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Stark's Conjecture and New Stickelberger Phenomena

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Abstract. We introduce a new conjecture concerning the construction of elements in the annihilator ideal associated to a Galois action on the higher-dimensional algebraic K-groups of rings of integers in number fields. Our conjecture is motivic in the sense that it involves the (transcendental) Borel regulator as well as being related to l-adic étale cohomology. In addition, the conjecture generalises the well-known Coates–Sinnott conjecture. For example, for a totally real extension when $r = -2, -4, -6, \ldots$ the Coates–Sinnott conjecture merely predicts that zero annihilates K_{-2r} of the ring of S-integers while our conjecture predicts a non-trivial annihilator. By way of supporting evidence, we prove the corresponding (conjecturally equivalent) conjecture for the Galois action on the étale cohomology of the cyclotomic extensions of the rationals.

1 Introduction

In 1890 Stickelberger [50] proved what might be called the first "equivariant motivic" result in number theory. Needless to say this aspect of Stickelberger's theorem was heavily disguised! Recall [57, p. 94] that one may construct, from the values of the Dirichlet L-function, a Stickelberger element in the rational group-ring of the Galois group of a cyclotomic field. Then the product of the annihilator ideal of the roots of unity with the principal fractional ideal generated by the Stickelberger element is integral and annihilates the class-group. Since Galois groups are involved it is clear how the adjective "equivariant" might be associated with Stickelberger's theorem. The purpose of this paper is to explain the association with "motivic" and to introduce, with supporting evidence, new conjectural Stickelberger-like phenomena.

In what follows, by a Galois representation of a field F we shall mean a continuous, finite-dimensional complex representation of the absolute Galois group of F, which amounts to saying that the representation factors through a finite Galois group $G = \mathrm{Gal}(E/F)$ of a Galois extension E/F.

We begin with the Stark conjecture, which asserts that the function assigning to a Galois representation of number fields the value of a regulator map divided by the leading term of the Artin L-function at $s = 0$ is always algebraic and is Galois equivariant. Stark's regulator is defined on K_1 of the ring of algebraic integers.

We assume that the higher-dimensional analogue of Stark's conjecture is true, that is, replace K_1 by K_{1-2r} for $r = -1, -2, -3, \ldots$ and the Dirichlet regulator by the Borel regulator. Having posed this higher-dimensional Stark conjecture in an earlier version of this paper, I learned from David Burns that it had long ago been mentioned by B. Gross [22].

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For a Galois extension E/F of number fields with abelian Galois group G we construct in §4 a "fractional ideal", a finitely generated $\mathbb{Z}[1/2][G]$-submodule $\mathcal{J}_E^r$ of the rational group-ring $\mathbb{Q}[G]$, for each $r = -1, -2, -3, \dots$. The construction of $\mathcal{J}_E^r$ is "motivic" in the sense that transcendental techniques (Borel's regulator, Deligne cohomology, etc.) and l-adic techniques are involved in its construction and the derivation of its properties (for example, Theorem 7.6).

When E/F is totally real and $r = -1, -3, -5, \dots$, the L-values at $s = r$ are non-zero, and in this case $\mathcal{J}_E^r$ is equal to the higher Stickelberger ideal which appears in the Brumer–Coates–Sinnott conjectures. We conjecture that $\mathcal{J}_E^r$ participates in a new Stickelberger phenomenon. Namely, for each odd prime l and a suitable Galois invariant set of primes S',

Conjecture 1.1

$$\big(\operatorname{ann}_{\mathbb{Z}_l[G]}(\operatorname{Tors} K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l) \cdot \mathcal{J}_E^r\big) \cap \mathbb{Z}_l[G] \subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}\big(K_{-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l\big),$$

where $\operatorname{ann}_{\mathbb{Z}_l[G]}(M)$ *denotes the annihilator ideal of M.*

The Quillen–Lichtenbaum conjecture relates K-groups to étale cohomology, predicting that the l-adic Chern classes yield natural isomorphisms of the form

$$c_{1-r,2-e}\colon K_{e-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l \xrightarrow{\cong} H^{2-e}_{\text{ét}}\big(\operatorname{Spec}(\mathcal{O}_{E,S'}); \mathbb{Z}_l(1-r)\big)$$

when $e = 0, 1$, $r = -1, -2, -3, \dots$, and l is an odd prime. I believe that recent unpublished work by Rost combined with recent unpublished work of Suslin–Voevodsky[1] is expected to establish the Quillen–Lichtenbaum conjecture for all odd primes. We shall not make use of these unpublished results, except by way of justification for concentrating on étale cohomology.

Conjecture 1.2

$$\begin{aligned}\big(\operatorname{ann}_{\mathbb{Z}_l[G]}(\operatorname{Tors} H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_{E,S'}); \mathbb{Z}_l(1-r))) \cdot \mathcal{J}_E^r\big) &\cap \mathbb{Z}_l[G] \\ &\subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}\big(H^2_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_{E,S'}); \mathbb{Z}_l(1-r))\big).\end{aligned}$$

For those familiar with such things I suggest that this conjectural phenomenon should be considered as "a kind of integral sharpening of a twisted Stark conjecture". I understand this to mean that this conjecture shows how to produce annihilator elements in the integral group-ring of the Galois group from *leading terms* of L-functions at negative integers. Furthermore, the rather unexpected form of the construction of the fractional ideal $\mathcal{J}_E^r$ is inspired by the Stark conjecture [53] concerning the Galois properties of these leading terms. As I mentioned above, when the leading terms are actually just the (non-zero) L-values, $\mathcal{J}_E^r$ is equal to the higher Stickelberger ideal which appears in the Brumer–Coates–Sinnott conjectures, which accounts for the occurrence of the phrase "new Stickelberger phenomena" in the title.

[1] See the very recent [56] as well as [19] and [32].

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In this paper we shall verify the second conjecture in the case of abelian extensions of the rationals. In fact, for this it suffices to treat the case of cyclotomic fields (Theorem 6.1). Even this simple case reveals a new phenomenon. Suppose that $E/\mathbb{Q}$ is a totally real, abelian Galois extension and that $r = -1, -3, -5, \dots$. In this case the Coates–Sinnott conjecture would predict that the higher Stickelberger ideal times the annihilator of Tors $K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l$ lies in the annihilator of $K_{-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l$. However, when $r = -2, -4, -6, \dots$, the Stickelberger ideal is zero and the Coates–Sinnott conjecture becomes trivial, but the conjecture which I have just introduced does not.

The paper is arranged in the following manner. In §2 we recall the Stark conjecture concerning the leading term at $s = 0$ of the Artin L-function and the Brumer conjecture, a generalisation of Stickelberger's theorem, concerning the relation between the value at $s = 0$ of the Artin L-function and the annihilator ideal of the S-class-group in an abelian extension of number fields. In §3 we describe the analogues of the Stark and Brumer conjectures, conjectures of Gross and Coates–Sinnott respectively, in which the algebraic K-groups K_1 and K_0 are replaced by K_{1-2r} and K_{-2r} for $r = -1, -2, -3, \dots$. In §4, assuming the validity of the higher-dimensional Stark conjecture, we construct a finitely generated, Galois invariant subgroup — the fractional ideal $\mathcal{J}^r_E$ — of $\mathbb{Q}[G]$ where $G = \mathrm{Gal}(E/F)$ is an abelian Galois group of number fields. We verify that $\mathcal{J}^r_E$ is welldefined and coincides with the higher Stickelberger ideal when the latter is defined and non-trivial. In §5 we introduce a new conjectural relationship between $\mathcal{J}^r_E$ and the annihilator ideals of higher-dimensional algebraic K-groups (or étale cohomology groups) of algebraic integers. In §6 we prove Conjecture 1.2, the étale cohomology version of Conjecture 1.1, for cyclotomic fields. This is sufficient to verify the conjecture for any abelian extension of the rationals. In §7 we use the technique of [49] (see also [47, Ch. 6 and 7]) together with results from [3,6] to establish the technical results which are needed in §6. Section 8 contains some concluding remarks about possible generalisations and the naturality of the fractional ideal and, most importantly, describes a method for constructing annihilator elements when the Galois group is non-abelian.

The first of the fractional ideal/annihilator relations of Conjecture 1.2 occurred in [49]. Several of the technical results used here were proved in that paper (building on results of [6]) in order to give proofs of the étale cohomology versions of the Coates–Sinnott conjecture and the Lichtenbaum conjecture for abelian extensions of the rationals. The importance of Conjectures 1.1 and 1.2 becomes apparent when the L-function vanishes. In the other case several authors, of whose work I learnt after completing the first version of this paper, can prove stronger results than mine [28, Theorem (0.7)], [7, Theorem 3.1], [37, Théorème 2.3]. In addition, the papers [1, 12, 13, 30] and the series [2, 25, 26], are closely related to this one.

2 Some Well-Known Conjectures

2.1 Let $\zeta_E(s)$ denote the Dedekind zeta function of a number field E. The analytic class number formula [53, p. 21] gives the residue at $s = 1$ in terms of the order of the class-group of $\mathcal{O}_E$, the algebraic integers of E, and the Dirichlet regulator $R_0(E)$. Let d_E denote the discriminant of E. In terms of algebraic K-groups of $\mathcal{O}_E$ the class-group is equal to the torsion subgroup $\operatorname{Tors} K_0(\mathcal{O}_E)$ of $K_0(\mathcal{O}_E)$ and the formula has the form

$$\operatorname{res}_{s=1}\ \zeta_E(s) \ = \ \frac{2^{r_1+r_2}\pi^{r_2}R_0(E)|\operatorname{Tors} K_0(\mathcal{O}_E)|}{|\operatorname{Tors} K_1(\mathcal{O}_E)|\sqrt{d_E}}.$$

The Dirichlet regulator $R_0(E)$, which is a real number, is the covolume of the lattice given by the image of the Dirichlet regulator homomorphism [53, p. 25]

$$R_E^0\colon \mathcal{O}_E^* = K_1(\mathcal{O}_E) \longrightarrow \mathbb{R}^{r_1+r_2-1}.$$

Here r_1 and $2r_2$ denote the number of real or complex embeddings of E respectively. Equivalently, by Hecke's functional equation [34], [53, p. 18], $\zeta_E(s)$ has a zero of order r_1+r_2-1 at $s=0$. Let $\zeta_E^*(s_0)$ denote the first non-zero coefficient in the Taylor series for ζ_E at $s=s_0$. Therefore at $s=0$ the functional equation yields

$$\zeta_E^*(0) = \lim_{s\longrightarrow 0}\frac{\zeta_E(s)}{s^{r_1+r_2-1}} = -\frac{R_0(E)|\operatorname{Tors} K_0(\mathcal{O}_E)|}{|\operatorname{Tors} K_1(\mathcal{O}_E)|}.$$

This form of the analytic class number formula prompted Lichtenbaum [33] to ask: Which number fields E satisfy the analogous equation for higher-dimensional algebraic K-groups

$$\zeta_E^*(r) = \pm 2^{\epsilon}\frac{R_r(E)|\operatorname{Tors} K_{-2r}(\mathcal{O}_E)|}{|\operatorname{Tors} K_{1-2r}(\mathcal{O}_E)|}$$

for $r=-1,-2,-3,\dots$ and some integer ϵ? Here $R_r(E)$ is the covolume of the Borel regulator homomorphism defined on $K_{1-2r}(\mathcal{O}_E)$ and to which we shall return shortly. This identity has become known as the Lichtenbaum conjecture and is known to be true in many cases. In particular, the étale cohomology version of Lichtenbaum's conjecture was proved for cyclotomic fields in [25] (see [49] for a different proof and the survey article [48] for an overview). The passage from cohomology to algebraic K-theory requires the deep results of Suslin–Voevodsky, Rost et al. which were referred to in the Introduction.

Next we shall recall how Stark [53] refined the analytic class number formula into a conjecture dealing with $L_F^*(0,V)$, the leading coefficient of the Taylor series at $s=0$ of the Artin L-function associated to a Galois representation V of F [34].

