = \int_Y \delta_y \times \mu_y \, d\nu(y)$$

In other words

$$\lambda(A) = \int_Y \mu_y(A \cap \{y\} \times X)$$

for $A$ Borel in $Y \times X$

1 5 *Disintegration of measures* We continue to suppose that $X$ and $Y$ are standard Suppose that $\mu$ and $\nu$ are Borel probabilities on $X$ and $Y$ and that $\phi \; X \to Y$ is a measure-preserving Borel map Then $\lambda$ may be disintegrated over the fibres of $\phi$, that is there is a measurable field $y \mapsto \mu_y$ such that $\mu_y$ is supported on $\phi^{-1}\{y\}$ and

$$\mu = \int_Y \mu_y \, d\nu(y)$$

Moreover $\mu_y$ is $\nu$-essentially unique (See Theorem 5 8 of [**Fu**] ) As a special case, if $\lambda$ is a Borel probability on $Y \times X$ projecting onto the measure $\nu$ on $Y$ then the distintegration takes the form

$$\lambda = \int_Y^{\oplus} \lambda_y \, d\nu(y)$$

where the $\lambda_y$ are measures on $X$ If $X$ and $Y$ are also $G$-spaces and $\phi$ is a $G$-factor map then the invariance of $\lambda$ together with uniqueness of the disintegration yields that for $g \in G$ $g\mu_y = \mu_{gy}$ for $\nu$ a a $y$ (the exceptional null set may depend on $g$)

1 6 *Relatively independent extension and relative product* Suppose now that $\phi_i$ $\mathscr{X}_i \to \mathscr{Y}_i$, $i = 1, \ldots, n$, are factor maps on $G$-actions and that $\mu_i$ has the disintegration

$$\mu_i = \int_{Y_i} \mu_{iy} \, d\nu_i(y)$$

If $\lambda \in J(\mathscr{Y}_1, \ldots, \mathscr{Y}_n)$ we define its relatively independent extension $\hat{\lambda} \in J(\mathscr{X}_1, \ldots, \mathscr{X}_n)$ by

$$\hat{\lambda} = \int_{Y_1 \times \cdots \times Y_n} \mu_{1y} \times \cdots \times \mu_{ny} \, d\lambda(y)$$

It is easy to check that $\hat{\lambda}$ is indeed a joining

We shall mainly use a special case of this construction, the relative product If $\phi_i$ $\mathscr{X}_i \to \mathscr{Y}$, $i = 1, \ldots, n$, are factor maps of $G$-actions we define the $Y$-relatively independent product of $\mathscr{X}_1, \ldots, \mathscr{X}_n$, $\lambda \in J(\mathscr{X}_1, \ldots, \mathscr{X}_n)$ by

$$\lambda = \int_Y \mu_{1y} \times \cdots \times \mu_{ny} \, d\nu(y),$$

in other words $\lambda$ is the relatively independent extension of the diagonal $n$-joining of $\mathscr{Y}$

1 7 *Ergodic decompositions* Suppose we have a Borel action of a locally compact group $G$ on the standard Borel space $X$ We denote by $M_G(X)$ the space of $G$-invariant Borel probability measures on $X$ and by $M_G^e(X)$ the space of ergodic $G$-invariant probabilities on $X$ $M_G(X)$ is convex and, as is well-known and easy to prove $\mathrm{ext}\,(M_G(X)) = M^e(X)$ The following theorem is from § 4 of [**Va2**]

THEOREM 1 7 *$M_G(X)$ and $M_G^e(X)$ are Borel subsets of $M(X)$ There exists a $G$-invariant Borel map $\beta$ $X \to M_G^e(X)$, called a decomposition map of the $G$-space $X$ such that $\mu\{x$ $\beta(x) = \mu\} = 1$ for all $\mu \in M_G^e(X)$ and for any $\lambda \in M_G(X)$, $\lambda(A) = \int_X \beta(x)(A)\, d\lambda(x)$ Moreover for each $\lambda \in M_G(X)$ there is a unique Borel probability measure $\nu$ on $M_G^e(X)$ such that $\lambda = \int_{M_G^e(X)} \sigma \, d\nu(\sigma)$*

We will mainly apply the ergodic decomposition to joinings If $\mathscr{X}$ and $\mathscr{Y}$ are, as always, ergodic and $\lambda \in J(\mathscr{X}, \mathscr{Y})$ then $\lambda$ has an ergodic decomposition

$$\lambda = \int_{M_G^e(X \times Y)} \sigma \, d\nu(\sigma),$$

where $\nu$ is a Borel probability on the space of $G$-invariant ergodic Borel probabilities

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

on $X\times Y$ Denoting projection on $X$ by $\pi$ we have

$$\mu=\pi\lambda=\int_{M_G^e(X\times Y)}\pi(\sigma)\,d\nu(\sigma)$$

By extremality (ergodicity) of $\mu$ we conclude that $\nu\{\sigma\ \ \pi(\sigma)=\mu\}=1$ (Note that $\{\sigma\ \ \pi\sigma=\mu\}$ is Borel ) Similarly $\nu$-a a $\sigma$ have marginal $\nu$ on $Y$, so we have established the following

COROLLARY 1 7 *If $\lambda\in J(\mathscr{X},\mathscr{Y})$ then there is a unique Borel probability on $J^e(\mathscr{X},\mathscr{Y})$ the (Borel) set of ergodic joinings of $\mathscr{X}$ and $\mathscr{Y}$, such that*

$$\lambda=\int_{J^e(\mathscr{X}\ \mathscr{Y})}\sigma\,d\nu(\sigma)$$

1 8 *Group extensions* Let $\mathscr{Y}$ be a $G$-action and $K$ a compact metric group Suppose that $a\ \ G\times Y\to K$ is Borel and that the prescription

$$g(y,k)=(gy,a(g,y)k)$$

defines an action of $G$ on $Y\times K$ This amounts to requiring that $a$ satisfy the cocycle equation

$$a(g_2g_1,y)=a(g_2,g_1y)a(g_1,y)$$

but we will have no occasion to use this The action is evidently Borel and preserves the measure $\mu=\nu\times dk$ ($dk$ denotes normalized Haar measure on $K$) We denote this $G$-action by $\mathscr{Y}\times_a K$ and refer to it as a group extension (or $K$-extension) of $\mathscr{Y}$ $K$ acts on the right on $\mathscr{Y}\times K$ by $(y,k)k_0=(y,kk_0)$ and the action of $K$ commutes with that of $G$ We denote the action of $k\in K$ by $R_k$ Finally the projection $Y\times K\to Y$ is a $G$-factor map and the $\sigma$-algebra it generates is the $\sigma$-algebra of (a e ) $K$-invariant Borel sets in $\mathscr{Y}\times K$

Now write $\mathscr{X}=\mathscr{Y}\times_a K$ and assume further that $\mathscr{X}$ is ergodic (The interested reader may refer to Corollary 3 8 of [**Zi1**] for a necessary and sufficient condition for ergodicity ) Then we have, abstractly, the following setup $\mathscr{X}$ is an ergodic $G$-action, $K$ is a subgroup of $C(\mathscr{X})$ which is compact in the weak topology and the $\sigma$-algebra of sets fixed by $K$ is the algebra generated by the projection onto $X$ The following well-known result asserts that all such abstract situations arise as $K$-extensions

If $\mathscr{X}$ is a $G$-action and $H$ is any subgroup of $C(\mathscr{X})$, the fixed algebra of $H$

$$\mathscr{G}(H)=\{A\ \ hA=A \text{ a e } \forall h\in H\}$$

is a factor algebra of $\mathscr{X}$ On the other hand if $\mathscr{G}$ is a factor algebra of $\mathscr{X}$ we define the closed subgroup of $C(\mathscr{X})$

$$H(\mathscr{G})=\{h\in C(\mathscr{X})\ \ hA=A\,\forall A\in\mathscr{G}\}$$

THEOREM 1 8 1 *Suppose $\mathscr{X}$ is an ergodic $G$-action and $K$ is a compact subgroup of $C(\mathscr{X})$ Then there is an isomorphism $\phi$ of $\mathscr{X}$ with a $K$-extension $\mathscr{X}'=\mathscr{Y}\times_a K$ which satisfies $\phi(kA)=R_k(\phi A)$ a e for each Borel $A\subset X$ In particular $\phi$ carries $\mathscr{G}(K)$ onto the factor algebra of $\mathscr{X}'$ generated by the projection onto $\mathscr{Y}$ Moreover $K=H(\mathscr{G}(K))$*

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

*Proof* Since this is well-known we do only the 'moreover' Note that $H(\mathscr{G}(K)\supset K$ trivially Suppose now that $h\in H(\mathscr{G}(K))$ By the first part of the theorem we may as well assume that $\mathscr{X}$ itself is $\mathscr{Y}\times_a K$ Since $h$ fixes $\mathscr{G}(K)\simeq\mathscr{Y}$ setwise the off-diagonal measure $\mu_h$ is concentrated on the set

$$\{(y,y,k_1,k_2)\ \ y\in Y, k_i\in K\}\subset(Y\times K)^2$$

The function $\theta(y,y,k_1,k_2)=k_1^{-1}k_2$ is $G$-invariant, so by ergodicity of $\mu_h$ (which is just ergodicity of $\mathscr{X}$) we have $\theta=k_0$, $\mu_h$-a e This means that $h(y,k)=(y,kk_0)$ $\mu$-a e □

In view of theorem 1 8 1, whenever $K$ is a compact subgroup of $C(\mathscr{X})$ we may, by passing to an isomorphic copy of $\mathscr{X}$, assume that there is a (pointwise) action of $K$ realizing its Boolean action and commuting (everywhere) with the $G$-action In this situation we will write $\mathscr{X}/\mathscr{K}$ for the space of $K$-orbits with the quotient $G$-action and quotient measure As Boolean spaces $\mathscr{X}/\mathscr{K}$ and $\mathscr{G}(K)$ are isomorphic

Suppose $\mathscr{X}=\mathscr{Y}\times_a K$ is an ergodic $K$-extension For $h\in K$ the off-diagonal joining $\mu_{R_h}\in J^e(\mathscr{X},\mathscr{X})$ has the disintegration

$$\mu_{R_h}=\int_{Y\times K}\delta_y\times\delta_k\times\delta_y\times\delta_{kh}\,d\mu(y)\,dk$$