Let $\Sigma(E)$ denote the set of embeddings of E into the complex numbers. For $r = -1,-2,-3,\dots$, set

$$Y_r(E) = \prod_{\Sigma(E)}(2\pi i)^{-r}\mathbb{Z} = \operatorname{Map}(\Sigma(E),(2\pi i)^{-r}\mathbb{Z})$$

endowed with the $G(\mathbf{C}/\mathbb{R})$-action diagonally on $\Sigma(E)$ and on $(2\pi i)^{-r}$. If c_0 denotes complex conjugation, $c_0((\dots,(2\pi i)^{-r}n_\sigma,\dots)_{\sigma\in\Sigma(E)})$ has $(-1)^r(2\pi i)^{-r}n_\sigma$ in the $c_0{\cdot}\sigma$-coordinate. Therefore the fixed points of $Y_r(E)$ under c_0, denoted by $Y_r(E)^+$, correspond to elements $\{(2\pi i)^{-r}n_\sigma\}_{\sigma\in\Sigma(E)}$ such that $(-1)^r n_\sigma = n_{c_0\cdot\sigma}$. Hence if $\sigma(E)\subset\mathbb{R}$ and r is odd then $n_\sigma=0$. When $r=0$ we define $Y_0(E)^+$ as the c_0-fixed points of

$$Y_0(E) = \operatorname{Ker}\Bigl(\alpha\colon \Bigl(\prod_{\Sigma(E)}\mathbb{Z}\Bigr)\to\mathbb{Z}\Bigr)$$

where α is the homomorphism defined by $\alpha((\dots,n_\sigma,\dots)_{\sigma\in\Sigma(E)}) = \sum_{\sigma\in\Sigma(E)} n_\sigma$.

This discussion shows that the rank of $Y_r(E)^+$ is given by

$$\operatorname{rank}_{\mathbb{Z}}(Y_r(E)^+) = \begin{cases} r_2 & \text{if } r \text{ is odd,} \\ r_1 + r_2 & \text{if } r > 0 \text{ is even,} \\ r_1 + r_2 - 1 & \text{if } r = 0. \end{cases}$$

where $|\Sigma(E)| = r_1 + 2r_2$ and r_1 is the number of real embeddings of E.

Now let G denote the Galois group of an extension of number fields E/F. Then, for $g \in G$, let $g((\dots,(2\pi i)^{-r}n_\sigma,\dots)_{\sigma\in\Sigma(E)}) \in Y_r(E)$ have $(2\pi i)^{-r}n_\sigma$ in the $\sigma\cdot g^{-1}$-coordinate. This defines a left G-action on $Y_r(E)$ which commutes with that of c_0 so that $Y_r(E)^+$ is a $\mathbb{Z}[G]$-lattice. The Dirichlet regulator homomorphism induces an $\mathbb{R}[G]$-module isomorphism of the form

$$R_E^0\colon K_1(\mathcal{O}_E)\otimes\mathbb{R} = \mathcal{O}_E^*\otimes\mathbb{R} \xrightarrow{\cong} Y_0(E)^+\otimes\mathbb{R} \cong \mathbb{R}^{r_1+r_2-1}.$$

The existence of this isomorphism implies that ([45, §12.1], [53, p. 26]) there exists at least one $\mathbb{Q}[G]$-module isomorphism of the form

$$f_{0,E}\colon K : K_1(\mathcal{O}_E)\otimes\mathbb{Q} \xrightarrow{\cong} Y_0(E)^+\otimes\mathbb{Q}.$$

For any choice of $f_{0,E}$ Stark forms the composition

$$R_E^0\cdot(f_{0,E})^{-1}\colon Y_0(E)^+\otimes\mathbb{C} \xrightarrow{\cong} Y_0(E)^+\otimes\mathbb{C}$$

which is an isomorphism of complex representations of G. Let V be a finite-dimensional complex representation of G whose contragredient is denoted by $V^\vee$. The Stark regulator is defined to be the exponential homomorphism, $(V\mapsto R(V,f_{0,E}))$, from representations to non-zero complex numbers given by

$$R(V,f_{0,E}) = \det\big((R_E^0\cdot f_{0,E}^{-1})_* \in \operatorname{Aut}_{\mathbb{C}}(\operatorname{Hom}_G(V^\vee, Y_0(E)^+\otimes\mathbb{C}))\big)$$

where $(R_E^0\cdot f_{0,E}^{-1})_*$ is composition with $R_E^0\cdot f_{0,E}^{-1}$.

Let $R(G)$ denote the complex representation ring of the finite group G, that is, $R(G) = K_0(\mathbb{C}[G])$. Since V determines a Galois representation of F, we have a non-zero complex number $L_F^*(0,V)$ given by the leading coefficient of the Taylor series at $s = 0$ of the Artin L-function associated to V [34]. We may modify $R(V,f_{0,E})$ to give another exponential homomorphism

$$\mathcal{R}_{f_{0,E}} \in \operatorname{Hom}(R(G),\mathbb{C}^*)$$

defined by

$$\mathcal{R}_{f_{0,E}}(V) = \frac{R(V,f_{0,E})}{L_F^*(0,V)}.$$

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of the rationals in the complex numbers and let $\Omega_{\mathbb{Q}}$ denote the absolute Galois group of the rationals, which acts continuously on $R(G)$ and $\overline{\mathbb{Q}}^*$. The Stark conjecture asserts that

$$\mathcal{R}_{f_{0,E}} \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*) \subseteq \mathrm{Hom}(R(G), \mathbb{C}^*).$$

In other words, $\mathcal{R}_{f_{0,E}}(V)$ is an algebraic number for each V and for all $z \in \Omega_{\mathbb{Q}}$ we have $z(\mathcal{R}_{f_{0,E}}(V)) = \mathcal{R}_{f_{0,E}}(z(V))$. Since any two choices of $f_{0,E}$ differ by multiplication by a $\mathbb{Q}[G]$-automorphism, the truth of the conjecture is independent of the choice of $f_{0,E}$.

We shall be particularly interested in the case when G is abelian in which case the following observation is important. Let $\hat{G} = \mathrm{Hom}(G, \overline{\mathbb{Q}}^*)$ denote the set of characters on G and let $\mathbb{Q}(\chi)$ denote the field generated by the character values of a representation χ.

Proposition 2.1 *Let G be a finite abelian group. Then there exists an isomorphism*

$$\lambda_G\colon \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*) \xrightarrow{\cong} \mathbb{Q}[G]^*$$

given by

$$\lambda_G(h) = \sum_{\chi \in \hat{G}} h(\chi) e_\chi$$

where

$$e_\chi = |G|^{-1} \sum_{g \in G} \chi(g) g^{-1} \in \mathbb{Q}(\chi)[G].$$

Proof This follows by combining the isomorphisms

$$\mathbb{Q}[G]^* \cong K_1(\mathbb{Q}[G]) \quad \text{and} \quad K_1(\mathbb{Q}[G]) \cong \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*),$$

which are part of Fröhlich's Hom-description machinery described in [14]. In fact the second isomorphism, proved originally in [40], is true for arbitrary finite groups G.

When G is abelian the proof is very simple. There is a well-known isomorphism of rings [29, p. 648]

$$\psi\colon \overline{\mathbb{Q}}[G] \longrightarrow \prod_{\chi \in \hat{G}} \overline{\mathbb{Q}} = \mathrm{Map}(\hat{G}, \overline{\mathbb{Q}})$$

given by $\psi\left(\sum_{g \in G} \lambda_g g\right)(\chi) = \sum_{g \in G} \lambda_g \chi(g)$. If $\Omega_{\mathbb{Q}}$ acts on $\overline{\mathbb{Q}}$ and $\hat{G}$ in the canonical manner then ψ is Galois equivariant and induces an isomorphism of $\Omega_{\mathbb{Q}}$-fixed units of the form

$$\mathbb{Q}[G]^* = (\overline{\mathbb{Q}}[G]^*)^{\Omega_{\mathbb{Q}}} \cong \mathrm{Map}_{\Omega_{\mathbb{Q}}}(\hat{G}, \overline{\mathbb{Q}}^*) \cong \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*).$$

It is straightforward to verify that this isomorphism is the inverse of λ_G. ■

2.2 Stickelberger Elements and Annihilators

Now we are going to turn our attention to some conjectures concerning annihilator ideals which appear in [4, 10, 59]. Suppose that E/F is a Galois extension of number fields with $G = \mathrm{Gal}(E/F)$ abelian. Suppose also that F is totally real and that E is totally real or is a CM field (*i.e.,* E is a totally imaginary quadratic extension of a totally real field (see [57, p. 38]). Let S be a finite set of primes of $\mathcal{O}_F$ including those which ramify in E/F. The reciprocity map of class field theory sends the class of a proper ideal prime to S, $\mathcal{A} \lhd \mathcal{O}_F$, to its Artin symbol $(\mathcal{A}, E/F) \in G$. The associated partial zeta function is defined for complex numbers s having $\mathcal{R}e(s) > 1$ by

$$\zeta_{F,S}(g, s) = \sum_{\substack{(\mathcal{A}, E/F) = g, \\ \mathcal{A} \text{ prime to } S}} N\mathcal{A}^{-s}.$$

Here $g \in G$ and the sum is over all ideals coprime to all primes in S. These functions have a meromorphic continuation to the whole complex plane and the corresponding Stickelberger elements are defined to be

$$\Theta_{E/F,S}(1 - r) = \sum_{g \in G} \zeta_{F,S}(g, r) \cdot g^{-1} \in \mathbb{C}[G]$$

for $r = 0, -1, -2, -3, \dots$. These elements are characterised by the relation

$$\chi(\Theta_{E/F,S}(1 - r)) = L_{F,S}(r, \chi^{-1})$$

for all one-dimensional complex representations χ of G, where $L_{F,S}(r, \chi^{-1})$ is the Artin L-function with all the Euler factors associated to elements of S removed. By a result of Klingen and Siegel [46] $\Theta_{E/F,S}(1 - r)$ lies in $\mathbb{Q}[G]$ for $r = 0, -1, -2, -3, \dots$.

Let $\mu(E)$ denote the roots of unity in E so that $\mu(E) = \mathrm{Tors}\, K_1(\mathcal{O}_E)$ in the notation of §2.1. The Stickelberger elements $\Theta_{E/F,S}(1)$ satisfy the integrality relation

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(E)) \cdot \Theta_{E/F,S}(1) \subseteq \mathbb{Z}[G]$$

where $\mathrm{ann}_{\mathbb{Z}[G]}(\mu(E)) \lhd \mathbb{Z}[G]$ denotes the annihilator ideal of $\mu(E)$. When $F = \mathbb{Q}$ this was proved in [27], for F real quadratic in [11] and in general in [8, 17].

The Brumer conjecture goes further than the mere integrality statement, asserting that

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Tors}\, K_1(\mathcal{O}_{E,S'})) \cdot \Theta_{E/F,S}(1) \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Tors}\, K_0(\mathcal{O}_{E,S'}))$$

where S' is the set of primes of E above those of S and $\mathcal{O}_{E,S'}$ denotes the S'-integers of E. When $F = \mathbb{Q}$ this is Stickelberger's Theorem ([9, p. 298]; [57, p. 94]). In general there are only partial results, for example, [21].

3 Analogous Conjectures for Higher K-Groups

3.1 Higher Dimensional Stark Conjectures

Lots of interesting progress has been made by simply taking some phenomenon involving class-groups or Picard groups, such as the analytic class number formula,

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and asking the question: What happens when K_0 is replaced by K_n? The Lichtenbaum conjecture of §2.1 is a prime example. It was B. H. Gross [22] who first asked this question about the Stark conjecture of §2.1.

For any negative integer $r < 0$ we have the Borel regulator [5, 24]

$$R_E^r : K_{1-2r}(\mathcal{O}_E) \otimes \mathbb{R} \xrightarrow{\cong} Y_r(E)^+ \otimes \mathbb{R}$$

which is an $\mathbb{R}[G]$-isomorphism. Now we mechanically imitate Stark's procedure with the Dirichlet regulator replaced by Borel's. We choose a $\mathbb{Q}[G]$-isomorphism of the form

$$f_{r,E} \colon K_{1-2r}(\mathcal{O}_E) \otimes \mathbb{Q} \xrightarrow{\cong} Y_r(E)^+ \otimes \mathbb{Q}$$

so that

$$R_E^r \cdot (f_{r,E})^{-1} \colon Y_r(E)^+ \otimes \mathbb{R} \xrightarrow{\cong} Y_r(E)^+ \otimes \mathbb{R}$$

is an $\mathbb{R}[G]$-isomorphism. Then, as in §2.1, we form the Stark regulator defined, for each representation V of G, by

$$R(V, f_{r,E}) = \det\bigl((R_E^r \cdot f_{r,E}^{-1})_* \in \operatorname{Aut}_{\mathbf{C}}(Hom_G(V^\vee, Y_r(E)^+ \otimes \mathbb{C}))\bigr) .$$

Let S be a finite set of primes of F which includes all the primes which ramify in E/F. Let $L^*_{F,S}(r, V)$ denote the leading term of the Taylor expansion of the Artin L-function associated to S and V at $s = r$. We define a function $\mathcal{R}_{f_{r,E}}$ given on a finite-dimensional complex representation V by

$$\mathcal{R}_{f_{r,E}}(V) = \frac{R(V, f_{r,E})}{L^*_{F,S}(r, V)}.$$

Then the higher-dimensional analogue of the Stark conjecture of §2.1 asserts that

$$\mathcal{R}_{f_{r,E}} \in \operatorname{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*) \subseteq \operatorname{Hom}(R(G), \mathbb{C}^*)$$

and the truth of this conjecture is independent of the choice of $f_{r,E}$.

The calculations of Beilinson [3] (see also [6, §4.2], [24, 38]) show that the higher-dimensional analogue of the Stark conjecture is true when E/F is a subextension of any abelian extension of the rationals (see the proof of Theorem 7.6).

3.2 Higher Dimensional Annihilator Conjectures

In this section we study the case E/F when the subfield F is the rational numbers. In this case it is traditional (and will also be convenient) to use Dirichlet L-functions [57]. As explained below, for our purposes this is equivalent to the use of Artin L-functions $L_{\mathbb{Q},S}(r, \chi)$ when S is the set of primes dividing the conductor of E.

Now let us examine the higher-dimensional analogues of the Brumer conjecture of §2.2. These analogues were first posed by Coates and Sinnott in the case of abelian extensions of the rationals and were expressed in terms of Stickelberger elements constructed from the Dirichlet L-function. Since we are going to return to this case as

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a source of crucial examples in §6.1 and Theorem 6.1 we shall recall the situation of [10].

Suppose that $E/\mathbb{Q}$ is a finite Galois extension of number fields with abelian Galois group G. Let f denote the non-archimedean part of the conductor (in the sense of class field theory) of $E/\mathbb{Q}$. That is, f is the smallest value of m such that $E \subseteq \mathbb{Q}(\xi_m)$ where $\xi_m = \exp(2\pi\sqrt{-1}/m)$. Then, for each negative integer $r = -1, -2, -3, \dots$, there is an element of the rational group-ring, called a higher Stickelberger element,

$$\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r) \in \mathbb{Q}[G]$$

which is defined in the following manner. Any character $\chi\colon G \longrightarrow \mathbb{C}^*$ may be considered as a Dirichlet character [57, p. 29]. If χ has conductor equal to f then

$$\chi(\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r)) = L(r, \chi^{-1})$$

where $L(s, \chi^{-1})$ denotes the Dirichlet L-function of the character χ^{-1} [57, Ch. 4]. More generally, if d is any divisor of f equal to the conductor of χ then

$$\chi(\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r)) = L(r, \chi^{-1}) \prod_{\substack{p \text{ prime},\ p|f \\ (p,d)=1}} ((1 - \chi(p)^{-1} p^{-r}).$$

The fact that higher Stickelberger elements exist in the rational group ring is a consequence of a result of Klingen and Siegel [46].

Let $c_0 \in G$ denote complex conjugation and write $e_r^{\pm} = (1 \pm (-1)^r c_0)/2$ for the idempotents (with the obvious convention when c_0 acts trivially) associated to complex conjugation in the group ring of the Galois group. Since $L(r, \chi^{-1}) = 0$ if and only if $\chi(c_0) = (-1)^r$ the element $\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r)$ is not uniquely defined but

$$e_r^- \cdot \Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r) \in \mathbb{Q}[G]$$

is uniquely characterised as the element taking the above non-zero values when $\chi(c_0) = (-1)^{r+1}$ and zero otherwise.

In particular, when E is totally real and $r = -1, -3, -5, \dots$, then $c_0 = 1$ and

$$\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r) \in \mathbb{Q}[G]^*.$$

The relation between these Stickelberger elements and those defined using the Artin L-function, as in §2.2, is described in the following manner. Suppose that $f = ml^{s+1}$ where l is an odd prime, $HCF(m, l) = 1$ and the conductor of χ is dl^n with $HCF(d, l) = 1$. When $n \geq 1$

$$L(r, \chi^{-1}) \prod_{\substack{p \text{ prime},\ p|f \\ (p,d)=1}} ((1 - \chi(p)^{-1} p^{-r}) = L_{\mathbb{Q}, S_{ml}}(r, \chi^{-1})$$

where S_{ml} is the set of finite primes of F which divide ml. However, when $n = 0$, this expression differs from $L_{\mathbb{Q}, S_{ml}}(r, \chi^{-1})$ by a factor $(1 - \chi^{-1}(l) l^{-r})$ which is an l-adic unit when $r = -1, -2, -3, \dots$.

Define $\mu_{1-r}(E)$ to be the $\mathbb{Z}[G]$-module given by

$$\mu_{1-r}(E) = \varinjlim_{M/\mathbb{Q}} (\mu(M)^{\otimes^{1-r}})^{G(M/E)}$$

where the limit is taken over Galois extensions $M/\mathbb{Q}$ containing E. Hence $\mu_1(E) = \mu(E) = \mathrm{Tors}\, K_1(\mathcal{O}_E)$ and the Quillen–Lichtenbaum conjectures in algebraic K-theory predict that $\mu_{1-r}(E) = \mathrm{Tors}\, K_{1-2r}(\mathcal{O}_E)$ [25, 42].

Assume now that E is totally real and that $r = -1, -3, -5, \dots$. Inspired by Stickelberger's Theorem [57, p. 94], the Coates–Sinnott conjecture [10] (see also [11]) asserts that for any prime l

$$\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r) \cdot \mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}(E) \otimes \mathbb{Z}_l) \subseteq \mathrm{ann}_{\mathbb{Z}_l[G]}(K_{-2r}(\mathcal{O}_E) \otimes \mathbb{Z}_l).$$

Actually the conjecture in [10] incorporated an extra factor denoted by $w_{n+1}(\mathbb{Q})$ which we have omitted because it was unnecessary (at least when l is odd; see [49, §1]). Also the annihilator of $\mu_{1-r}(E)$ is known ([47, Proposition 7.2.5]; [9]).

When $r = -2, -4, -6, \dots$ and E is a CM field a similar conjecture is posed in [10] in which $K_{-2r}(\mathcal{O}_E) \otimes \mathbb{Z}_l$ and $\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r)$ are replaced by $e_r^- \cdot K_{-2r}(\mathcal{O}_E) \otimes \mathbb{Z}_l$ and $e_r^- \cdot \Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r)$, respectively.

The higher-dimensional analogue of the Brumer conjecture, posed and discussed in [47, Ch. 6, 7], asserts for $r = -1, -2, -3, \dots$ and $E/F, S, \Theta_{E/F,S}(1-r)$ as in §2.2 that

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Tors}\, K_{1-2r}(\mathcal{O}_{E,S'})) \cdot \Theta_{E/F,S}(1-r) \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(K_{-2r}(\mathcal{O}_{E,S'}))$$

where S' is the set of primes of E above those in S and $\mathcal{O}_{E,S'}$ denotes the S'-integers of E. When $F = \mathbb{Q}$ this is equivalent to the conjecture of [10] mentioned above (see also [1]). Note that $K_{-2r}(\mathcal{O}_{E,S'}) = \mathrm{Tors}\, K_{-2r}(\mathcal{O}_{E,S'})$, being a finite group.

4 The Canonical Fractional Ideal

4.1 As in §2.2, let E/F be a Galois extension of number fields with abelian Galoup group G. Let S be a finite set of primes of $\mathcal{O}_F$ including those which ramify in E/F. Throughout this section we shall assume that the higher-dimensional Stark conjecture of §3.1 is true. Therefore, by Proposition 2.1, we have an element

$$\mathcal{R}_{f_{r,E}} \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R(G), \overline{\mathbb{Q}}^*) \cong \mathbb{Q}[G]^*$$

which depends upon the choice of a $\mathbb{Q}[G]$-isomorphism $f_{r,E}$ in §3.1 where $r = -1, -2, -3, \dots$.

The following result is an observation concerning the naturality of the Stark conjecture.

Proposition 4.1 *Suppose that E/F is an abelian extension for which the higher-dimensional Stark conjecture* §3.1 *holds and suppose that E/F_1 is a subextension. If we choose for F_1 the set of primes S_1 over those of S then the conjecture holds for E/F_1 also.*

Let M/F be an intermediate Galois extension of E/F. Then the conjecture is also true for M/F and S if it is true for E/F and S in §3.1.

Proof Let χ denote a character of G. We may choose the same $f_{r,E}$ for F and F_1 then

$$L_{F_1,S_1}(r, \mathrm{Res}^G_{\mathrm{Gal}(E/F_1)}(\chi)) = L_{F,S}(r, \chi \otimes \mathrm{Ind}^G_{\mathrm{Gal}(E/F_1)}(1))$$

and

$$\mathrm{Hom}_{\mathrm{Gal}(E/F_1)}(\mathrm{Res}^G_{\mathrm{Gal}(E/F_1)}(\chi)^\vee, Y_r(E)) \cong \mathrm{Hom}_G((\chi \otimes \mathrm{Ind}^G_{\mathrm{Gal}(E/F_1)}(1))^\vee, Y_r(E)).$$

Therefore, since $(R^r_E \cdot f^{-1}_{r,E})_*$ is the same for F and F_1, we find that

$$\mathcal{R}_{f_{r,E}}(\mathrm{Res}^G_{\mathrm{Gal}(E/F_1)}(\chi)) = \mathcal{R}_{f_{r,E}}(\chi \otimes \mathrm{Ind}^G_{\mathrm{Gal}(E/F_1)}(1)).$$

Therefore the conjecture of §3.1 holds for E/F_1 if it holds for E/F because

$$\mathrm{Res}^G_{\mathrm{Gal}(E/F_1)} \colon R(G) \longrightarrow R(\mathrm{Gal}(E/F_1))$$

is surjective.

The proof for intermediate extensions M/F is similar and will be left to the reader. ■

4.2 $\mathrm{Det}_P(\alpha)$

Here is a simple, probably familiar, algebraic construction. Let l be a prime and G a finite abelian group. For $a, b \in \mathbb{Q}_l[G]$ write $a \simeq b$ in $\mathbb{Q}_l[G]$ if and only if $a = ub$ for some $u \in \mathbb{Z}_l[G]^*$. Suppose that P is a finitely generated projective $\mathbb{Z}_l[G]$-module and that $\alpha \in \mathrm{End}_{\mathbb{Q}_l[G]}(P \otimes \mathbb{Q}_l)$. Choosing a finitely generated $\mathbb{Z}_l[G]$-module R such that $P \oplus R$ is free and taking the determinant of $\alpha \oplus 1$ with respect to a $\mathbb{Z}_l[G]$-basis yields a well-defined element

$$\mathrm{Det}_P(\alpha) \in \mathbb{Q}_l[G]/\simeq .$$

Sometimes it will be convenient to replace $\mathbb{Z}_l[G]$, $\mathbb{Q}_l$ and $\mathbb{Q}_l[G]$ by $\mathbb{Z}[1/2][G]$, $\mathbb{Q}$ and $\mathbb{Q}[G]$, respectively, in the construction of $\mathrm{Det}_P(\alpha)$. Write $e^{\pm}_r = ((1 \pm (-1)^r c_0)/2)$. The proof of the following result is straightforward.