Let $\lambda\in J(\mathscr{X},\mathscr{X})$ denote the $\mathscr{Y}$-relatively independent product of $\mathscr{X}$ with itself, that is

$$\lambda=\int_Y\delta_y\times dk\times\delta_y\times dk\,d\mu(y)$$

Then the ergodic decomposition of $\lambda$ is evidently

$$\lambda=\int_K\mu_{R_h}\,dh$$

Thus if $\mathscr{X}$ is a group extension of $\mathscr{Y}$ the ergodic decomposition of the relative product is supported on off-diagonal measures, i e measures which identify the two co-ordinate $\sigma$-algebras in $X\times X$ The following theorem is the converge, stated in terms of factor algebras It is essentially proved in [Ve], although not explicitly stated there We include the proof for completeness

When $\mathscr{G}$ is a factor algebra of $\mathscr{X}$ the $\mathscr{G}$-relatively independent product of $\mathscr{X}$ with itself is the 2-joining $\lambda=\mu\times_{\mathscr{G}}\mu$ of $\mathscr{X}$ defined on rectangles by

$$\lambda(A\times B)=\int_X P(A|\mathscr{G})P(B|\mathscr{G})\,d\mu$$

Of course if $\mathscr{G}$ is generated by a factor map this coincides with the definition in § 1 6 Note that $\lambda(A\times A^c)=0$ if and only if $A\in\mathscr{G}$

We observe that the map $h\mapsto\mu_h$ from $C(\mathscr{X})$ into $J(\mathscr{X},\mathscr{X})$ is a Borel isomorphism from $C(\mathscr{X})$ onto a Borel subset of $J(\mathscr{X},\mathscr{X})$ Thus the assumption that the ergodic decomposition of $\mu\times_{\mathscr{G}}\mu$ is supported on off-diagonal measures is meaningful and amounts to saying that

$$\mu\times_{\mathscr{G}}\mu=\int_{C(\mathscr{X})}\mu_h\,d\tau(h)$$

for some Borel probability $\tau$ on $C(\mathscr{X})$

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

THEOREM 1 8 2 (Veech) *Suppose that $\mathscr{X}$ is a G-action, $\mathscr{G}$ is a factor algebra and that the ergodic decomposition of $\lambda = \mu \times_{\mathscr{G}} \mu$ is supported on off-diagonals, that is*

$$\lambda = \int_{C(\mathscr{X})} \mu_h \, d\tau(h)$$

*for some Borel probability $\tau$ on $C(\mathscr{X})$ Then $H(\mathscr{G})$ is a compact subgroup of $C(\mathscr{X})$, $\tau$ is Haar measure on $H(\mathscr{G})$ and $\mathscr{G} = \mathscr{G}(H(\mathscr{G}))$*

*Proof* For $A \in \mathscr{G}$ we have

$$0 = \lambda(A \times A^c) = \int_{C(\mathscr{X})} \mu(A \cap h^{-1}A^c) \, d\tau(h),$$

that is $\tau\{h \ \ \mu(A \cap h^{-1}A^c) = 0\} = 1$ Choosing a countable family $\{A_i\}$ dense in $\mathscr{G}$ and observing that

$$H(\mathscr{G}) = \bigcap_i \{h \ \ \mu(A_i \cap h^{-1}A_i^c) = 0\},$$

we conclude that $\tau(H(\mathscr{G})) = 1$

Now for $h_0 \in H(\mathscr{G})$, $P(h_0^{-1}B \mid \mathscr{G}) = P(B \mid \mathscr{G})$ so $\lambda(A \times h_0^{-1}B) = \lambda(A \times B)$, that is $(\mathrm{id} \times h_0)\lambda = \lambda$ But

$$\begin{aligned}(\mathrm{id} \times h_0)\lambda &= \int_{C(\mathscr{X})} (\mathrm{id} \times h_0)\mu_h \, d\tau(h) \\ &= \int_{H(\mathscr{G})} \mu_{h_0 h} \, d\tau(h) = \lambda \\ &= \int_{H(\mathscr{G})} \mu_h d\tau(h)\end{aligned}$$

By uniqueness of the ergodic decomposition of $\lambda$ we conclude that $\tau$ is invariant under left multiplication by $h_0$ Since $h_0 \in H(\mathscr{G})$ was arbitrary, $\tau$ is a left-invariant probability measure on $H(\mathscr{G})$ The proof of Proposition 4 5 of **[Ve]** includes a short proof that if a separable metric topological group admits a left-invariant Borel probability then the group is compact Thus $H(\mathscr{G})$ is compact Finally $\mathscr{G} \subset \mathscr{G}(H(\mathscr{G}))$ is trivial On the other hand if $A \in \mathscr{G}(H(\mathscr{G}))$ then

$$\lambda(A \times A^c) = \int_{H(\mathscr{G})} \mu(A \cap h^{-1}A^c) \, d\tau(h) = 0,$$

so $A \in \mathscr{G}$ □

## 2 *The non-weakly-mixing case*

Let $K$ be a compact group with normalized Haar measure $dk$, $G$ a locally compact group and $\phi \ \ G \to K$ a Borel homomorphism onto a dense sugroup of $K$ $K$ is then an ergodic $G$-space under the action $gk = \phi(g)k$ This action is a $K$-extension of the trivial (one-point) action of $G$ Such an action is simple, in a rather trivial way, as product measure is not an ergodic joining We will need the more general fact that ergodic group extensions are 'relatively simple' in a sense made precise by the following theorem

THEOREM 2 1 *Suppose $\mathscr{X} = \mathscr{Y} \times_a K$ is an ergodic group extension and $\lambda$ is an ergodic joining of $\mathscr{X}$ with itself which is diagonal on $\mathscr{Y}$, that is the projection of $\lambda$ on $Y \times Y$ is diagonal measure Then $\lambda$ is an off-diagonal measure of the form $\mu_{R_{k_0}}$ where $\mu = \nu \times dk$ on $Y \times K$ and $R_{k_0}$ denotes right multiplication by $k_0$ on $Y \times K$*

*In particular the action of G on K by left multiplications described above is simple and its centralizer consists of all the right multiplications by K It has* MSJ *if and only if $\phi(G) = K$ and K is abelian*

*Proof* The hypothesis that $\lambda$ is diagonal on $\mathscr{Y}$ means that $\lambda$ is supported on $\{(y, y, k_1, k_2) \ \ y \in Y, k_i \in k\}$ It is invariant under the action

$$g(y, y, k_1, k_2) = (gy, gy, a(g, y)k_1, a(g, y)k_2)$$

and has marginal $\mu$ on both $Y \times K$ factors Now the function

$$\theta(y, y, k_1, k_2) = k_1^{-1}k_2$$

is evidently $G$-invariant so it is equal to a constant, say $k_0$, $\lambda$-a e In other words $\lambda$ is supported on $\{y, y, k, kk_0 \ \ y \in Y, k \in K\}$ and since $\lambda$ has marginal $\mu$ it is of the desired form

Specializing to the case where $\mathscr{Y}$ is trivial it follows that the action of $G$ on $K$ by left multiplications is 2-fold simple with centralizer consisting of all the $R_k$, $k \in K$ $n$-fold simplicity can be deduced directly Thus if $\phi(G) = K$ and $K$ is abelian the action evidently has MSJ since it is its own centralizer On the other hand if it has MSJ then for each $k \in K$, $R_k$ agrees a e with some left multiplication $L_{\phi(g)}$ and hence, by continuity, agrees everywhere with $L_{\phi(g)}$ In particular

$$\phi(g) = L_{\phi(g)}e = R_k e = k$$

Thus $\phi(G) = K$ and $R_k = L_k$ so $K$ is abelian □

THEOREM 2 2 *If a simple action $\mathscr{X}$ is not weakly mixing then it is isomorphic to an action by left multiplications as in Theorem* 2 1

*Proof* Since $\mathscr{X}$ is simple and product measure is not ergodic, the hypotheses of Theorem 1 8 2 are satisfied with $G$ the trivial factor algebra By theorems 1 8 1 and 1 8 2 $\mathscr{X}$ is isomorphic to a $K$-extension of the trivial (one-point) $G$-action This means $X = K$ and $g \in G$ acts via left multiplication by $\phi(g) \in K$ $\phi$ must be a homomorphism and it is easy to see that ergodicity forces $\overline{\phi(G)} = K$

## 3 *Factors of simple actions*

Recall the definitions of $H(\mathscr{G})$ and $\mathscr{G}(H)$ (§ 1 8) For the case of $\mathbb{Z}$-actions the following theorem is due to Veech (Theorem 1 2 of [**Ve**])

THEOREM 3 1 *Suppose $\mathscr{X}$ is 2-fold simple and $\mathscr{G}$ is a non-trivial factor algebra Then $H(\mathscr{G})$ is compact and $\mathscr{G}(H(\mathscr{G})) = \mathscr{G}$*

*Proof* Let $\lambda$ denote the $\mathscr{G}$-relative product of $\mathscr{X}$ with itself As in 1 7, $\lambda$ can be expressed as an integral of ergodic joinings Since the only ergodic joinings are $\mu \times \mu$ and off-diagonals we have

$$\lambda = c(\mu \times \mu) + \int_{C(\mathscr{X})} \mu_h \, d\tau(h)$$

for some Borel sub-probability $\tau$ on $C(\mathscr{X})$ Choosing any non-trivial $A \in \mathscr{G}$ we have

$$0 = \lambda(A \times A^c) = c\mu(A)\mu(A^c) + \int_{C(\mathscr{X})} \mu_h(A \times A^c)\, d\tau(h)$$

Since $\mu(A)\mu(A^c) \neq 0$ we have $c = 0$, so we are in the situation of Theorem 1 8 2 □

*Remark* $\mathscr{X}$ is called prime if its only factor algebras are the algebra of Borel null or co-null sets and the full Borel algebra $B(X)$ It follows from Theorem 3 1 that a simple weakly mixing $\mathscr{X}$ is prime if and only if $C(\mathscr{X})$ has no compact subgroups other than {id} The 'if' direction is clear For the 'only if' if $K \neq \{\mathrm{id}\}$ is a compact subgroup of $C(\mathscr{X})$ then $\mathscr{G}(K)$ is not $B(X)$, so $\mathscr{G}(K)$ is the null, co-null algebra, that is, $K$ acts ergodically This means $\mathscr{X}$ is a $K$-extension of the trivial one-point $G$-action, which contradicts weak mixing

Alternately, a simple weakly mixing $\mathscr{X}$ is prime if and only if each $S \in C(\mathscr{X})$ such that $S \neq \mathrm{id}$ is ergodic On the one hand if $S \in C(\mathscr{X})$ is not ergodic $\{A \ SA = A\}$ is a non-trivial factor algebra On the other hand if $\mathscr{G}$ is a non-trivial factor algebra then $H(\mathscr{G})$ is non-trivial since $\mathscr{G} = \mathscr{G}(H(\mathscr{G}))$ Any $S \in H(\mathscr{G})$ fixes each set $A \in \mathscr{G}$, so $S$ is non-ergodic