Theorem 4.2

(i) *In* §7.1 $\mathrm{Det}_P(\alpha)$ *depends only on* α.

(ii) *If* α *is an automorphism then* $\det_P(\alpha)$ *defines an element of*

$$\mathbb{Q}_l[G]^*/\mathbb{Z}_l[G]^* \cong K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$$

corresponding to $[P, \alpha, P]$. *Here the isomorphism is the one described in* §7.1.

(iii) *If* l *is an odd prime and* $c_0 \in G$ *has order two with* $P = e^{\pm}_r \cdot \mathbb{Z}_l[G]^n \cong (\mathbb{Z}_l[G]/e^{\mp}_r)^n$ *then under the canonical map*

$$\mathbb{Q}_l[G]/\simeq \longrightarrow \big((\mathbb{Q}_l[G]/((1 \mp c_0)/2))/\simeq\big) \times \big(\mathbb{Q}_l[G]/((1 \pm c_0)/2))/\simeq\big)$$

$\mathrm{Det}_P(\alpha)$ *maps to* $(\det(\alpha), 1)$, *where* $\det(\alpha)$ *is the determinant of* α *computed with respect to a* $(\mathbb{Z}_l[G]/((1 \mp c_0)/2))$*-basis for* P.

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(iv) *Parts* (i) *and* (iii) *remain true if* $\mathbb{Z}_l[G]$, $\mathbb{Q}_l$ *and* $\mathbb{Q}_l[G]$ *are replaced by* $\mathbb{Z}[1/2][G]$, $\mathbb{Q}$ *and* $\mathbb{Q}[G]$, *respectively.*

Definition 4.3 (The fractional ideal $\mathcal{J}_E^r$) Let E/F be a Galois extension of number fields with abelian Galois group G. When $r = -1, -2, -3, \dots$ the lattice $Y_r(E)$ is a free $\mathbb{Z}[G]$-module as is seen by identifying $Y_r(E)$ with the $G(L/E)$-fixed elements of $Y_r(L)$ for some Galois extension of the rationals L containing E. If $c \in G(\mathbb{C}/\mathbb{R})$ denotes complex conjugation acting on $Y_r(E) = \sum_{\Sigma(E)} (2\pi i)^{-r}\mathbb{Z}$ by acting simultaneously on $\Sigma(E)$ and on $(2\pi i)^{-r}\mathbb{Z}$ then $Y_r(E)$ becomes a $\mathbb{Z}[G \times G(\mathbb{C}/\mathbb{R})]$-module. Therefore $Y_r(E)^+ \otimes \mathbb{Z}[1/2]$ is a finitely generated, projective $\mathbb{Z}[1/2][G]$-module. By the construction of §4.2, each $\mathbb{Q}[G]$-endomorphism of $Y_r(E)^+ \otimes \mathbb{Q}$ gives rise to an element $\mathrm{Det}_{Y_r(E)^+\otimes\mathbb{Z}[1/2]}(\alpha) \in \mathbb{Q}[G]$ which is well defined up to multiplication by a unit of $\mathbb{Z}[1/2][G]$.

Define $\mathcal{I}_{f_{r,E}}$ to be the (finitely generated) $\mathbb{Z}[1/2][G]$-submodule of $\mathbb{Q}[G]$ generated by all the elements $\mathrm{Det}_{Y_r(E)^+\otimes\mathbb{Z}[1/2]}(\alpha)$ where $\alpha \in \mathrm{End}_{\mathbb{Q}[G]}(Y_r(E)^+ \otimes \mathbb{Q})$ satisfies the integrality condition

$$\alpha \cdot f_{r,E}(K_{1-2r}(\mathcal{O}_E)) \subseteq Y_r(E)^+.$$

Define $\mathcal{J}_E^r$ to be the finitely generated $\mathbb{Z}[1/2][G]$-submodule of $\mathbb{Q}[G]$ given by

$$\mathcal{J}_E^r = \mathcal{I}_{f_{r,E}} \cdot \tau(\mathcal{R}_{f_{r,E}}^{-1})$$

where τ is the automorphism of the group-ring induced by sending each $g \in G$ to its inverse. Recall that throughout this section (see §4.1) we are assuming the validity of the higher-dimensional Stark conjecture of §3.1 in order for $\mathcal{R}_{f_{r,E}} \in \mathbb{Q}[G]^*$ to be defined.

Example 4.4 (i) In the situation of §4.1 and Definition 4.3 suppose that E is totally real and $r = -1, -2, -3, \dots$. In this case $Y_r(E)^+$ is a free $\mathbb{Z}[G]$-module of rank equal to $[F : \mathbb{Q}]$ when r is even and is trivial when r is odd.

Assume for the moment that $r = -1, -3, -5, \dots$. Bearing in mind that the determinant of the zero automorphism of the zero module is 1, for each character χ we have

$$1 = R(\chi, f_{r,E}) = \det((R_E^r \cdot f_{r,E}^{-1}))_* = \det\left(0 \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{Hom}_G(\chi^{-1}, 0))\right).$$

Next we form

$$\mathcal{R}_{f_{r,E}}(\chi) = \frac{R(\chi, f_{r,E})}{L_{F,S}^*(r, \chi)} = L_{F,S}(r, \chi)^{-1},$$

since $L_{F,S}(r, \chi)$ is non-zero.

Therefore $\mathcal{J}_E^r$ is the $\mathbb{Z}[1/2][G]$-submodule of $\mathbb{Q}[G]$ given by

$$\mathcal{J}_E^r = \mathbb{Z}[1/2][G] \cdot \tau(\mathcal{R}_{f_{r,E}}^{-1}) = \mathbb{Z}[1/2][G] \cdot (\chi \mapsto L_{F,S}(r, \chi^{-1})).$$

Hence $\mathcal{J}_E^r$ is the principal $\mathbb{Z}[1/2][G]$-submodule generated by the Stickelberger element $\Theta_{E/F,S}(1 - r)$ of §2.2

$$\mathcal{J}_E^r = \mathbb{Z}[1/2][G]\langle\Theta_{E/F,S}(1 - r)\rangle \in \mathbb{Q}[G].$$

This fact can also be deduced directly from Theorem 4.2(iii) and (iv).

(ii) In the situation of §4.1 and Definition 4.3 suppose that E is a CM field, F totally real and $c_0 \in G$ denotes complex conjugation. Then c_0 acts on $Y_r(E)^+$ like multiplication by $(-1)^r$. Hence $e_r^- \cdot Y_r(E)^+ = 0$. Remembering once again that the determinant of the zero automorphism of the zero module is 1 or by means of Theorem 4.2(iii) and (iv) we find that

$$\mathfrak{I}_{f_{f,E}} \cdot e_r^- = \mathbb{Z}[1/2][G] \cdot e_r^- .$$

Therefore, since $1 = R(\chi, f_{r,E})$ for all characters χ such that $\chi(c_0) = (-1)^{r+1}$, we find that

$$\begin{aligned}\mathcal{J}_E^r \cdot e_r^- &= \mathfrak{I}_{f_{r,E}} e_r^- \cdot \tau(\mathcal{R}_{f_{r,E}}^{-1})\\ &= \mathbb{Z}[1/2][G] e_r^- \cdot \tau(\mathcal{R}_{f_{r,E}}^{-1})\\ &= \mathbb{Z}[1/2][G] e_r^- \cdot \Theta_{E/F,S}(1-r).\end{aligned}$$

Notice that $\Theta_{E/F,S}(1-r))e_r^- \in \mathbb{Q}[G]$ is characterised by the relation that

$$\chi(\Theta_{E/F,S}(1-r))e_r^- = L_{F,S}(r, \chi^{-1})$$

for all characters of G satisfying $\chi(c_0) = (-1)^{r+1}$ and is zero otherwise.

Proposition 4.5 *Let E/F be a Galois extension of number fields with abelian Galois group G. Then, assuming that the higher-dimensional Stark conjecture of* §3.1 *holds for E/F, the finitely generated $\mathbb{Z}[1/2][G]$-submodule of $\mathbb{Q}[G]$, $\mathcal{J}_E^r$ defined in* §4.2*, is independent of the choice of $f_{r,E}$.*

Proof Changing $f_{r,E}$ to another $\mathbb{Q}[G]$-module isomorphism $g_{r,E}$ in §3.1 changes $R_E^r(f_{r,E})^{-1}$ to $R_E^r(f_{r,E})^{-1} f_{r,E}(g_{r,E})^{-1}$ and

$$f_{r,E}(g_{r,E})^{-1} \in \mathrm{Aut}_{\mathbb{R}[G]}(Y_r(E)^+ \otimes \mathbb{R})$$

is a scalar extension of a $\mathbb{Q}[G]$-module isomorphism u of $Y_r(E)^+ \otimes \mathbb{Q}$. Hence

$$R(V, g_{r,E}) = R(V, f_{r,E}) \det(u_* \in \mathrm{Aut}_{\mathbb{C}}\big(\mathrm{Hom}_G(V^\vee, Y_r(E)^+ \otimes \mathbb{C})\big)\,.$$

Now suppose that $P = Y_r(E)^+ \otimes \mathbb{Z}[1/2]$, which is a finitely generated, projective $\mathbb{Z}[1/2][G]$-module (see Definition 4.3) and suppose that R is another finitely generated projective $\mathbb{Z}[1/2][G]$-module such that there is a $\mathbb{Z}[1/2][G]$-module isomorphism of the form

$$\phi\colon P \oplus R \xrightarrow{\cong} \mathbb{Z}[1/2][G]^n$$

as in §4.2. Then

$$\begin{aligned}&\det\big(u_* \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{Hom}_G(V^\vee, Y_r(E)^+ \otimes \mathbb{C}))\big)\\ &\qquad = \det\big((u \oplus 1)_* \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{Hom}_G(V^\vee, (P \oplus R) \otimes \mathbb{C}))\big)\\ &\qquad = \det\big((\phi(u \oplus 1)\phi^{-1})_* \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{Hom}_G(V^\vee, \mathbb{Z}[1/2][G]^n \otimes \mathbb{C}))\big)\,.\end{aligned}$$

Now consider the set of elements $\alpha \in \mathrm{End}_{\mathbb{Q}[G]}(P \otimes \mathbb{Q})$ as in Definition 4.3 such that

$$\alpha \cdot f_{r,E}(K_{1-2r}(\mathcal{O}_E)) \subseteq Y_r(E)^+ \subseteq Y_r(E)^+ \otimes \mathbb{Q}.$$

This set is exactly those elements $\beta = \alpha \cdot u$ such that

$$\beta \cdot g_{r,E}(K_{1-2r}(\mathcal{O}_E)) \subseteq Y_r(E)^+ \subseteq Y_r(E)^+ \otimes \mathbb{Q},$$

since $\alpha f_{r,E} = (\alpha \cdot u)(g_{r,E})^{-1}$. Therefore $\det_P(\beta) = \det_P(\alpha)\det_P(u)$ and by Theorem 4.2(ii) the difference between the images of $\det_P(\beta)$ and $\det_P(\alpha)$ in $\frac{\mathbb{Q}[G]^*}{\mathbb{Z}[1/2][G]^*}$ corresponds to the factor

$$V \mapsto \det((\phi(u \oplus 1)\phi^{-1})_* \in \mathrm{Aut}_{\mathbb{C}}\big(\mathrm{Hom}_G(V, \mathbb{Z}[1/2][G]^n \otimes \mathbb{C})\big)$$

under the isomorphism λ_G of Proposition 2.1. The automorphism τ on $\mathbb{Q}[G]^*$ transforms this function (via λ_G) to

$$V \mapsto \det((\phi(u \oplus 1)\phi^{-1})_* \in \mathrm{Aut}_{\mathbb{C}}\big(\mathrm{Hom}_G(V^\vee, \mathbb{Z}[1/2][G]^n \otimes \mathbb{C})\big)$$

so that

$$\begin{aligned} \mathcal{I}_{g_{r,E}} \cdot \tau(\mathcal{R}_{g_{r,E}}^{-1}) &= \mathcal{I}_{f_{r,E}} \det_P(u)\tau(\det(u_*)^{-1})\tau(\mathcal{R}_{f_{r,E}}^{-1}) \\ &= \mathcal{I}_{f_{r,E}}\tau(\mathcal{R}_{f_{r,E}}^{-1})\tau(\det(u_*))\tau(det(u_*)^{-1}) \\ &= \mathcal{I}_{f_{r,E}}\tau(\mathcal{R}_{f_{r,E}}^{-1}), \end{aligned}$$

as required. Here we have used the fact that λ_G is the inverse of the reduced norm homomorphism which is sends the group-ring element u to the function $\det(u_*)$, as explained in [14, II pp. 332–333]. ∎

5 Annihilators, A New Conjecture

5.1 As in §2.2, let E/F be a Galois extension of number fields with abelian Galois group G. Let S be a finite set of primes of $\mathcal{O}_F$ including those which ramify in E/F and let S' denote the primes of E above those of S. Throughout this section we shall assume that the higher-dimensional Stark conjecture of §3.1 is true. Therefore from §4.2 we have a "fractional ideal" (that is, a well-defined, finitely generated $\mathbb{Z}[1/2][G]$-module)

$$\mathcal{J}_E^r \subseteq \mathbb{Q}[G]$$

for each negative integer $r = -1, -2, -3, \ldots$.

Now we come to the most important part of the paper, a conjecture for which some supporting evidence will be presented in §6.1 and Theorem 6.1.

Conjecture 5.1 *Let l be an odd prime. Then, in the situation and notation of* §5.1,

$$\big(\mathrm{ann}_{\mathbb{Z}_l[G]}(\mathrm{Tors}\, K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l) \cdot \mathcal{J}_E^r\big) \cap \mathbb{Z}_l[G] \subseteq \mathrm{ann}_{\mathbb{Z}_l[G]}\big(K_{-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l\big).$$

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Remark 5.2 In Conjecture 5.1 when E is totally real and $r = -1, -3, -5, \ldots$ then, by Example 4.4(i),

$$\mathcal{J}_E^r = \mathbb{Z}[G]\langle \Theta_{E/F,S}(1-r)\rangle.$$

In addition, the Quillen–Lichtenbaum conjecture (see §6.1 and the footnote following Conjecture 1.1 in the Introduction) predicts that

$$\mu_{1-r}(E) \otimes \mathbb{Z}_l \cong \operatorname{Tors} K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l$$

so that

$$\operatorname{ann}_{\mathbb{Z}_l[G]}(\operatorname{Tors} K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l) \cdot \mathcal{J}_E^r$$

is expected to contain

$$\Theta^{\mathrm{Dir}}_{E/\mathbb{Q}}(1-r) \cdot \operatorname{ann}_{\mathbb{Z}_l[\mathrm{Gal}(E/\mathbb{Q})]}(\mu_{1-r}(E) \otimes \mathbb{Z}_l)$$

of §3.2. By [17] (see also [9]) this finitely generated $\mathbb{Z}_l[G]$-submodule of $\mathbb{Q}_l[G]$ actually lies in $\mathbb{Z}_l[G]$.

In any case, this discussion shows that Conjecture 5.1 coincides with one of the well-known conjectures of §2.2 when E is totally real and $r = -1, -3, -5, \ldots$.

On the other hand, if E is totally real in §5.1 and $r = -2, -4, -6, \ldots$ then $\mu_{1-r}(E)$ is trivial because the action of complex conjugation on the $\mathbb{Z}_l$-module $\mu(M)^{\otimes^{1-r}} \otimes \mathbb{Z}_l$ of §3.2 is multiplication by $(-1)^{1-r}$. In this case the Quillen–Lichtenbaum conjecture predicts that $\operatorname{Tors} K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l$ is trivial and Conjecture 5.1 reduces to

$$\mathcal{J}_E^r \cap \mathbb{Z}_l[G] \subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}(K_{-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l).$$

Question 5.3 (Integrality) Perhaps, by analogy with the totally real case when r is odd,

$$\operatorname{ann}_{\mathbb{Z}_l[G]}(\operatorname{Tors} K_{1-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l) \cdot \mathcal{J}_E^r \lhd \mathbb{Z}_l[G]$$

in general? That is, perhaps the intersection with $\mathbb{Z}_l[G]$ is unnecessary in Conjecture 5.1? Perhaps

$$\mathcal{J}_E^r \subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}(K_{-2r}(\mathcal{O}_{E,S'}) \otimes \mathbb{Z}_l)$$

when E is totally real in §5.1 and $r = -2, -4, -6, \ldots$?

6 Supporting Evidence

6.1 Throughout this section let l be an odd prime, m a positive integer prime to l and $r = -1, -2, -3, -4, -5, \ldots$. We are going to study Conjecture 5.1 for the cyclotomic extension $\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}$ where $s \geq 0$ and $\xi_t = e^{2\pi\sqrt{-1}/t}$. We shall study the case where $\mathcal{O}_{E,S}$ is equal to $\mathbb{Z}[\xi_{ml^{s+1}}][1/ml]$, to get from this case to a larger set S is straightforward using the localisation exact sequence.

For $\epsilon = 1, 2$ there are étale cohomology Chern classes [15, 16] of the form

$$K_{2-2r-\epsilon}(\mathbb{Z}[\xi_{ml^{s+1}}][1/ml]) \otimes \mathbb{Z}_l \xrightarrow{c_{1-r,\epsilon}} H^{\epsilon}_{\text{ét}}\big(\operatorname{Spec}(\mathbb{Z}[\xi_{ml^{s+1}}][1/ml]); \mathbb{Z}_l(1-r)\big)$$

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which the Quillen–Lichtenbaum conjecture predicts to be isomorphisms. This was proved for K_2 in [52] and for K_3 in [31, 36]. As a corollary of the fundamental results of Voevodsky [54, 55], the corresponding Chern classes when $l = 2$ are nearly isomorphisms in all dimensions [42]. Voevodsky's method requires the existence of suitable "norm varieties" which is not yet established for all odd primes (however, see footnote following Conjecture 1.1 in the Introduction).

Observe that, if $c_{1-r,1}$ is an isomorphism, we have

$$\begin{aligned}\mathrm{Tors}(K_{1-2r}(\mathbb{Z}[\xi_{ml^{s+1}}]) \otimes \mathbb{Z}_l) &\cong \mathrm{Tors}\, H^1_{\text{ét}}\big(\mathrm{Spec}(\mathbb{Z}(\xi_{ml^{s+1}})[1/ml]); \mathbb{Z}_l(1-r)\big)\\ &\cong H^0_{\text{ét}}\big(\mathrm{Spec}(\mathbb{Z}(\xi_{ml^{s+1}})[1/ml]); (\mathbb{Q}_l/\mathbb{Z}_l)(1-r)\big)\\ &\cong \mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}}))\end{aligned}$$

where $\mu_{1-r}(E)$ is as in §3.2 and Question 5.2.

We are now almost ready to state our main result. Since we are studying cyclotomic fields I shall follow the example of [10] and state the result in terms of Dirichlet L-functions. We shall need to recall the corresponding higher Stickelberger elements and leading terms.

Suppose that $L/\mathbb{Q}$ is a finite Galois extension of number fields with abelian Galois group, G. Recall from §3.2, if c_0 denotes complex conjugation and $e_r^{\pm} = (1 \pm (-1)^r c_0)/2$, for each negative integer $r = -1, -2, -3, \dots$, there is a unique element of the rational group-ring (*cf.* §2.2)

$$\Theta^{\mathrm{Dir}}_{L/\mathbb{Q}}(1-r) \in e_r^{-1} \cdot \mathbb{Q}[G]$$

characterised by the relation

$$\chi(\Theta^{\mathrm{Dir}}_{L/\mathbb{Q}}(1-r)) = L(r, \chi^{-1}) \prod_{\substack{p \ \text{prime},\ p|f\\ (p,d)=1}} (1 - \chi(p)^{-1} p^{-r}).$$

Here $\chi\colon G \to \mathbb{C}^*$ is a character whose conductor, considered as a Dirichlet character, is equal to d, f is the conductor of the abelian field L and $L(s, \chi^{-1})$ denote the Dirichlet L-function of χ^{-1}.

We have $0 = L(r, \chi^{-1})$ precisely when $\chi(c_0) = (-1)^r$, c_0 being complex conjugation as in Question 5.2. In this case there is a zero of order one and the leading term is defined by the formula

$$L^*(r, \chi) = \frac{d}{dz} L(z, \chi)|_{z=r} \cdot \prod_{\substack{p \ \text{prime}\ p|ml,\\ (p, f(\chi))=1}} (1 - \chi(\sigma_p)^{-1} p^{-r}).$$

Now let $f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}$, $\mathfrak{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}}$ and $R(\chi, f_{r,\mathbb{Q}(\xi_{ml^{s+1}})})$ be as in §2, §4.1 and Definition 4.3. Imitating Definition 4.3, set

$$\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}}(\chi) = \frac{R(\chi, f_{r,\mathbf{Q}(\xi_{ml^{s+1}})})}{L^*(r, \chi)}$$

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and

$$\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} = \mathcal{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot \tau((\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}})^{-1}).$$

It is important to realise that, although in general the fractional ideal of Definition 4.3 was constructed under the assumption of the higher-dimensional Stark conjecture of §3.1, in the case of abelian extensions of the rationals the calculations of [3] (see also [6] and [24]) show that the higher-dimensional Stark conjecture is true (see the proofs of Theorem 7.6 and [49, Theorem 4.9]).

Theorem 6.1 *Let l be an odd prime. Then, in the situation and notation of* §6.1,

$$(\mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}) \cdot \mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})}) \cap \mathbb{Z}_l[G] \subseteq \mathrm{ann}_{\mathbb{Z}_l[G]}\big(H^2_{\text{ét}}\big(\mathrm{Spec}(\mathbb{Z}[\xi_{ml^{s+1}}][1/ml]); \mathbb{Z}_l(1-r)\big)\big)$$

where $\mu_{1-r} = \mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l$ *as in* §6.1.

Remark 6.2 It is possible to deduce from Theorem 6.1 the corresponding result in which $\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})}$ is replaced by $\mathcal{J}^r_{\mathbb{Q}(\xi_{ml^{s+1}})}$, in the spirit of Conjecture 5.1. The passage from Dirichlet L-functions to Artin L-functions uses the relation between $L_{\mathbb{Q},S_{ml}}(r,\chi^{-1})$ and $L(r,\chi^{-1})$ which is explained in §3.2.

Proof of Theorem 6.1 Arguing as in Example 4.4(ii) we have

$$\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} = \mathbb{Z}[1/2][G] \cdot e_r^- \cdot \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1-r) + \mathcal{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+ \cdot \tau(\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbf{Q}(\xi_{ml^{s+1}})}})^{-1}.$$

Furthermore it is shown in the proof of Theorem 7.6 that $f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}$ may be chosen so that

$$\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+ = e_r^+.$$

Making this choice ensures that

$$\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} = \mathbb{Z}[1/2][G] \cdot e_r^- \cdot \Theta^{\mathrm{Dir}}_{\mathbf{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1-r) + \mathcal{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+.$$

Since c_0 acts on μ_{1-r} like multiplication by $(-1)^{r+1}$, we have

$$\mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}) = \mathbb{Z}_l[G] \cdot e_r^+ + \mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}) \cdot e_r^-$$

and therefore

$$(\mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}) \cdot \mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})}) \cap \mathbb{Z}_l[G]$$

is generated by

$$e_r^- \cdot \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbf{Q}}(1-r)\, \mathrm{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r})) \cap \mathbb{Z}_l[G]$$

and

$$(\mathcal{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+) \cap \mathbb{Z}_l[G]$$

which both lie in $\mathrm{ann}_{\mathbb{Z}_l[G]}(H^2_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[\xi_{ml^{s+1}}][1/ml]); \mathbb{Z}_l(1-r)))$, by §7.2 and Theorem 7.6. ∎

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7 Annihilators and $K_0(\mathbb{Z}[G], \mathbb{Q}_l)$

The results of this section extend the results for the totally real subfield of a cyclotomic field, proved in [49], to the full cyclotomic field in order to establish the results which were required in the proof of Theorem 6.1.