We now fix for the remainder of this section for each compact subgroup $K$ of $C(\mathscr{X})$ a factor map generating $\mathscr{G}(K)$ and denote it $\phi_K \ \mathscr{X} \to \mathscr{X}/K$ (Note that here $\mathscr{X}/K$ is not the space of $K$-orbits - strictly speaking $K$ has no orbits ) We will identify $\mathscr{G}(K)$ with $\mathscr{X}/K$, via the map $\phi_K$

Now suppose $K_1, \quad, K_k$ are compact subgroups of $C(\mathscr{X})$ and $S_1, \quad, S_k \in C(\mathscr{X})$ The off-diagonal $k$-joining $(S_1 \times \quad \times S_k)\mu_\Delta$ of $\mathscr{X}$ projects onto a joining $\lambda$ of $\mathscr{X}/K_1, \quad, \mathscr{X}/K_k$ We call such a joining rigid A rigid joining need not be off-diagonal as will be clarified by the following results

Another way to describe the above joining $\lambda$ is as follows $\phi_{K_i} S_i$ is a factor map generating the factor algebra $S_i^{-1}\mathscr{G}(K_i) = \mathscr{G}(S_i^{-1} K_i S_i)$ and $\lambda$ is evidently the joining induced by the imbeddings via $\phi_{K_i} S_i$ of the actions $\mathscr{X}/K_i$ in $\mathscr{X}$ From this point of view the results we are about to describe bear an interesting formal similarity to Ratner's results on joinings of horocycle flows, [**Ra**]

THEOREM 3 2 *If $\mathscr{X}$ is a simple action and $K_1, \quad, K_k$ are compact subgroups of $C(\mathscr{X})$ then every ergodic joining of $\mathscr{X}/K_1 \quad \mathscr{X}/K_k$ is a product of rigid joinings*

*Proof* If $\lambda$ is an ergodic joining of $\mathscr{X}/K_1, \quad, \mathscr{X}/K_k$ denote by $\hat{\lambda}$ the relatively independent extension (§ 1 6) of $\lambda$ to a $k$-joining of $\mathscr{X}$ $\hat{\lambda}$ may be decomposed as an integral $\hat{\lambda} = \int \tau\, d\sigma(\tau)$ of ergodic $k$-joinings of $\mathscr{X}$, which, by simplicity, are all POOD's Denoting the map $\phi_{K_1} \times \quad \times \phi_{K_k}$ by $\pi$ we have

$$\lambda = \int \pi\tau\, d\sigma(\tau)$$

By extremality of $\lambda$ we have $\pi\tau = \lambda$ for $\sigma$-a a $\tau$ In particular there is at least one POOD $\tau$ such that $\pi\tau = \lambda$ so $\lambda$ is a product of rigid joinings □

If $K_1 \subset K_2$ are compact subgroups of $C(\mathscr{X})$ then $\mathscr{G}(K_1) \supset \mathscr{G}(K_2)$ so there is a natural factor map from $\mathscr{X}/K_1$ to $\mathscr{X}/K_2$ More generally if $S \in C(\mathscr{X})$ and $S^{-1}K_1S \subset K_2$ then $S$ induces a Boolean isomorphism from $\mathscr{G}(K_1)$ to $S^{-1}\mathscr{G}(K_1) = \mathscr{G}(S^{-1}K_1S) \supset \mathscr{G}(K_2)$ and thus a (pointwise) factor map denoted $S_{K_1,K_2}$ $\mathscr{X}/K_1 \to \mathscr{X}/K_2$

COROLLARY 3 3 *If $K_1$ and $K_2$ are compact subgroups of $C(\mathscr{X})$ each factor map $\mathscr{X}/K_1 \to \mathscr{X}/K_2$ is an $S_{K_1 K_2}$ for some $S \in C(\mathscr{X})$ such that $S^{-1}K_1S \subset K_2$ If the factor map is an isomorphism then $S^{-1}K_1S = K_2$ $C(\mathscr{X}/K_1) = \{S_{K_1 K_1}$ $S^{-1}K_1S = K_1\}$*

*Proof* If $\phi$ $\mathscr{X}/K_1 \to \mathscr{X}/K_2$ is a factor map the corresponding joining $\lambda$ of $\mathscr{X}/K_1$ and $\mathscr{X}/K_2$ is, by theorem 3 2, the projection of an off-diagonal Lifting $\phi$ to a set map $\hat{\phi}$ $\mathscr{G}(K_2) \to \mathscr{G}(K_1)$ we thus have that there is an $S \in C(\mathscr{X})$ such that

$$\mu(A \cap \hat{\phi}B) = \mu(A \cap SB)$$

for $A \in \mathscr{G}(K_1)$, $B \in \mathscr{G}(K_2)$ Taking $A = \hat{\phi}(B)$ shows that $\hat{\phi}(B) = S(B)$ (a e ) for each $B$ In particular $S\mathscr{G}(K_2) = \mathscr{G}(SK_2S^{-1}) \subset \mathscr{G}(K_1)$ whence, by Theorem 3 1 $SK_2S^{-1} \supset K_1$ so $S^{-1}K_1S \subset K_2$ If $\phi$ is an isomorphism we have $\hat{\phi}$ $\mathscr{G}(K_2) = \mathscr{G}(K_1)$ so $\mathscr{G}(SK_2S^{-1}) = \mathscr{G}(K_1)$ and $S^{-1}K_1S = K_2$ The last statement is an immediate consequence since $C(\mathscr{X}/K_1)$ is the automorphism group of $\mathscr{X}/K_1$ □

The following lemma will be technically useful

LEMMA 3 4 *Suppose $\mathscr{Y}_1 = \mathscr{X}/K_1$ and $\mathscr{Y}_2 = \mathscr{X}/K_2$ are factors of the simple G-action $\mathscr{X}$ and $\lambda$ is a rigid joining of $\mathscr{X}/K_1$ and $\mathscr{X}/K_2$ Then the extension*

$$(Y_1 \times Y_2, \lambda) \to (Y_1, \nu_1)$$

*has relatively discrete spectrum in the sense of* [**Zi1**]

*Proof* Since $\lambda$ is rigid it is the projection of an off-diagonal 2-joining $\mu_S$, $S \in C(\mathscr{X})$ Now the extension $\mathscr{X} \to \mathscr{X}/K_1$ has relatively discrete spectrum by theorem 1 8 and Example 4 1 of [**Zi1**] It is isomorphic as an extension to the extension $(X \times X, \mu_S) \to \mathscr{X}/K_1$ so this extension also has relatively discrete spectrum But this extension is the composition

$$(X \times X, \mu_s) \to (Y_1 \times Y_2, \lambda) \to (Y_1, \nu_1)$$

Thus the second extension must also have relatively discrete spectrum by the following well-known lemma, whose proof we omit □

LEMMA 3 5 *If $\mathscr{X} \to \mathscr{Z}$ is an extension with relatively discrete spectrum, which factors as $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ then $\mathscr{Y} \to \mathscr{Z}$ also has relatively discrete spectrum*

COROLLARY 3 6 *If $\mathscr{X}$ is simple and $K$ is a compact subgroup of $C(\mathscr{X})$ then $\mathscr{X}/K$ is simple if and only if $K$ is normal in $C(\mathscr{X})$*

*Proof* Suppose $K$ is normal By Theorem 3 2 it will suffice to show that each rigid $k$-joining of $\mathscr{X}/K$ is off-diagonal Such a joining is the projection of an off-diagonal $k$-joining $(S_1 \times \quad \times S_k)\mu_\Delta$ of $\mathscr{X}$ Since $S_i^{-1}KS_i = K$, $S_{iK,K} \in C(\mathscr{X}/K)$(Corollary 3 3) and the joining in question is the off-diagonal $(S_{1K,K} \times \quad \times S_{kK,K})\bar{\mu}_\Delta$, where $\bar{\mu}_\Delta$ is the diagonal $k$-joining of $\mathscr{X}/K$

Now suppose $\mathscr{X}/K$ is simple and consider the rigid 2-joining $\lambda$ of $\mathscr{X}/K$ which is the projection of the off-diagonal 2-joining $\mu_S$ of $\mathscr{X}$, $S \in C(\mathscr{X})$ If $\mathscr{X}/K$ is not weakly mixing then $\lambda$ is an off-diagonal On the other hand if $\mathscr{X}/K$ is weakly mixing then product measure is a weakly mixing extension of $\mathscr{X}/K$ while $\lambda$, by 3 4, is an extension of $\mathscr{X}/K$ with relatively discrete spectrum which is incompatible with relative weak mixing by Lemma 8 11 and Theorem 8 7 of [**Zi2**] ($\mathscr{X}/K$ is not trivial!) Thus in the weakly mixing case we can also conclude that $\lambda$ is an off-diagonal This means, by Corollary 3 3, that there is a $T \in \mathscr{C}(\mathscr{X})$ such that $T^{-1}\mathscr{G}(K) = \mathscr{G}(K)$ and

$$\mu_T(A \times B) = \mu_S(A \times B)$$

for $A, B \in \mathscr{G}(K)$ Taking $B = TA \in \mathscr{G}(K)$ we have

$$\mu(A \cap T^{-1}TA) = \mu(A \cap S^{-1}TA)$$

whence $TA = SA$ for each $A \in \mathscr{G}(K)$ In particular $S^{-1}\mathscr{G}(K) = \mathscr{G}(K)$ so, as in the proof of Corollary 3 3, $S^{-1}KS = K$ As $S$ was arbitrary, $K$ is normal □

We mention here that it is now possible to carry over much of the analysis in [**Ru1**] of a map with MSJ to the case of a general simple group action, at least to the extent that the results in [**Ru1**] deal with constant powers of $T$ Denoting by $\mathscr{X}^k$ the cartesian product action $(X^k, \mu^k, G)$, we state the following result as a sample

PROPOSITION 3 7 *If $\mathscr{X}$ is weakly-mixing and simple then $C(\mathscr{X}^k)$ is generated by the maps $S_1 \times \cdots \times S_k$, $S_i \in C(\mathscr{X})$, and the co-ordinate permutations Moreover if $\mathscr{G}$ is a factor algebra of $\mathscr{X}^k$ which is not contained in any of the σ-algebras generated by a strict subset of the co-ordinate projections, then $H(\mathscr{G})$ is compact in $C(\mathscr{X}^k)$ and $\mathscr{G} = \mathscr{G}(H(\mathscr{G}))$*

## 4 *Joinings of a simple action with another action*

In this section we study joinings of a simple action $\mathscr{X}$ with an arbitrary (ergodic) action $\mathscr{Y}$ When are two such actions not disjoint, that is when does there exist a joining other than product measure? One possibility is that $\mathscr{X}$ and $\mathscr{Y}$ have a non-trivial common factor – then the relatively independent joining over that factor is not product measure