7.1 Let l be a prime, G a finite group and let $f\colon \mathbb{Z}_l[G] \to \mathbb{Q}_l[G]$ denote the homomorphism of group-rings induced by the inclusion of the l-adic integers into the fraction field, the l-adic rationals. Write $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$ for the relative K-group of f, denoted by $K_0(\mathbb{Z}_l[G], f)$ in [51, p. 214] (see also [47, Definition 2.1.5]). By [51, Lemma 15.6] elements of $K_0(\mathbb{Z}_l[G], \mathbf{Q}_l)$ are represented by triples $[A, g, B]$, subject to relations described in [51], where A, B are finitely generated, projective $\mathbb{Z}_l[G]$-modules and g is a $\mathbb{Q}_l[G]$-module isomorphism of the form $g\colon A \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{\cong} B \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. This group fits into a localisation sequence of the form [41, §5 Theorem 5] (see also [20, p. 233])

$$K_1(\mathbb{Z}_l[G]) \xrightarrow{f_*} K_1(\mathbb{Q}_l[G]) \xrightarrow{\partial} K_0(\mathbb{Z}_l[G], \mathbb{Q}_l) \xrightarrow{\pi} K_0(\mathbb{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbb{Q}_l[G]).$$

Assume now that G is abelian. In this case $K_1(\mathbb{Q}_l[G]) \cong \mathbb{Q}_l[G]^*$ because $\mathbb{Q}_l[G]$ is a product of fields and $K_1(\mathbb{Z}_l[G]) \cong \mathbb{Z}_l[G]^*$ [14, I, p. 179, Theorem (46.24)]. Under these isomorphisms f_* is identified with the canonical inclusion.

The homomorphism, $K_0(\mathbb{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbb{Q}_l[G])$, is injective for all finite groups G [45, Theorem 34, p. 131], [14, II, p. 47, Theorem 39.10]. Thus the localisation sequence yields an isomorphism of the form

$$K_0(\mathbb{Z}_l[G], \mathbb{Q}_l) \cong \frac{\mathbb{Q}_l[G]^*}{\mathbb{Z}_l[G]^*}$$

when G is abelian. From the explicit description of ∂ [51, p. 216] this isomorphism sends the coset of $\alpha \in \mathbb{Q}_l[G]^*$ to $\left[\mathbb{Z}_l[G], (\alpha \cdot -), \mathbb{Z}_l[G]\right]$. The inverse isomorphism sends $[A, g, B]$, where A and B may be assumed to be free $\mathbb{Z}_l[G]$-modules, to the coset of $\det(g) \in \mathbb{Q}_l[G]^*$ with respect to any choice of $\mathbb{Z}_l[G]$-bases for A and B.

We shall be particularly interested in the following source of elements of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$. Let l be a prime and let G be a finite abelian group. Suppose that

$$0 \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

is a bounded complex of finitely generated, projective $\mathbb{Z}_l[G]$-modules (*i.e.,* a *perfect* complex of $\mathbf{Z}_l[G]$-modules), having all its homology groups finite. As usual, let $Z_t = \operatorname{Ker}(d_t\colon F_t \to F_{t-1})$ and $B_t = d_{t+1}(F_{t+1}) \subseteq F_t$ denote the $\mathbb{Z}_l[G]$-modules of t-dimensional cycles and boundaries, respectively. We have short exact sequences of the form

$$0 \to B_i \xrightarrow{\phi_i} Z_i \to H_i(F_*) \to 0$$

and

$$0 \to Z_{i+1} \xrightarrow{\psi_{i+1}} F_{i+1} \xrightarrow{d_{i+1}} B_i \to 0.$$

Applying $(- \otimes \mathbb{Q}_l)$ we obtain isomorphisms

$$\phi_i \colon B_i \otimes \mathbb{Q}_l \xrightarrow{\cong} Z_i \otimes \mathbb{Q}_l$$

and we may choose $\mathbb{Q}_l[G]$-module splittings of the form

$$\eta_i \colon B_i \otimes \mathbb{Q}_l \longrightarrow F_{i+1} \otimes \mathbb{Q}_l$$

such that $(d_{i+1} \otimes 1)\eta_i = 1 \colon B_i \otimes \mathbb{Q}_l \longrightarrow B_i \otimes \mathbb{Q}_l$. Then, using these splittings, we form a $\mathbb{Q}_l[G]$-module isomorphism of the form

$$X \colon \ \oplus_j F_{2j} \otimes \mathbb{Q}_l \xrightarrow{\cong} \oplus_j F_{2j+1} \otimes \mathbb{Q}_l.$$

The explicit formula for X on $(w_0, w_2, \dots) \in \oplus_j F_{2j}$ is given by

$$X(w_0, w_2, \dots) = \Big(\eta(w_0) + d_2(w_2), \eta_2\big(w_2 - \eta_1(d_2(w_2))\big) + d_4(w_4), \dots \\ \eta_{2t}\big(w_{2t} - \eta_{2t-1}(d_{2t}(w_{2t}))\big) + d_{2t+2}(w_{2t+2}), \dots\Big).$$

This construction defines a class, $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}]$, in $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$ which is well known to be independent of the choices of the splittings used to define X [51, Ch. 15] (see also [47, Propositions 2.5.35, 7.1.8]).

We shall denote by

$$\det(X) \in \frac{\mathbb{Q}_l[G]^*}{\mathbb{Z}_l[G]^*}$$

the element which corresponds to $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$ under the isomorphism mentioned above. We may modify the F_i to be free finitely generated $\mathbb{Z}_l[G]$-modules without changing the homology modules or the class in $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$. Then $\det(X)$ is explicitly represented by the determinant of the $\mathbb{Q}_l[G]$-isomorphism X with respect to any $\mathbb{Z}_l[G]$-basis for F_*.

Let us recall [35, Appendix] (see also [58]) the properties of the Fitting ideal (referred to as the initial Fitting invariant in [39]). Let R be a commutative ring with identity and let M be a finitely presented R-module. In our applications M will actually be finite. Suppose that M has a presentation of the form

$$R^a \xrightarrow{f} R^b \longrightarrow M \longrightarrow 0$$

with $a \geq b$. Then the Fitting ideal of the R-module M, denoted by $F_R(M)$, is the ideal of R generated by all $b \times b$ minors of any matrix representing f.

The Fitting ideal $F_R(M)$ is independent of the presentation chosen for M and is contained in the annihilator ideal of M, $F_R(M) \subseteq \mathrm{ann}_R(M)$. If M is generated by n elements then $\mathrm{ann}_R(M)^n \subseteq F_R(M)$ and if $\pi \colon M \longrightarrow M'$ is a surjection of finitely presented R-modules then $F_R(M) \subseteq F_R(M')$.

The following result yields relations between the annihilator ideals and Fitting ideals of the homology modules in previous example in the special case when each $H_i(F_*)$ is finite and zero except for $i = 0, 1$.

Theorem 7.1 ([49, Theorem 2.4]) *Let G be a finite abelian group and l a prime. Suppose that*

$$0 \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

is a bounded, perfect complex of $\mathbb{Z}_l[G]$-modules, as in Example 2.1*, having $H_i(F_*)$ finite for $i = 0, 1$ and zero otherwise. Let*

$$[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbb{Z}_l[G], \mathbb{Q}_l) \cong \frac{\mathbb{Q}_l[G]^*}{\mathbb{Z}_l[G]^*}$$

be as in §7.1*.*

(i) *If $t_i \in \mathrm{ann}_{\mathbb{Z}_l[G]}(H_i(F_*))$,*

$$\det(X)^{(-1)^i} t_i^{m_i} \in \mathrm{ann}_{\mathbb{Z}_l[G]}(H_{1-i}(F_*)) \lhd \mathbb{Z}_l[G]$$

for $i = 0, 1$. Here m_0, m_1 is the minimal number of generators required for the $\mathbb{Z}_l[G]$-module $H_0(F_*)$, $\mathrm{Hom}(H_1(F_*), \mathbb{Q}_l/\mathbb{Z}_l)$, respectively.

(ii) *If the Sylow l-subgroup of G is cyclic, then in* (i) *$\mathrm{ann}_{\mathbb{Z}_l[G]}(H_{1-i}(F_*))$ may be replaced by $F_{\mathbb{Z}_l[G]}(H_{1-i}(F_*))$.*

The following result gives a perfect complex to which we may apply Theorem 7.1.

Theorem 7.2 ([49, Theorem 4.3, Proposition 4.9]) *Let l be an odd prime, m a positive integer not divisible by l, $s = 0, 1, 2, \dots$ and $r = -1, -2, -3, \dots$. Let $X_{l,m,s}$ denote* $\mathrm{Spec}(\mathbf{Z}[\xi_{ml^{s+1}}][1/ml])$ *and $G = \mathrm{Gal}(\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q})$.*

(i) *There exists a bounded perfect cochain complex $P(r)^*$ of $\mathbb{Z}_l[G]$-modules such that $H^i(P(r)^*)$ is finite and is zero when $i \neq 1, 2$.*

(ii) *Furthermore, in the notation of Theorem* 7.1*, the determinant of this complex satisfies*

$$\det(X)^{-1} = e_r^+ - e_r^- \cdot \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1 - r) \in \frac{\mathbb{Q}_l[G]^*}{\mathbb{Z}_l[G]^*}.$$

(iii) *There is a $\mathbb{Z}_l[G]$-module isomorphism*

$$H^2(P(r)^*) \cong H^2_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1 - r)).$$

(iv) *There is a short exact sequence*

$$0 \to \mathbb{Z}_l[G]/(e_r^-) \xrightarrow{\phi} H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1 - r)) \to H^1(P(r)^*) \to 0.$$

Elements in the image of ϕ are usually called "cyclotomic elements".

Remark 7.3 Here is a sketch of how Theorem 7.2 is derived from [6]. This "descent" construction is quite delicate and is given in considerable detail in [49, Theorem 4.3, Proposition 4.9]. The point of this digressionary remark is to make clear the obstructions to passing from perfect complexes over the Iwasawa algebra to perfect complexes over $\mathbb{Z}[G]$ while still controlling their determinant and cohomology.

If $\xi_t = e^{2\pi i/t}$, we have a canonical projection of the form $\mathrm{Gal}(\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\xi_{ml^s})/\mathbb{Q})$ and, taking the inverse limit over the induced homomorphisms of l-adic group-rings, we define

$$\Lambda_m = \varprojlim_n \mathbb{Z}_l[\mathrm{Gal}(\mathbb{Q}(\xi_{ml^{n+1}})/\mathbb{Q})].$$

Write $Q(\Lambda_m)$ for the total quotient ring of Λ_m [18, p. 60]. In [6, §7 (48)] a bounded perfect complex of Λ_m-modules, denoted by $C_m(r)$, is constructed for which $C_m(r) \otimes_{\Lambda_m} Q(\Lambda_m)$ is acyclic [6, Lemma 7.2]. This complex defines a class in $K_0(\Lambda_m, Q(\Lambda_m))$. Furthermore, [6, Theorem 7.1] calculates the associated determinant, $\det(X)^{-1} \in Q(\Lambda_m)^*/\Lambda_m^*$. Unfortunately, there is no canonical homomorphism from $Q(\Lambda_m)^*/\Lambda_m^*$ to $\mathbb{Q}_l[G]^*/\mathbb{Z}_l[G]^*$, because non-divisors of zero in Λ_m may map to divisors of zero in $\mathbb{Q}_l[G]$.

Let $S_m \subset \Lambda_m$ denote the multiplicative subset of elements whose images in $\mathbb{Z}_l[\mathrm{Gal}(\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q})]$ are non-divisors of zero for all $s \geq 0$. In [49, §4.6–4.8] it is shown that $C_m(r) \otimes_{\Lambda_m} \Lambda_m S_m^{-1}$ is acyclic, which means that the associated determinant $\det(X)^{-1}$ actually lies in $(\Lambda_m S_m^{-1})^*/\Lambda_m^*$.