To describe the other possibility we need the notion of a symmetric product We denote by $\mathscr{X}^n$ the action $(X^n, \mu^n, G)$ The symmetric group $S_n$ acts on $X^n$ in a natural way by co-ordinate permutation We denote by $X^{n\bigcirc}$ the quotient space $X^n/S_n$, which is a standard Borel space The quotient map $\pi: X^n \to X^{n\bigcirc}$ is equivariant with respect to the action of $G$ so we have a factor map

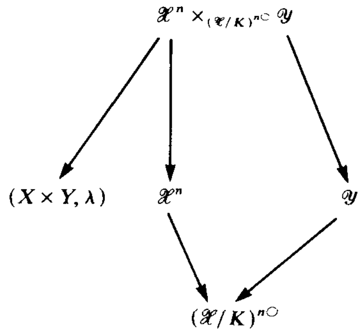
$$\pi: \mathscr{X}^n \to \mathscr{X}^{n\bigcirc}$$

where $\mathscr{X}^{n\bigcirc}$ denotes the quotient $G$-action equipped with the quotient Borel structure and the projection of $\mu^n$

Now suppose that $K$ is a compact subgroup of $C(\mathscr{X})$ and $\phi: \mathscr{Y} \to (\mathscr{X}/K)^{n\bigcirc}$ is a factor map $(\mathscr{X}/K)^{n\bigcirc}$ is also a factor of $\mathscr{X}^n$ so we may form the relatively independent joining of $\mathscr{X}^n$ and $\mathscr{Y}$ over $(\mathscr{X}/K)^{n\bigcirc}$ Restricting to $X \times Y$ we get a joining $\lambda$ of $\mathscr{X}$ and $\mathscr{Y}$ (It is easy to see that this joining does not depend on which of the $n$ copies

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

of $X$ is chosen ) In pictures




As a simple example of this construction, when $K$ is trivial and $\mathscr{Y}=\mathscr{X}^{2\circ}$, $\lambda$ is the joining arising from the common embedding of $\mathscr{X}$ and $\mathscr{X}^{2\circ}$ in $\mathscr{X}^2$ via the natural factor maps When $\mathscr{X}$ is a $\mathbb{Z}$-action with MSJ this gives an example [**Ru1**] of two $\mathbb{Z}$-actions without common factors which are not disjoint

The main result of this section is the following theorem which asserts that every ergodic joining of a simple $\mathscr{X}$ with an arbitrary $\mathscr{Y}$ arises in this way

THEOREM 4 1 *If $\mathscr{X}$ is a simple action and $\mathscr{Y}$ is any action then every ergodic joining of $\mathscr{X}$ and $\mathscr{Y}$ which is not product measure arises as described above, namely it is the projection on $X\times Y$ of the relatively independent joining of $\mathscr{X}^n$ and $\mathscr{Y}$ over $(\mathscr{X}/K)^{n\circ}$ for some compact subgroup K of $C(\mathscr{X})$ and some factor map $\phi$ $\mathscr{Y}\to(\mathscr{X}/K)^{n\circ}$ If $\mathscr{X}$ is not weakly mixing n must be* 1

*Proof* Suppose $\lambda$ is an ergodic joining of $\mathscr{Y}$ and $\mathscr{X}$ which is not product measure $\nu\times\mu$ Set $\lambda_0=\lambda$ and define inductively $\lambda_{n+1}$ on $Y\times X^{2^n}\times X^{2^n}=Y\times X^{2^{n+1}}$ to be an ergodic component of the relative product $\lambda_n\times_{\mathscr{Y}}\lambda_n$ for which the algebra corresponding to $X^{2^n}\times X^{2^n}$ is strictly bigger than the algebra corresponding to the first $X^{2^n}$ factor, if such an ergodic component exists Several remarks are in order Firstly, $\lambda_0\times_{\mathscr{Y}}\lambda_0$ is formally a measure on $(Y\times X)^2$ but since it is diagonal on $Y\times Y$ it is canonically identified with a measure on $Y\times X^2$ Secondly by the usual extremality argument almost all ergodic components of $\lambda_n\times_{\mathscr{Y}}\lambda_n$ will have both $Y\times X^{2^n}$-marginals equal to $\lambda_n$, so we choose $\lambda_{n+1}$ to satisfy this condition In particular every $Y\times X$-marginal of $\lambda_n$ is $\lambda$ Thirdly the $X^{2^{n+1}}$-marginal of $\lambda_{n+1}$ (or any candidate for $\lambda_{n+1}$) is a POOD by simplicity of $\mathscr{X}$ Thus with respect to $\lambda_{n+1}$ any $X$-factor among the second $X^{2^n}$ in $Y\times X^{2^n}\times X^{2^n}$ is either independent of the full first $X^{2^n}$-factor or identified with one of these factors via a map in $C(\mathscr{X})$ This means that if $\lambda_{n+1}$ can in fact be chosen then in $X^{2^n}\times X^{2^n}$ at least one of the $X$-factors among the second $X^{2^n}$-factor is independent of the full first $X^{2^n}$-factor On the other hand if $\lambda_{n+1}$ cannot be chosen then the ergodic decomposition of $\lambda_n\times_{\mathscr{Y}}\lambda_n$ is supported on measures which identify each $X$-factor in the second $X^{2^n}$-factor with some $X$-factor in the first $X^{2^n}$-factor, via a map in $C(\mathscr{X})$

Now we claim that it is not, in fact, possible to continue indefinitely choosing the $\lambda_n$'s as described Indeed if it were possible we would obtain a joining of $\mathscr{Y}$ and infinitely many copies of $\mathscr{X}$ for which infinitely many of the $\mathscr{X}$-factors are jointly

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

independent Projecting on these independent factors we obtain a joining $\hat{\lambda}$ of $\mathscr{Y}$ and $\mathscr{X}^{\mathbb{N}}$ which has each $Y\times X$-marginal equal to $\lambda$ Now let $\mathscr{F}_n$ denote the $\sigma$-algebra in $Y\times X^{\mathbb{N}}$ corresponding to the $X$-factor with index $n$ and $\mathscr{F}_n^\infty$ the $\sigma$-algebra $\bigvee_{N\geq n}\mathscr{F}_n$ $\bigcap_n \mathscr{F}_n^\infty$ is $\lambda$-trivial by the 0–1 law Thus for $A\in\mathscr{G}$, the $\sigma$-algebra corresponding to the $Y$-factor, we have

$$\|E((1_A-\nu(A))\,|\,\mathscr{F}_n)\|_2^2\leq \|E((1_A-\nu(A))\,|\,\mathscr{F}_n^\infty)\|_2^2\to 0$$

for some Borel probability $\tau$ supported on the subgroup $H'$ of $C(Y\times X^n,\hat{\lambda})$ $\lambda$ Thus $E(1_A|F_n)=\nu(A)$ for all $n$ which means $\mathscr{G}$ and $F_n$ are independent, contradicting our initial assumption

Thus there is a $k\geq 0$ for which it is not possible to choose $\lambda_{k+1}$ as described We now choose a maximal set of $X$-factors in $Y\times X^{2^k}$ which are $\lambda_k$-independent and project $\lambda_k$ on the product of $Y$ and these independent $\mathscr{X}$-factors to obtain an ergodic joining $\hat{\lambda}$ of $\mathscr{Y}$ and $\mathscr{X}^n$ for some $n\leq 2^k$ $\hat{\lambda}$ inherits from $\lambda_k$ the following properties each $Y\times X$-marginal of $\hat{\lambda}$ is $\lambda$ and the ergodic decomposition of $\hat{\lambda}\times_{\mathscr{Y}}\hat{\lambda}$ is supported on measures identifying each $X$-factor in the second $X^n$-factor in $Y\times X^n\times X^n$ with some $X$-factor in the first $X^n$-factor, via a map in $C(\mathscr{X})$ That is,

$$\hat{\lambda}\times_{\mathscr{Y}}\hat{\lambda}=\int_{C(Y\times X^n\,\hat{\lambda})}\hat{\lambda}_k\,d\tau(h)$$

for some Borel probability $\tau$ supported on the subgroup $H'$ of $C(Y\times X^n,\hat{\lambda})$ consisting of $\hat{\lambda}$-preserving maps $h$ of the form $\mathrm{id}\times(h_1\times\quad\times h_n)U_\pi$, $h_i\in C(\mathscr{X})$, $U_\pi$ a co-ordinate permutation of $X^n$ (In this context, by a natural abuse of notation $\hat{\lambda}_h(A\times B\times C)=\hat{\lambda}(A\times B\times h^{-1}(A\times C))$ for $A\subset Y$, $B\subset X^n$, $C\subset X^n$) By Theorem 1 8 2 $(Y\times X^n,\hat{\lambda})$ is a group extension of $\mathscr{Y}$ by the compact subgroup

$$H=H(\mathscr{Y})=\{h\in C(Y\times X^n,\hat{\lambda})\ \ h(A)=A\ \forall A\in\mathscr{Y}\}$$

(We regard $\mathscr{Y}$ as a sub-$\sigma$-algebra of $(Y\times X^n,\hat{\lambda})$ ) Moreover since $\tau$ is Haar measure on $H$, $H'\subset H$ and $\tau(H')=1$ we have $H'=H$

Now let $K$ be the subgroup of $C(\mathscr{X})$ consisting of those $k\in C(\mathscr{X})$ such that $(\mathrm{id}\times k)\lambda=\lambda$ (Note that $(\mathrm{id}\times k)\lambda$ is meaningful, even though $k$ is only a $\mu$-equivalence class of maps, because $\lambda$ has marginal $\mu$ ) We claim that for each $k\in K$ there is an $h\in H$ such that for each $B\subset Y\times X$

$$h(B\times X^{n-1})=[(\mathrm{id}\times k)B]\times X^{n-1}\quad \hat{\lambda}\text{ a e },$$

that is the action of $\mathrm{id}\times k$ on the $\sigma$-algebra $Y\times X$ in $Y\times X^n$ is the restriction to $Y\times X$ of some $h\in H$ To see this consider the off-diagonal joining $\lambda_{\mathrm{id}\times k}$ of $(Y\times X,\lambda)$ with itself This is an ergodic joining and hence can be extended to an ergodic joining $\sigma$ of $(Y\times X^n,\hat{\lambda})$ with itself (e g take an ergodic component of the relatively independent extension of $\lambda_{\mathrm{id}\times k}$) Since $\lambda_{\mathrm{id}\times k}$ is diagonal on $\mathscr{Y}$, so is $\sigma$ Since $(Y\times X^n,\hat{\lambda})$ is an $H$-extension of $\mathscr{Y}$, by theorem 2 1 $\sigma$ has the form $\hat{\lambda}_h$ for some $h\in H$, which establishes the claim We conclude from this that $K$ is compact, since $H$ is We shall assume henceforth that we are working with a version of $\mathscr{X}$ on which $K$ acts (pointwise) as described following theorem 1 8 1 We set

$$\hat{K}=\{(k_1\times\quad\times k_n)U_\pi,\ k_i\in K,\ U_\pi\text{ a co-ordinate permutation}\}\subset C(\mathscr{X}^n)$$

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

$\hat{K}$ acts pointwise on $X^n$, preserving $\mu^n$ It also acts pointwise on $Y\times X^n$, fixing the $Y$-co-ordinate, and commuting with $G$, but does not necessarily preserve $\hat{\lambda}$

Since $H=H'$ any $h\in H$ has the form $\mathrm{id}\times(h_1\times \quad \times h_n)U_\pi$, $h_i = C(\mathscr{X})$ Because $h$ preserves $\hat{\lambda}$ and every $Y\times X$-marginal of $\hat{\lambda}$ is $\lambda$, each $\mathrm{id}\times h_i$ must preserve $\lambda$, that is each $h_i\in K$ In other words each $h\in H$ has the form $\mathrm{id}\times\hat{k}$, that is $H\subset\hat{K}$ when we regard $\hat{K}$ as acting on $Y\times X^n$ Now if $\hat{k}=(k_1\times \quad \times k_n)U_\pi\in\hat{K}$ each $Y\times X$-marginal of $\hat{k}\hat{\lambda}$ is again $\lambda$, because each $(\mathrm{id}\times k_i)\lambda=\lambda$ Thus the same is true of

$$\tilde{\lambda}=\int_{\hat{K}}\hat{k}\hat{\lambda}\,d\hat{k}$$

Moreover $\tilde{\lambda}$ is a $\hat{K}$-invariant not necessarily ergodic joining of $\mathscr{Y}$ and $\mathscr{X}^n$ Thus we regard $\hat{K}$ as a subgroup of $C(Y\times X^n,\tilde{\lambda})$ which acts pointwise We denote by $\mathscr{Y}$ and $\mathscr{X}^n$ the corresponding subalgebras of $(Y\times X^n,\tilde{\lambda})$ and note that $\mathscr{Y}=\mathscr{G}(\hat{K})$, the fixed algebra of $\hat{K}$ if $\hat{k}A=A$ for all $\hat{k}\in\hat{K}$ then in particular $A$ is $H$-invariant Since $\mathscr{Y}=\mathscr{G}(H)$ (with respect to the measure $\hat{\lambda}$), $A=A'$ $\hat{\lambda}$-a e for some $A'\in\mathscr{Y}$ Since $A$ and $A'$ are both $\hat{K}$ invariant

$$\tilde{\lambda}(A\,\Delta A')=\int_{\hat{K}}\hat{\lambda}(\hat{k}(A\,\Delta A'))\,d\hat{k}=0$$

so $A=A'$ $\hat{\lambda}$-a e

This yields a $\sigma$-Boolean homomorphism $\psi$ $(B(X^n/\hat{K}),\overline{\mu^n})\to(B(Y),\nu)$ each $A\in\mathscr{X}^n/\hat{K}$, when regarded as sitting in the joining $(Y\times X^n,\tilde{\lambda})$, is a $\hat{K}$-invariant set and hence agrees $\tilde{\lambda}$ a e with a ($\nu$-a e ) unique $A'=\psi(A)\in B(Y)$ Thus we obtain a (pointwise) factor map $\mathscr{Y}\to\mathscr{X}^n/\hat{K}$ and using the $\hat{K}$-invariance of $\tilde{\lambda}$ one readily sees that $\tilde{\lambda}$ is the $\mathscr{X}^n/\hat{K}$-relative product of $\mathscr{Y}$ and $\mathscr{X}^n$ Since $\mathscr{X}^n/\hat{K}$ is canonically isomorphic to $(\mathscr{X}/K)^{n\bigcirc}$ this concludes the proof

In case $\mathscr{X}$ is not weakly mixing the inductive definition of the $\lambda_k$'s is obstructed at the first step, so $n=1$ Alternately one can use the structure provided by theorem 2 2 to show that $(X/K)^{n\bigcirc}$ is non-ergodic for $n>1$ Since $(X/K)^{n\bigcirc}$ is a factor of $\mathscr{Y}$ and $\mathscr{Y}$ is ergodic (since $\lambda$ is) we must have $n=1$

COROLLARY 4 2 *If $\mathscr{X}$ and $\mathscr{Y}$ are simple, any ergodic joining of $\mathscr{X}$ and $\mathscr{Y}$ is given as in Theorem* 4 1, *but with* $n=1$

*Proof* If $\mathscr{X}$ is not weakly mixing we are done by Theorem 4 1, so we suppose $\mathscr{X}$ is weakly mixing By Theorem 4 1 it now suffices to show that for $n>1$, $(\mathscr{X}/K)^{n\bigcirc}$ cannot be a factor of the simple action $\mathscr{Y}$ We set $\mathscr{Z}=\mathscr{X}/K$ and all we shall use about $\mathscr{Z}$ is that it is weak-mixing and non-trivial We show that $\mathscr{Z}^{n\bigcirc}$ cannot be a factor of a simple action by exhibiting an ergodic 2-joining of $\mathscr{Z}^{n\bigcirc}$ which is not product measure but which does not have relatively discrete spectrum over $\mathscr{Z}^{n\bigcirc}$ (see Lemma 3 4)

Consider the 2-joining $\sigma$ of $\mathscr{Z}^n$ obtained by linking the first co-ordinates in each of the copies of $\mathscr{Z}^n$ diagonally to each other Precisely

$$\sigma(A_1\times\quad\times A_n\times B_1\times\quad\times B_n)=\lambda(A_1\cap B_1)\lambda(A_2)\quad\lambda(A_n)\lambda(B_2)\quad\lambda(B_n),$$

where $\lambda$ denotes the measure on $Z$ We also denote by $\sigma$ the projection of $\sigma$ on

$Z^{n\bigcirc}\times Z^{n\bigcirc}$ We remark that the extension

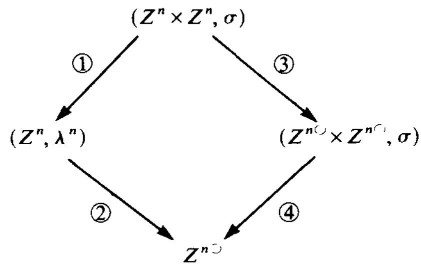
$$(Z^n\times Z^n,\sigma)\to(Z^n,\lambda^n)$$

is naturally isomorphic to the extension

$$(Z^{2n-1},\lambda^{2n-1})\to(Z^n,\lambda^n)$$

This last extension is a direct product and hence is a weakly mixing extension (in the sense of Definition 7 9 of [**Zi2**]) by Corollary 7 11 of [**Zi2**], since $\mathscr{X}$ is weak mixing

Now consider the following commutative diagram of extensions




(1) is a weakly mixing extension as we have just remarked Suppose that (4) had relatively discrete spectrum Since the finite extension (3) certainly has discrete spectrum we would conclude that the composition of (3) and (4) or equivalently (1) and (2) has generalized discrete spectrum (Definition 8 4 of [**Zi2**]) It follows that (1) would also have generalized discrete spectrum (use the equivalence of generalized discrete spectrum and a relatively separating sieve, together with Proposition 8 6 of [**Zi2**]) This is incompatible with the weak mixing of (1) (Lemma 8 11 of [**Zi2**]) Thus (4) cannot have discrete spectrum, completing the proof □

COROLLARY 4 3 *If $\mathscr{X}$ and $\mathscr{Y}$ are simple G-actions with no common factor then $\mathscr{X}$ and $\mathscr{Y}$ are disjoint*

*Proof* This is immediate from Corollary 4 2

## 5 *Weakly mixing group extensions*

Our main aim in this section is to prove that a weakly-mixing group extension of an action with MSJ is simple The following general lemma, which is similar to Proposition 3 10 of [**Fu**], will be our main tool

LEMMA 5 1 *Let $\mathscr{Y}=\mathscr{X}\times_a K$ be an ergodic group extension Let $\lambda$ be any G-invariant measure on $Y=X\times K$ which projects onto $\mu$ Then $\lambda=\mu\times dk$ where $dk$ denotes normalized Haar measure*

*Proof* We denote by $R_k$ the action of $k\in K$ on $X\times K$ by right translation For $A$ Borel in $Y$, $(R_k\lambda)A$ is a measurable function of $k$ (we leave the proof as an exercise) Thus we may define

$$\bar\lambda=\int_K R_k\lambda\,dk$$

Evidently $R_k\bar\lambda=\bar\lambda$ for each $k\in K$ and $\bar\lambda$ projects on $\mu$ By disintegrating $\bar\lambda$ over $X$ it follows immediately that $\bar\lambda=\mu\times dk$ Since each $R_k\lambda$ is $G$-invariant, ergodicity of

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

$\bar{\lambda}$ gives that $R_k\lambda = \bar{\lambda}$ for a a $k$ and hence for all $k$ by continuity In particular $\lambda = \bar{\lambda} = \mu \times dk$. □

Before proceeding to the main result we introduce an auxiliary concept which is of some interest in its own right Let us say an action $\mathscr{X}$ is pairwise independently determined (PID) if for all $n$ any $n$-joining of $\mathscr{X}$ which is pairwise independent (that is, its projection on the product of any two copies of $X$ in $X^n$ is product measure) must be product measure $\mu^n$

Note that for weakly mixing $\mathscr{X}$ it suffices to require this for ergodic $n$-joinings if $\lambda$ is an arbitrary pairwise independent joining then almost all of its ergodic components must also be pairwise independent because $\mu \times \mu$ is $G$-ergodic

We observe that a weakly mixing $\mathscr{X}$ is simple (has MSJ) iff it is 2-fold simple (has 2-fold MSJ) and is PID Indeed if $\mathscr{X}$ is 2-fold simple and PID and $\lambda$ is any ergodic $n$-joining, split $X^n$ as a product of maximal factors on each of which $\lambda$ is off-diagonal Each of these factors is isomorphic to $\mathscr{X}$ itself and any two of them are independent since they are not off-diagonally linked and $\mathscr{X}$ is 2-fold simple Thus these factors are jointly independent and $\lambda$ is a POOD

LEMMA 5 2 *A weakly mixing group extension* $Y = X \times_a K$ *of a* PID *action is again* PID