Furthermore, as explained in [49, Theorem 4.], one may pass to the quotient complex $P(r)^* = C_m(r) \otimes_{\Lambda_m} \mathbb{Z}_l[G]$, which is a perfect complex of $\mathbb{Z}_l[G]$-modules with finite cohomology groups, which are calculated in [49, Theorem 4.9]. The cohomology calculation is not absolutely straightforward because, of course, taking cohomology does not commute with descent. The determinant of $P(r)^*$ is computed by naturality from the calculations of [6].

Now we shall apply Theorem 7.1 to the complex $P(r)^*$ of Theorem 7.2.

Theorem 7.4 *Let l be an odd prime, m a positive integer prime to l, and $r = -1, -2, -3, -4, -5, \dots$ as in* §6.1. *Then, in the notation of* §7.1 *and* §7.2*:*

(i) *If $t_i \in \mathrm{ann}_{\mathbb{Z}_l[G]}(H^{2-i}(P(r)^*))$, the element*

$$\left(e_r^+ - \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1-r)e_r^-\right)^{(-1)^{i+1}} t_i^{m_i}$$

lies in $\mathrm{ann}_{\mathbb{Z}_l[G]}(H^{1+i}(P(r)^*))$ for $i = 0, 1$.
Here $m_1 = \max(m_{1,+}, m_{1,-})$ and $m_0, m_{1,\pm}$ is the minimal number of generators required for the $\mathbb{Z}_l[G]$-module

$$H^2(P(r)^*), \quad \mathrm{Hom}(H^1(P(r)^*)^{\pm}, \mathbb{Q}_l/\mathbb{Z}_l),$$

respectively, and $A^{\pm}$ is the ± 1-eigenspace of complex conjugation.

(ii) *If l does not divide $m-1$ then in* (i) *$\mathrm{ann}_{\mathbb{Z}_l[G]}(H^{1+i}(P(r)^*))$ may be replaced by the Fitting ideal $F_{\mathbb{Z}_l[G]}(H^{1+i}(P(r)^*))$.*

(iii) *Furthermore, $m_1 = 1$.*

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Proof Without changing $\det(X)$, we may replace $P(r)^*$ by an equivalent perfect complex of $\mathbb{Z}_l[G]$-modules of the form (see [49, §2.10])

$$0 \to \underline{P}(r)^0 \to \underline{P}(r)^1 \to \underline{P}(r)^2 \to 0.$$

Then we may apply Theorem 7.1 to each of the complexes $F_* = \underline{P}(r)^{2-*,\pm}$ to prove parts (i) and (ii) immediately.

It remains to prove part (iii). Since l is odd, it suffices to show that the Pontrjagin duals $\mathrm{Hom}(H^1(P(r)^*)^{\pm}, \mathbb{Q}_l/\mathbb{Z}_l)$ of each of the eigenspaces of complex conjugation are generated as a $\mathbb{Z}_l[G]$-module by one element.

From the long exact étale cohomology sequence associated to

$$\mathbb{Z}_l(1-r) \to \mathbb{Q}_l(1-r) \to (\mathbb{Q}_l/\mathbb{Z}_l)(1-r)$$

we see that

$$\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r)) \cong H^0_{\text{ét}}(X_{l,m,s}, (\mathbb{Q}_l/\mathbb{Z}_l)(1-r)) \cong \mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l$$

on which complex conjugation acts as multiplication by $(-1)^{1-r}$. Dividing out the short exact sequence of §7.2 by $\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))$ we obtain a short exact sequence of the form

$$0 \to \mathbb{Z}_l[G(\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q})]/(e_r^-) \xrightarrow{\phi} \frac{H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))}{\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbf{Z}_l(1-r))}$$
$$\to \frac{H^1(P(r)^*)}{\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))} \to 0.$$

The homomorphism induced by ϕ of §7.2 is denoted by $c_m(r)$ in [6]. Since complex conjugation acts like multiplication by $(-1)^r$ on the left-hand module it must also act by $(-1)^r$ on the central module, since ϕ induces a rational isomorphism. Hence if we set

$$\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s}) = \frac{H^1(P(r)^*)}{\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}; \mathbb{Z}_l(1-r))}$$

(the notation arises from the image of ϕ consisting of the so-called cyclotomic elements) then $\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s})$ is the $(-1)^r$ eigenspace of the finite group $H^1(P(r)^*)$ and the finite cyclic group $\mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l$ is the $(-1)^{1-r}$-eigenspace. Of course, the Pontrjagin dual of a finite cyclic group is again a cyclic group, so it remains to show that

$$\mathrm{Ext}^1_{\mathbb{Z}_l}(\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s}), \mathbb{Z}_l) \cong \mathrm{Hom}(\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s}), \mathbb{Q}_l/\mathbb{Z}_l)$$

is generated as a $\mathbb{Z}_l[G]$-module by one element.

We have a short exact sequence of $\mathbb{Z}_l[G]/(e_r^-)$-modules

$$0 \to \mathrm{Hom}_{\mathbb{Z}_l}\Big(\frac{H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))}{\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))}, \mathbb{Z}_l\Big)$$
$$\to \mathrm{Hom}_{\mathbb{Z}_l}(\mathbb{Z}_l[G]/(e_r^-), \mathbb{Z}_l) \to \mathrm{Ext}^1_{\mathbb{Z}_l}(\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s}), \mathbb{Z}_l) \to 0$$

in which the central $\mathbb{Z}_l[G]$-module is generated by one element, because $\mathbb{Z}_l[G]/(e_r^-)$ is self-dual. Hence the right-hand $\mathbb{Z}_l[G]$-module is also generated by one element, as required.

To see that $\mathbb{Z}_l[G]/(1 \pm c) \cong \mathbb{Z}_l[G](1 \mp c)$ is a self-dual $\mathbb{Z}_l[G]$-module when G is abelian, l is an odd prime and $c^2 = 1$, recall that $\mathbb{Z}_l[G]$ is self-dual as a module over itself by means of the bilinear form

$$\mathbb{Z}_l[G] \times \mathbb{Z}_l[G] \longrightarrow \mathbb{Z}_l$$

which sends (x, y) to the coefficient of the identity element in the product $x\tau(y) \in \mathbb{Z}_l[G]$. Here $\tau(\sum_{g\in G} n_g g) = \sum_{g\in G} n_g g^{-1}$. The submodules

$$\mathbb{Z}_l[G](1 + c) \quad \text{and} \quad \mathbb{Z}_l[G](1 - c)$$

are orthogonal complements with respect to this bilinear form, which easily implies the required self-duality. ■

Corollary 7.5 *In the situation and notation of Theorem* 7.4

$$\begin{aligned}\{t_0^{m_0} \mid t_0 \in \operatorname{ann}_{\mathbb{Z}_l[G]}(H^2(P^*(r)))\} &\subseteq z_{m,r} \cdot \operatorname{ann}_{\mathbb{Z}_l[G]}(H^1(P^*(r)))\\ &\subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}(H^2(P^*(r)))\end{aligned}$$

where, in $\mathbb{Q}_l[G]^*$,

$$z_{m,r} = e_r^+ - \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1 - r) \cdot e_r^- .$$

7.2 From the proof of Theorem 7.4

$$H^1(P^*(r)) = \mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s}) \oplus (\mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l)$$

where complex conjugation acts like multiplication by $(-1)^r$ on the first summand and by $(-1)^{r+1}$ on the second.

Now suppose that $t \in \operatorname{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l)$ then

$$t \cdot e_r^- \in \operatorname{ann}_{\mathbb{Z}_l[G]}(H^1(P^*(r)))$$

and, by Corollary 7.5,

$$t \cdot e_r^- \left(e_r^+ - \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbf{Q}}(1 - r)e_r^- \right) = -t \cdot e_r^- \cdot \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbf{Q}}(1 - r)$$

lies in $\operatorname{ann}_{\mathbb{Z}_l[G]}(H^2(P^*(r)))$. Therefore

$$\begin{aligned}\operatorname{ann}_{\mathbb{Z}_l[G]}(\mu_{1-r}(\mathbb{Q}(\xi_{ml^{s+1}})) \otimes \mathbb{Z}_l) \cdot \Theta^{\mathrm{Dir}}_{\mathbb{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1 - r) \cdot e_r^-&\\ \subseteq \operatorname{ann}_{\mathbb{Z}_l[G]}\left(H^2_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1 - r))\right).&\end{aligned}$$

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Similarly, if $t' \in \mathrm{ann}_{\mathbb{Z}_l[G]}(\mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s}))$ then

$$t' \cdot e_r^+ \in \mathrm{ann}_{\mathbb{Z}_l[G]}\left(H^1(P^*(r))\right)$$

and

$$t' \cdot e_r^+\left(e_r^+ - \Theta^{\mathrm{Dir}}_{\mathbf{Q}(\xi_{ml^{s+1}})/\mathbb{Q}}(1-r)e_r^-\right) = t' \cdot e_r^+$$

lies in $\mathrm{ann}_{\mathbb{Z}_l[G]}(H^2(P^*(r)))$. Therefore

$$\mathrm{ann}_{\mathbb{Z}_l[G]}(\mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s})) \cdot e_r^+ \subseteq \mathrm{ann}_{\mathbb{Z}_l[G]}\left(H^2_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))\right).$$

The following result explains the relation between the fractional ideal of Definition 4.3 and the annihilator ideal of $\mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s})$.

Theorem 7.6 *In the notation of* §4.2 *and Theorem* 7.4, *Corollary* 7.5 *and* §7.2

$$\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} \cap \mathbb{Z}_l[G] \cdot e_r^+ \subseteq \mathrm{ann}_{\mathbb{Z}_l[G]}(\mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s})).$$

Proof First we show that $f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}$ may be chosen so that

$$\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+ = e_r^+$$

in $(\mathbb{Q}[G]/(e_r^-)^*$.

It is known that $K_{1-2r}(X_l) \otimes \mathbb{Q}$ is the free $\mathbb{Q}[G]/(e_r^-)$-module on the Beilinson element, which is denoted by $(-r)!\,(ml^{s+1})^{-r} b_r(ml^{s+1})$ in the notation of [6]. Therefore we may choose f_r to satisfy

$$f_r((-r)!\,(ml^{s+1})^{-r} b_r(ml^{s+1})) = y_r \in Y_r(\mathbb{Q}(\xi_{ml^{s+1}}))^+ \otimes \mathbb{Q}$$

where, in the notation of [6, §8]), $Y_r(\mathbb{Q}(\xi_{ml^{s+1}}))^+ \otimes \mathbb{Z}[1/2]$ is a free $\mathbb{Z}[1/2][G]/(e_r^-)$-module of rank one with a generator y_r.

However, as recapitulated in [6, §4.2], Beilinson proved [38, Part I, Theorem 4.3(ii); Part II, Theorem 1.1] that

$$\begin{aligned} R^r_{\mathbb{Q}(\xi_{ml^{s+1}})^+}(f_r^{-1}(y_r)) &= R^r_{\mathbb{Q}(\xi_{ml^{s+1}})^+}((-r)!(ml^{s+1})^{-r} b_r(ml^{s+1})) \\ &= \sum_{\chi(c_0)=(-1)^r} L^*_{\mathbb{Q}}(r, \chi^{-1}) e_\chi y_r \end{aligned}$$

which establishes the claim concerning $\mathcal{R}^{\mathrm{Dir}}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+$.