*Proof* Let $\lambda$ be an $n$-joining of $\mathscr{Y}$ which is pairwise independent The projection of $\lambda$ on $X^n$ is an $n$-joining of $\mathscr{X}$ which is again pairwise independent and hence must be product measure Now $\mathscr{Y}^n$ is a group extension of $\mathscr{X}^n$ by the group $K^n$, which is ergodic since $\mathscr{Y}$ is weakly mixing Moreover $\lambda$ is a $G$-invariant measure on $X^n \times K^n$ which projects on $\mu^n$ as we have just seen Thus by Lemma 5 1 $\lambda = \mu^n \times (dk)^n = (\mu \times dk)^n$ □

The following lemma says that joinings of different PID actions obey the same rule as joinings of a single PID action pairwise independence implies independence We will use this result in § 6

PROPOSITION 5 3 *Let* $\lambda$ *be a joining of the* PID *actions* $\mathscr{X}_1, \quad, \mathscr{X}_k$ *If* $\lambda$ *is pairwise independent then* $\lambda$ *is the product joining* $\mu_1 \times \quad \times \mu_k$

*Proof* We begin with the following general observation Suppose that $\lambda$ is a joining of actions $\mathscr{X}$ and $\mathscr{Y}$ and that in the relative product $\lambda \times_{\mathscr{Y}} \lambda$ the two copies of $\mathscr{X}$ are independent Then $\lambda$ is the product joining To see this let $\lambda = \int_Y \lambda_y \, d\nu(y)$ and for $A \subset X$

$$\int_Y (\lambda_y(A) - \mu(A))^2 \, d\nu(y) = \int_Y \lambda_y(A)^2 \, d\nu(y) - 2\mu(A) \int_Y \lambda_y(A) \, d\nu(y) + \mu(A)^2$$
$$= (\lambda \times_{\mathscr{Y}} \lambda)(A \times A \times Y) - 2\mu(A)^2 + \mu(A)^2$$
$$= \mu(A)^2 - 2\mu(A)^2 + \mu(A)^2 = 0$$ □

Next we turn to a special case of the proposition, namely $\mathscr{X}_1 = \mathscr{X}$, $\mathscr{X}_2 = \mathscr{X}_3 = \quad = \mathscr{X}_k = \mathscr{Y}$ We form the $\mathscr{X}_1$-relative product $\hat{\lambda}$ of $\lambda$ with itself $\hat{\lambda}$ is a joining of $\mathscr{X}_1, \mathscr{X}_2, \quad, \mathscr{X}_k, \mathscr{X}_2, \quad, \mathscr{X}_k$ and with respect to $\hat{\lambda}$ any single copy of $\mathscr{Y}$ is independent

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

of $\mathscr{X}_1$, since $\hat{\lambda}$ projects on $\lambda$ which is pairwise independent For the same reason any two copies of $\mathscr{Y}$ both coming from the first group $\mathscr{X}_2, \quad , \mathscr{X}_k$, or both from the second are independent If we consider copies of $\mathscr{Y}$ taken from the first and second groups respectively they are also independent, because they are independent conditionally on $\mathscr{X}_1$ (definition of $\hat{\lambda}$) and each is independent of $\mathscr{X}_1$ Thus the family of $\sigma$-algebras $\mathscr{X}_2, \quad , \mathscr{X}_k, \mathscr{X}_2, \quad , \mathscr{X}_k$ is pairwise independent, hence independent, since $\mathscr{Y}$ is PID By the observation at the beginning of the proof $\mathscr{X}_1 \perp (\mathscr{X}_2 \vee \quad \vee \mathscr{X}_k)$ so $\mathscr{X}_1, \quad , \mathscr{X}_k$ is jointly independent as desired

For the general case we proceed by induction on the number of distinct actions among $\mathscr{X}_1, \quad , \mathscr{X}_k$ We may as well assume, for simplicity of notation, that each action occurs the same number of times, say $r$, in $\mathscr{X}_1, \quad , \mathscr{X}_k$ If $\lambda$ is a pairwise independent joining of $\mathscr{X}_1, \quad , \mathscr{X}_k$ all the copies of a fixed system sit jointly independent, so gathering together like copies and relabelling we may assume $\lambda$ is a joining of $\mathscr{X}_1^r, \quad , \mathscr{X}_k^r$ in which any pair $\mathscr{X}_i, \mathscr{X}_j$ sit independently Moreover by our special case each $\mathscr{X}_i$, $i > 1$ is independent of $\mathscr{X}_1^r$ Form the relatively independent product $\hat{\lambda}$ of $\lambda$ with itself over $\mathscr{X}_1^r$, considered as a measure on $X_1^r \times X_2^r \times \quad \times X_k^r \times X_2^r \times \quad \times X_k^r$

We claim that with respect to $\hat{\lambda}$ any single factor $\mathscr{X}_i$ is independent of any other single factor $\mathscr{X}_j$ ($i$ and $j$ may be identical) To see this we need only consider the case where $\mathscr{X}_i$ comes from the first group $\mathscr{X}_2^r \times \quad \times \mathscr{X}_k^r$ and $\mathscr{X}_j$ from the second for on each group $\lambda$ is product measure by our induction hypothesis But then $\mathscr{X}_i$ and $\mathscr{X}_j$ are conditionally independent given $\mathscr{X}_1^r$ and each is independent of $\mathscr{X}_1^r$, so they are independent of each other

Now projecting $\hat{\lambda}$ on $\mathscr{X}_2^r \times \quad \times \mathscr{X}_2^r \times \mathscr{X}_2^r \quad \times \mathscr{X}_k^r$ we have a pairwise independent joining $\bar{\lambda}$ of copies of only $k-1$ distinct systems By induction (and when $k=2$ by the PID property of $\mathscr{X}_2$) we conclude that $\bar{\lambda}$ is product measure As we have already seen this implies that $\lambda$ is product measure

THEOREM 5 4 *Suppose $\mathscr{X}$ is a G-action with* MSJ, *satisfying $gx = x$ a e implies $g = e$ Suppose further that $\mathscr{Y} = \mathscr{X} \times_a K$ is a weakly mixing group extension Then $\mathscr{Y}$ is simple and $C(\mathscr{Y})$ is the group generated by the right multiplications $R_k$, together with the action of $Z(G)$, the center of G Moreover, the natural action of $G \times K$ on $X \times K$ has* MSJ

*Proof* By Lemma 5 2 it suffices to show that any ergodic two-joining of $\mathscr{Y}$ is off-diagonal or product measure Let $\lambda$ be an ergodic 2-joining of $\mathscr{Y}$, that is a $G$-invariant measure on $X \times K \times X \times K$ whose projection on each $X \times K$ is $\mu \times dk$. The projection $\bar{\lambda}$ of $\lambda$ on $X \times X$ is an ergodic 2-joining of $\mathscr{X}$, hence it is product measure or an off-diagonal If $\bar{\lambda}$ is $\mu \times \mu$ then as in the proof of Lemma 5 2, $\lambda$ is $(\mu \times dk)^2$

Suppose now that $\bar{\lambda}$ is an off-diagonal

$$\mu_g(A \times B) = \mu(g^{-1}A \cap B)$$

for some $g \in G$ (recall that $\mathscr{X}$ has MSJ) Because the action of $g$ belongs to $C(\mathscr{X})$ our hypothesis on $\mathscr{X}$ implies that $g \in Z(G)$ Let $\lambda'$ be the image of $\lambda$ under the map

$\mathrm{id}\times g^{-1}$ of $Y\times Y$ to itself Then $\lambda'$ is again a $G$-joining (because $g\in Z(G)$) and its projection on $X\times X$ is diagonal measure By Theorem 2 1 $\lambda'$ has the form $\nu_{R_{k_0}}$ ($\nu=\mu\times dk$) for some $k_0\in K$, whence $\lambda=\nu_{gR_{k_0}}$ Thus we have shown that $\mathscr{Y}$ is simple and the centralizer is as claimed

If $G\times K$ acts on $X\times K$ then the map $g_{R_{k_0}}$ belongs to the action so the action has MSJ □

*Remark.* It is perhaps worth highlighting why simplicity of $\mathscr{X}$ would not suffice for the proof of theorem 5 4 We assumed that $\lambda$ projected onto $(\mathrm{id}\times g)\mu_\Delta$ and then worked with $(\mathrm{id}\times g^{-1})\lambda$ If $\lambda$ projected on $(\mathrm{id}\times S)\mu_\Delta$, $S\in C(\mathscr{X})$, $S$ need not extend to a map belonging to $C(\mathscr{Y})$ If it did the proof would go through

*Example* 5 5 Theorem 5 4 and Corollary 3 6 allow us to find an example of a simple $\mathscr{Y}$ with a non-simple factor It suffices to let $\mathscr{Y}=\mathscr{X}\times_a K$ where $\mathscr{X}$ is free with MSJ, $\mathscr{Y}$ is weakly mixing and $K$ is a compact group with a closed non-normal subgroup $K'$ Since $K'$ is not normal in $K$ it is a fortiori non-normal as a subgroup of $C(\mathscr{Y})$, so $\mathscr{Y}/K'$ is not simple (We remark that it is well-known that for an arbitrary weakly mixing $\mathscr{X}$ and compact group $K$ there is an abundance of cocycles $a$ such that $\mathscr{X}\times_a K$ is again weakly mixing )

It is natural to ask whether the assumption of MSJ in theorem 5 4 can be weakened to simplicity The following counterexample is due to S Glasner (Proposition 1 7 of [Gl]) It replaces our original more complicated construction The fact that it is not simple is implicit in Proposition 1 7 of [Gl] but we sketch a proof here without using the language of quasifactors

*Example* 5 6 A weakly mixing group extension of a simple $\mathbb{Z}$-action which is not simple

When $\mathscr{X}$ is a $\mathbb{Z}$-action we write $\mathscr{X}=T$ where $T$ is the map generating the action Let $T$ be any weakly mixing map with MSJ and $\phi$ a cocycle into the circle group $K$ such that $S=T\times_\phi K$ is weakly mixing (We will identify $\phi$ with the function $\phi(1,\ )$) $S$ is simple by Theorem 5 4 Now define a $K$-extension $R$ of $S$ by

$$\begin{aligned}R(x,k_1,k_2)&=(S(x,k_1),k_1k_2)\\&=(Tx,\phi(x)k_1,k_1k_2)\end{aligned}$$

$R$ is weakly mixing by Proposition 1 7 of [Gl]

We exhibit an ergodic 2-joining of $R$ which is neither product measure nor an off-diagonal Consider the measures $\lambda_1$ and $\lambda_2$ on $(X\times K\times K)^2$ defined by the disintegrations

$$\lambda_1=\int_{X\times K\times K}^{\oplus}\delta_{(x\ -k_1,k_2)}\,d\mu(x)\,dk_1\,dk_2$$

$$\lambda_2=\int_{X\times K\times K}^{\oplus}\delta_{(x,-k_1,-k_2)}\,d\mu(x)\,dk_1\,dk_2$$

$\lambda_1$ and $\lambda_2$ each have both marginals on $X\times K\times K$ equal to $d\mu\,dk_1\,dk_2$, but they are not 2-joinings of $R$ Indeed

$$(R\times R)((x,k_1,k_2),(x,-k_1,k_2))=((Tx,a(x)k_1,k_1k_2),(Tx,-a(x)k_1,-k_1k_2)),$$

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

whence it follows that $(R\times R)\lambda_1=\lambda_2$ and similarly $(R\times R)\lambda_2=\lambda_1$ Thus $\frac{1}{2}(\lambda_1+\lambda_2)$ is a 2-joining of $R$ which is not product measure and not off-diagonal (it has 2-point fibres over $X\times K\times K$) Moreover it is ergodic (but not weak-mixing) as it is isomorphic in an obvious way to $R\times f$, where $f$ denotes the interchange map on $\{-1,1\}$

We conclude this section with some general remarks about PID $\mathbb{Z}$-actions It is easy to see that Bernoulli shifts are not PID Since every positive entropy map has a Bernoulli factor it follows, via relatively independent extension, that positive entropy maps are not PID On the other hand it is not hard to see that translation by $e^{2\pi i\alpha}$, $\alpha$ irrational, on the circle group is not PID The translation by 1 on $\mathbb{Z}/m\mathbb{Z}$ is also not PID It follows, again by extension, that any non-weakly mixing map is not PID However we know of no weakly mixing 0-entropy counterexample It is not hard to see that if a map is 2-mixing but not 3-mixing then it is not PID so a proof that 0-entropy weak mixing implies PID will not be easily found A more specific problem is does 2-fold simplicity (MSJ) imply simplicity (MSJ)?

Passing to $\mathbb{Z}^2$-actions we observe that Ledrappier's example ([**Le**]) of a 2-mixing but not 3-mixing action furnishes an example of a non-PID, mixing, 0-entropy $\mathbb{Z}_2$-action This example is also not 2-fold simple there is a natural 2-1 factor map from it to itself

## 6 *The action of a co-compact subgroup*

THEOREM 6 1 *Let $\mathscr{X}$ be a weakly mixing simple G-action and H a closed, normal, co-compact subgroup of G Then H acts simply and $C(X,H)=C(X,G)$*

*Proof* First we observe that the action of $H$ is weakly mixing, which can be seen by showing that the only functions $f\in L_2(X)$ such that $Hf$ is precompact in the norm topology of $L_2(X)$ are the constants (See, for example [**Zi2**], Theorem 7 1 and Theorem 7 8 specialized to the case where $Y$ is trivial )

Now let $\pi\ G\to G/H$ denote the canonical projection, choose a Borel cross-section $\sigma\ G/H\to G$ (Theorem 8 11 of [**Va1**]) We denote normalized Haar measure on $G/H$ by $d\xi$

Suppose that $\lambda$ is an ergodic $k$-joining of $(X,H)$ Note that the field of measures $\{g\lambda\}_{g\in G}$ is measurable for $A$ Borel in $X^k$

$$(g\lambda)(A)=\lambda(g^{-1}A)=\int_{X^k}1_A(gx)\,d\lambda$$

is a measurable function of $g$ by Fubini's theorem Thus $\{\sigma(\xi)\lambda\}_{\xi\in G/H}$ is also a measurable field so we may define

$$\bar\lambda=\int_{G/H}\sigma(\xi)\lambda\,d\xi$$

$\bar\lambda$ has marginals $\mu$, since each $\sigma(\xi)\lambda$ has marginals $\mu$ Moreover for $g_0\in G$

$$g_0\bar\lambda=\int_{G/H}(g_0\sigma(\xi)\lambda\,d\xi$$

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

Now $g_0\sigma(\xi)$ and $\sigma(g_0\xi)$ both belong to the coset $g_0\xi$, so they differ by multiplication on the right by an element of $H$ Since $\lambda$ is $H$-invariant, $(g_0\sigma(\xi))\lambda = \sigma(g_0\xi)\lambda$ Thus

$$g_0\bar{\lambda} = \int_{G/H} \sigma(g_0\xi)\lambda \, d\xi$$
$$= \int_{G/H} \sigma(\xi)\lambda \, d\xi = \bar{\lambda},$$

by invariance of $d\xi$ We have shown $\bar{\lambda}$ is a $G$-joining We claim $\bar{\lambda}$ is also $G$-ergodic Indeed if $A \subset X^n$ is $G$-invariant (literally, not $\bar{\lambda}$-a e ) then $\lambda(A) = 0$ or 1 by $H$-ergodicity of $\lambda$ Since $A$ is $G$-invariant $\lambda((\sigma(\xi))^{-1}A) = \lambda(A)$ so $\bar{\lambda}(A) = 0$ or 1 according as $\lambda(A) = 0$ or 1

Thus by simplicity of $(X, G)$ we now have that $\bar{\lambda}$ is a POOD (with respect to $C(X, G)$) Since $(X, H)$ is weakly mixing a POOD is also ergodic with respect to $H$ Thus $\bar{\lambda}$ is an $H$-ergodic average of the $H$-invariant measures $\sigma(\xi)\lambda$ so by extremality we conclude that $\sigma(\xi)\lambda = \bar{\lambda}$ for a a $\xi \in G/H$ In particular there is at least one $g \in G$ such that $g\lambda = \bar{\lambda}$ whence $\lambda = g^{-1}\bar{\lambda} = \bar{\lambda}$ We already know that $\bar{\lambda}$ is a POOD with respect to $C(X, G)$ so this completes the proof □

COROLLARY 6 2 *With the hypotheses of Theorem 6 1 every H-invariant factor algebra of $\mathscr{X}$ is G-invariant If $(X, G)$ is prime so is $(X, H)$*

*Proof* Follows immediately from Theorems 3 1 and 6 1

PROPOSITION 6 3 *Suppose that H is a closed, normal, co-compact subgroup of G and that $\mathscr{X}$ and $\mathscr{Y}$ are weakly mixing simple G-actions such that every ergodic G-joining of $\mathscr{X}$ and $\mathscr{Y}$ is weakly mixing (For example this is true if either $\mathscr{X}$ or $\mathscr{Y}$ is prime by Corollary 4 2 ) Then any H-joining of $\mathscr{X}$ and $\mathscr{Y}$ is a G-joining In particular any H-factor map $\mathscr{X} \to \mathscr{Y}$ is a G-factor map*

*Proof* It suffices to prove this for an ergodic $H$-joining $\lambda$ Then, as in the proof of Theorem 6 1, we form the $G$-invariant and ergodic joining

$$\bar{\lambda} = \int_{G/H} \sigma(\xi)\lambda \, d\xi$$

By hypothesis $\bar{\lambda}$ is weakly mixing as a $G$-action, hence as in the proof of theorem 6 1, also weakly mixing as an $H$-action, hence also $H$-ergodic One concludes as in 6 1 that $\lambda = \bar{\lambda}$ □

We now apply the above results to $\mathbb{Z}$- and $\mathbb{R}$-actions

COROLLARY 6 4 *If $\{T_t\}$ is a weakly mixing simple prime flow then $T_a$ is a prime map for $a \neq 0$ If $a, b \neq 0$ then $T_a$ and $T_b$ are either disjoint or isomorphic $T_1$ and $T_a$ are isomorphic if and only if the flows $\{T_t\}_{t\in\mathbb{R}}$ and $\{T_{at}\}_{t\in\mathbb{R}}$ are isomorphic More generally, if $\{T_t\}$ and $\{S_t\}$ are weakly-mixing simple prime flows then the maps $T_1$ and $S_1$ are either disjoint or isomorphic according as $\{T_t\}$ and $\{S_t\}$ are disjoint or isomorphic*

*Proof* This follows from Corollary 6 2, Proposition 6 3 and Corollary 4 2

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

This is a good place to observe that a weak mixing flow with MSJ (examples are provided by [J, P] and [**Ra**]) is prime, so that the above results apply In particular its time-one map is simple and prime

PROPOSITION 6 5 *A weakly mixing flow with minimal self-joinings is prime*

*Proof* Since each non-zero time in a weakly mixing flow is again weakly mixing and a fortiori ergodic this follows immediately from the remark following the proof of theorem 3 1

For $\mathbb{Z}$-actions with MSJ Corollary 6 4 may be sharpened

COROLLARY 6 5 *If $T$ is a weakly mixing map with* MSJ *and $|n|>|m|>0$ then $T^n$ and $T^m$ are disjoint*

*Proof* It suffices by Corollary 6 4 to show that $T^n$ and $T^m$ are not isomorphic We claim that $T^m$ has no $n$th root (while $T^n$, of course, does) Indeed if $S$ were an $n$th root of $T^m$ then $S \in C(T^m) = C(T)$ (Theorem 6 1) so $S = T^l$ Thus $T^{ln} = T^m$ and $ln = m$ which is impossible when $|n|>|m|$ □

We observe that Proposition 6 4 cannot be similarly strengthened for flows with MSJ Indeed it is shown in [**Ra**] that certain horocycle flows $\{T_t\}$ have MSJ, providing examples where $T_a$ and $T_b$ are isomorphic for all $a, b > 0$

We are now in a position to clarify the relation between our definition of minimal self-joinings in the case of $\mathbb{Z}$-actions and the original apparently much stronger one used in [**Ru1**] Let's say that a map $T$ has minimal power joinings (MPJ) if any ergodic joining of possibly different non-zero powers of $T$ is a POOD (with respect to $T$) (Warning not all POOD's are joinings now Off-diagonal links can occur only between co-ordinates which are acted on by the same power of $T$) This is what was called minimal self-joinings in [**Ru1**]

PROPOSITION 6 7 *A weak mixing map $T$ has* MPJ *if and only if it has* MSJ *and $T$ and $T^{-1}$ are not isomorphic*

*Proof* The 'only if' direction is obvious Suppose that $T$ is weakly mixing with MSJ First observe that $T^m$ and $T^{-m}$ are disjoint by Theorem 6 1 and Corollary 6 4 applied to the simple $\mathbb{Z}$-actions $T^m$ and $T^{-m}$ Combining this with 6 6 we have that any two non-zero powers of $T$ are disjoint

Suppose now that $\lambda$ is a joining of powers of $T$ (with multiplicities) Grouping together co-ordinates on which like powers of $T$ act we have that on any group the marginal of $\lambda$ on that group is a POOD (for $T$) because $T^n$ is simple and $C(T^n) = C(T)$ Furthermore on any group $\lambda$ is isomorphic to a cartesian power of $T^n$, since an off-diagonal factor is isomorphic to $T^n$ Thus we may assume that $\lambda$ is a joining of copies of $T^n$ (for various $n$) in which any two like copies sit independently Copies of different powers automatically sit independently since they are disjoint Since each copy is weak-mixing and simple, and hence PID, Proposition 5 3 implies that $\lambda$ is the product joining which completes the proof □

We remark that there are weakly mixing maps $T$ with MSJ such that $T$ and $T^{-1}$ are isomorphic, as in the examples of [**Ju1**] A symmetrized version of Chacón's

https://doi.org/10.1017/S0143385700004193 Published online by Cambridge University Press

example (see **[JRS]**) gives easier examples (Use, for example the substitution $0 \to 00100$, $1 \to 1$ ) Even when $T$ and $T^{-1}$ are isomorphic one can explicitly describe all ergodic joinings of powers of $T$ For if $\phi T = T^{-1}\phi$ then $\phi$ may be used to replace negative powers of $T$ by positive ones After the relabelling the joining is a POOD which means that the original joining is a product of off-diagonals 'skewed' by $\phi$ By a skewed off-diagonal we mean a joining of copies of $T^n$ and $T^{-n}$ (for a fixed $n$) where any two $T^n$'s are linked by a $T^l$ and a $T^n$ is linked with a $T^{-n}$ by a $T^l\phi$ Note moreover that $\phi$ must be an involution, $\phi^2 = \mathrm{id}$ Indeed $\phi^2 \in C(T)$, so $\phi^2 = T^n$ for some $n$ If $n \neq 0$ we have $\phi \in C(T^n) = C(T)$ so $\phi = T^m$, which is impossible

Say a flow $\{T_t\}$ has minimal re-scaling joinings (MRJ) if for all $k$ and $a_1, \quad , a_k \in \mathbb{R} - \{0\}$ every ergodic joining of the flows $\{T_{a_1 t}\}_{t\in\mathbb{R}}, \quad , \{T_{a_k t}\}_{t\in\mathbb{R}}$ is a POOD The following result was applied in **[J, P]** to conclude that the weakly mixing flow with MSJ constructed in that paper actually has MRJ The proof is similar to the proof of Proposition 6 7

PROPOSITION 6 8 *A weakly mixing flow has* MRJ *if and only if it has* MSJ *and for all* $a \in \mathbb{R} - \{1\}$, $\{T_t\}$ *and* $\{T_{at}\}$ *are non-isomorphic*

The following example of weakly mixing simple maps $S$ and $\bar{S}$ such that $S^2$ and $\bar{S}^2$ are isomorphic but $S$ and $\bar{S}$ are not shows that 6 3 may fail when the actions in question have ergodic joinings which are not weakly mixing

*Example* 6 9 Let $T$ be a weakly mixing map with MSJ, $K = \{-1, 1\}$ and $\phi$ a cocycle into $K$ such that $S = T \times_\phi K$ is weak mixing As is well known weak mixing of $S$ is equivalent to the requirement that $\phi$ not be cohomologous to a constant function, that is the equation

$$\phi(x) = b(Tx)b(x)^{-1}k_0 \qquad \text{a e} \tag{1}$$

has no measurable solution $b \quad X \to K$ for $k_0 = -1$ or $1$ (As usual we identify $\phi$ with the function $\phi(1, \ )$ )

It follows that $\bar{S} = T \times_{-\phi} K$ is also weakly mixing Both $S$ and $\bar{S}$ are simple by Theorem 5 4 Note that the relatively independent joining of $S$ and $\bar{S}$ over the common factor $T$ is in a natural way isomorphic to $T \times_{\phi \times -\phi} K \times K$ where $(\phi \times -\phi)(x) = (\phi(x), -\phi(x))$ Defining $\theta(x, k_1, k_2) = (x, k_1, k_1 k_2)$ we have the following diagram

$$\begin{array}{ccc} (x, k_1, k_2) & \xrightarrow{T \times_{\phi \times -\phi} K \times K} & (Tx, \phi(x)k_1, -\phi(x)k_2) \\ \Big\downarrow \theta & & \Big\downarrow \theta \\ (x, k_1, k_1 k_2 & \xrightarrow{T \times_{\phi \times -1} K \times k} & (Tx, \phi(x)k_1, -k_2) \end{array}$$

Thus $T \times_{\phi \times -\phi} K \times K$ is isomorphic to $T \times_{\phi \times -1} K \times K$ which is ergodic but not weakly mixing

Now $S^2 = \bar{S}^2$ so id $\ X \times K \to X \times K$ is an isomorphism of $S^2$ and $\bar{S}^2$ but not of $S$ and $\bar{S}$ Moreover it is easy to see that $S$ and $\bar{S}$ are non-isomorphic Indeed an isomorphism $\phi$ would also have to be an isomorphism of $S^2$ and $\bar{S}^2$, that is $\phi \in C(S^2)$ By theorem 6 1 $C(S^2) = C(S)$ so $\phi$ would commute with $S$, a contradiction

We conclude by mentioning a few open problems, restricting ourselves for the most part to $\mathbb{Z}$-actions It is natural to ask how prevalent the class of simple maps is While we now have a fairly wide variety of examples, they are all of a very special nature For one thing they are all constructed from something with MSJ, either by group extension or by taking a non-zero time in a $\mathbb{Z}$- or $\mathbb{R}$-action with MSJ The class of maps with MSJ is small in the precise sense that it is meagre in the weak topology This is because in general (that is, for a residual set) a map is rigid, that is $\exists n_i \to \infty$ such that $T^{n_i} \to \mathrm{id}$, which implies that $C(T)$ has the cardinality of the continuum

Elsewhere we will show how Chacón's map can be modified to give a rigid simple prime map This is a step in the right direction as it shows there is at least one simple prime map in the rigid class, which is generic Moreover the construction has nothing to do with MSJ Of course it leaves open the question of whether the simple maps form a residual class One may ask the same question about the prime maps It is interesting to note that every example of primality so far known derives more or less directly from simplicity Is there an essentially different sufficient condition for primality?

We have been unable to answer the following question does every weakly mixing simple map have a non-trivial prime factor? The only examples we have of such maps are either themselves prime or group extension of maps with MSJ

Can one say something about joinings of $\mathscr{X}_1, \ldots, \mathscr{X}_k$, $\mathscr{X}_i$ simple, in the spirit of Corollary 4 5? If the $\mathscr{X}_i$ are all prime then they are pairwise disjoint or isomorphic so Corollary 4 5 and Proposition 5 3 describe all joinings of $\mathscr{X}_1, \ldots, \mathscr{X}_k$

In § 5 we already raised the questions Does 0 entropy weak mixing imply PID? Does 2-fold MSJ imply 3-fold MSJ?

Many of the results of this paper can almost surely be relativized The natural definition of relative 2-fold simplicity of the extension $\mathscr{X} \to \mathscr{Y}$ has already been given by Veech (for $\mathbb{Z}$-actions) every ergodic 2-joining of $\mathscr{X}$ which is diagonal on $\mathscr{Y}$ is either the $\mathscr{Y}$-relative product or an off-diagonal He has shown (Theorem 4 8 of [**Ve**]) that if $\mathscr{X} \to \mathscr{Y}$ is relatively simple and the $\mathscr{Y}$-relative centralizer of $\mathscr{X}$ (namely those $S \in C(\mathscr{X})$ which fix each set in the factor algebra $\mathscr{G}$ corresponding to $Y$) has no non-trivial compact subgroups then $\mathscr{X} \to \mathscr{Y}$ is relatively prime, that is there are no factor algebras strictly between $\mathscr{G}$ and $B(\mathscr{X})$

The group $SL_2(\mathbb{Z})$ acts as automorphisms of the 2-torus We conjecture that this action has MSJ and that its centralizer is trivial so the only 2-joinings are product measure and diagonal measure minimal self-joinings in the strongest possible sense!

## REFERENCES


[**Bo**] N Bourbaki *Elements of Mathematics, General topology*, Part 2 Hermann, Paris (1966)

[**Fu**] H Furstenberg *Recurrence in Ergodic Theory and Combinatorial Number Theory* Princeton University Press, Princeton (1981)

[**Gl**] S Glasner Quasi-factors in ergodic theory *Israel J Math* **45** (1983), 198–208

[**Ju2**] A del Junco A family of counterexamples in ergodic theory *Israel J Math* **44** (1983), 160–188

[**Ju2**] A del Junco *Prime Systems in Ergodic Theory and Toplogical Dynamics, Report of the Department of Mathematics* Technische Hogeschool Delft, Delft (1982)

[**J, K**] A del Junco & M Keane On generic points in the cartesian square of Chacón's transformation To appear in *Ergod Th & Dynam Sys*

[**J, P**] A del Junco & K Park An example of a measure-preserving flow with minimal self-joinings *J d'Analyse Math* **42** (1983), 199-211

[**J, R**] A del Junco & D Rudolph An example of a rigid, prime, simple map Preprint

[**JRS**] A del Junco, M Rahe & L Swanson Chacón's automorphism has minimal self-joinings *J d'Analyse Math* **37** (1980), 276-284

[**Le**] F Ledrappier Un champ markovien peut etre d'entropie nulle et melangeant *C R Acad Sci Paris Ser A* 287 (1978), 561-563

[**Ma1**] G W Mackey Point representations of transformation groups *Illinois J Math* **6** (1962), 327-335

[**Ma2**] G W Mackey Borel structures in groups and their duals *Trans Amer Math Soc* **85** (1957), 134-165

[**Ph**] R R Phelps *Lectures on Choquet's Theorem* Van Nostrand, Princeton (1966)

[**Ram**] A Ramsay Virtual groups and group actions *Advances in Math* **6** (1971), 253-322

[**Ra**] M Ratner Joinings of horocycle flows Preprint

[**Ru1**] D Rudolph An example of a measure-preserving map with minimal self-joinings, and applications *J d'Analyse Math* **35** (1979), 97-122

[**Ru2**] D Rudolph The second centralizer of a Bernoulli shift is just its powers *Israel J Math* **29** (1978), 167-178

[**Va1**] V S Varadarajan *Geometry of Quantum Theory*, Vol II Van Nostrand Reinhold, New York (1970)

[**Va2**] V S Varadarajan Groups of automosphisms of Borel spaces *Trans Amer Math Soc* **109** (1963), 191-220

[**Ve**] W A Veech A criterion for a process to be prime *Monatshefte Math* **94** (1982), 335-341

[**Zi1**] R Zimmer Extensions of ergodic actions *Illinois J Math* **20** (1976), 373-409

[**Zi2**] R Zimmer Ergodic actions with generalized discrete spectrum *Illinois J Math* **20** (1976), 555-588