For the remainder of the proof, assume that $f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}$ has been chosen in this manner. Hence

$$\begin{aligned} \mathcal{H}^1_{\mathrm{cyclo}}(X_{l,m,s}) &= \frac{H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))}{\mathrm{Tors}\, H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r)) + \mathrm{Im}(c_m(r))} \\ &\cong \frac{H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r)) \cdot e_r^+}{\mathrm{Im}(c_m(r))} \end{aligned}$$

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and, by Definition 4.3,

$$\mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} \cdot e_r^+ = \mathcal{I}_{f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}} \cdot e_r^+$$

is generated as a $\mathbb{Z}[1/2][G]$-module by the elements $\alpha \in \mathbb{Q}[G] \cdot e_r^+$ such that

$$\alpha \cdot f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}(K_{1-2r}(X_{l,m,s})) \subseteq Y_r(\mathbb{Q}(\xi_{ml^{s+1}}))^+.$$

Let $x \in H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))$ satisfy $c_0(x) = (-1)^r x$, every element in $\mathcal{H}^1_{\text{cyclo}}(X_{l,m,s})$ may be represented by such an x, and let

$$\alpha \in \mathcal{J}^r_{\mathrm{Dir},\mathbb{Q}(\xi_{ml^{s+1}})} \cap \mathbb{Z}_l[G] \cdot e_r^+.$$

We must show that $\alpha x \in \operatorname{Im}(\phi) \subseteq H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))$, where ϕ is the homomorphism of Theorem 7.2(iv). The image of the Chern class

$$c_{1-r,1} \colon K_{1-2r}(X_{l,m,s}) \to H^1_{\text{ét}}(X_{l,m,s}, \mathbb{Z}_l(1-r))$$

(denoted by $c^r_{\mathbb{Q}(\xi_{ml^{s+1}})}$ in [6, Lemma 8.16]) is dense. Choose $x_m \in K_{1-2r}(X_{l,m,s})$ so that $\varinjlim_m c_{1-r,1}(x_m) = x$. Then $f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}(\alpha x_m)$ lies in

$$Y_r(\mathbb{Q}(\xi_{ml^{s+1}}))^+ = f_{r,\mathbb{Q}(\xi_{ml^{s+1}})}(\mathbb{Z}[G]\langle(-r)!\,(ml^{s+1})^{-r} b_r(ml^{s+1})\rangle)$$

so that

$$\alpha x_m \in \mathbb{Z}[G]\langle(-r)!\,(ml^{s+1})^{-r} b_r(ml^{s+1})\rangle.$$

However, $c_{1-r,1}((-r)!(ml^{s+1})^{-r} b_r(ml^{s+1}))$ is equal to an element $\eta_m(r)$ which generates $\operatorname{Im}(\phi)$ [6, Lemma 8.16]; [24, proof of Theorem 6.4]. Hence $\varinjlim_m \alpha c_{1-r,1}(x_m) = \alpha x \in \operatorname{Im}(\phi)$, since ϕ is continuous in the l-adic topology. ■

8 Concluding Observations

8.1 Galois Descent

Whenever one predicts a new phenomenon concerning Galois actions on some number theoretic Mackey functor such as a cohomology group or an algebraic K-group, the question of Galois descent arises. For example, in the case of the Brumer conjecture, Hayes, Popescu and Sands have shown that the Stickelberger ideal is natural with respect to passing from E/F to a subextension E/L (for further details see [23, 43, 44]).

As explained in [47, Ch. VI–VII, p. 287] the method of proof of Theorem 6.1, given in §7, would predict nice functorial behaviour for Conjecture 5.1 under all types of passage to Galois subextensions. This is because the perfect complexes from which the annihilator relations were derived arose first in the construction of Chinburg-type invariants (see [47, Ch. III]) and are natural with respect to change of fields.

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I do not know very much about the functorial behaviour of the fractional ideal $\mathcal{J}_F^r$. Here is a modest example of the difficulties.

Consider the special case when $E = \mathbb{Q}(\xi_{ml^{s+1}})^+$, $F = \mathbb{Q}$ and $F_1 \subset E$ is quadratic over $\mathbb{Q}$. Let $G = \mathrm{Gal}(E/\mathbb{Q})$ and $H = \mathrm{Gal}(E/F_1)$. Suppose also that $r = -2, -4, -6, \ldots$ since otherwise the fractional ideal is just one of the higher Stickelberger ideals. In this case we may choose $f_{r,E}$ so that $\mathcal{R}_{f_{r,E}}(\chi) = 1$ for all characters χ of G (see the proof Theorem 7.6). Hence, using the same $f_{r,E}$ for E/F_1 we see that $\mathcal{R}_{f_{r,E}}$ is trivial for E/F_1 also.

There exists $g \in G$ such that $g^2 \in H$ and

$$Y_r(E)^+ = \mathbb{Z}[G]\langle\sigma_0\rangle \cong \mathbb{Z}[H]\langle\sigma_0\rangle \oplus \mathbb{Z}[H]\langle g\sigma_0\rangle.$$

Now suppose that $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{Q}[H]$ are such that for all $z \in K_{1-2r}(\mathcal{O}_E)$ with $f_{r,E}(z) = f_1(z)\sigma_0 \oplus f_2(z)g\sigma_0$ and $f_i(z) \in \mathbb{Z}[H]$ both $y_1(z) = a_{1,1}f_1(z) + a_{1,2}f_2(z)$ and $y_2(z) = a_{2,1}f_1(z) + a_{2,2}f_2(z)$ lie in $\mathbb{Z}[H]$. In other words, the $\mathbb{Q}[H]$-module endomorphism of $Y_r(E)^+ \otimes \mathbb{Q}$ with matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

satisfies the integrality condition of Definition 4.3 for the extension F/E_1. Therefore we have

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_1(z) = a_{2,2}y_1(z) - a_{1,2}y_2(z),$$
$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_2(z) = -a_{2,1}y_1(z) + a_{1,1}y_2(z).$$

In this totally real situation, Conjecture 5.1 for E/F_1 (see [49, Theorem 4.9]) predicts, in the notation of Theorem 6.1, that when $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})$ lies in $\mathbb{Z}_l[H]$, it annihilates $H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_E[1/ml]); \mathbb{Z}_l(1-r))$. If we could show for all z that the expressions for $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_1(z)$ and $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_2(z)$ both lie in $\mathbb{Z}[H]$ then $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_{r,E}(K_{1-2r}(\mathcal{O}_E)) \in Y_r(E)^+$ and then $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \in \mathbb{Q}[G]$ is one of the generators for $\mathcal{J}_E^r$ for $E/\mathbb{Q}$. Then, by Theorem 6.1, if $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \in \mathbb{Z}_l[G]$, it would annihilate $H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_E[1/ml]), \mathbb{Z}_l(1-r))$.

However, such integrality does not happen automatically. Here is a purely algebraic example.

Let $G = \mathbb{Z}/2 = \langle g\rangle$ and let $f_r \in \mathrm{Aut}_{\mathbb{Q}[G]}(\mathbb{Q}[G])$ be $f_r = ((a+bg)^{-1} \cdot -)$ where $a + bg \in \mathbb{Z}[G] \cap \mathbb{Q}[G]^*$. Hence we clearly have

$$\mathcal{I} = \{\alpha \in \mathbb{Q}[G] \mid \alpha f_r(\langle 2, 1-g\rangle) \subseteq \mathbb{Z}[G]\} = \langle a+bg, (a+bg)(1-g)/2\rangle.$$

Now let $\beta\colon \mathbb{Q}[G] \to \mathbb{Q}[G]$ be a homomorphism of $\mathbb{Z}$-modules satisfying $\beta f_r(\mathbb{Z}[G]) \subseteq \langle 2, 1-g\rangle$ and suppose that

$$\beta = \begin{pmatrix} u & w \\ v & z \end{pmatrix},$$

meaning $\beta(1) = u + vg$, $\beta(g) = w + zg$. Taking the values $a = 1, b = 2$ with $u = v = 9/2, w = 3/2 = -z$ one finds that β satisfies the integrality condition but $\det(\beta) = -27/2$ which does not lie in $\mathcal{I} = \langle 1+2g, (1+2g)(1-g)/2\rangle$ because $\mathbb{Z} \cap \mathcal{I} = 3\mathbb{Z}$.

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8.2 The Non-Abelian Case

Now suppose that we have a tower of number fields $F \subseteq K \subseteq L \subseteq E$ where E/F is Galois with not necessarily abelian Galois group, which we shall denote by $G(E/F)$. Suppose that L/K is Galois with $G(L/K)$ abelian. Hence we have a group extension of the form

$$\{1\} \to G(E/L) \to G(E/K) \xrightarrow{\pi} G(L/K) \to \{1\}.$$

Let $\hat{g} \in G(E/K)$ and $g = \pi(\hat{g}) \in G(L/K)$. If $\mathrm{Tr}_{E/L}$ denotes the transfer on algebraic K-theory we have a commutative diagram

$$\begin{array}{ccc} K_i(E) & \xrightarrow[\mathrm{Tr}_{E/L}]{} & K_i(L) \\ \downarrow \hat{g}_* & & \downarrow g_* \\ K_i(E) & \xrightarrow{\mathrm{Tr}_{E/L}} & K_i(L) \end{array}$$

and the composition

$$K_i(E) \xrightarrow{Tr_{E/L}} K_i(L) \longrightarrow K_i(E)$$

is equal to $\sum_{h \in G(E/L)} h_* \colon K_i(E) \to K_i(E)$. Since the algebraic K-groups of the S-integers $\mathcal{O}_{E,S}$ embed into those of E in dimensions greater than or equal to one we may replace $K_i(L)$ and $K_i(E)$ by $K_i(\mathcal{O}_{L,S'})$ and $K_i(\mathcal{O}_{E,S})$ in the above discussion, where S is the set of primes of E above those in S' and $i \geq 1$. Finally, let l be an odd prime and suppose that S' contains all the primes above l. Since the Chern class from $K_{e-2r}(\mathcal{O}_{E,S}) \otimes \mathbb{Z}_l$ to $H^e_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S}), \mathbb{Z}_l(1-r))$ is surjective for $r < 0$ and $e = 1, 2$ we may replace $K_i(L)$ and $K_i(E)$ by $H^e_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{L,S'}), \mathbb{Z}_l(1-r))$ and $H^e_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S}), \mathbb{Z}_l(1-r))$, respectively.

Now let us define a finitely generated $\mathbb{Z}[1/2][G(E/F)]$-submodule of $\mathbb{Q}[G(E/F)]$ by the following procedure. Suppose that the Gross conjecture holds for E/F, which as shown in Proposition 4.1 implies that the conjecture holds for all intermediate fields. This means that the fractional ideal $\mathcal{J}^r_L \subseteq \mathbb{Q}[G(L/K)]$ is defined for each intermediate extension L/K (as above $F \subseteq K \subseteq L \subseteq E$) with abelian Galois group. In fact, write $\mathcal{J}^r_{L/K}$ rather than $\mathcal{J}^r_L$. Now form the two-sided $\mathbb{Z}[1/2][G(E/F)]$-submodule of $\mathbb{Q}[G(E/F)]$ generated by

$$\Big(\sum_{h \in G(E/L)} h\Big) \pi^{-1}(\mathcal{J}^r_{L/K}) \subseteq \mathbb{Q}[G(E/K)] \subseteq \mathbb{Q}[G(E/F)].$$

We would like to say that

$$\mathbb{Z}_l[G(E/F)] \cap \Big(\mathcal{J}^r_{E/F} \cdot \mathrm{ann}_{\mathbb{Z}_l[G(E/F)]}\big(\mathrm{Tors}\, H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S}), \mathbb{Z}_l(1-r))\big)\Big) \\ \subseteq \mathrm{ann}_{\mathbb{Z}_l[G(E/F)]}\big(H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S}), \mathbb{Z}_l(1-r))\big)$$

if the same is true for all intermediate Galois extensions with abelian Galois group. Unfortunately at the moment we do not know the integrality of

$$\mathcal{J}^r_{E/F} \cdot \mathrm{ann}_{\mathbb{Z}_l[G(E/F)]}\big(\mathrm{Tors}\, H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S}), \mathbb{Z}_l(1-r))\big)$$

which makes such a result difficult to state. For the moment we shall content ourselves with the following result.

Theorem 8.1 *Let S denote a finite set of primes of F including all Archimedean primes and all which ramify in E/F and for each intermediate field L let S(L) denote those primes of L above the ones in S. Suppose also for each intermediate extension L/K, as in* §8.2, *with G(L/K) abelian that*

$$\mathbb{Z}_l[G(L/K)] \cap \left(\mathcal{J}^r_{L/K} \cdot \mathrm{ann}_{\mathbb{Z}_l[G(L/K)]}\left(\mathrm{Tors}\, H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{L,S(L)}), \mathbb{Z}_l(1-r)) \right) \right) \subseteq \mathrm{ann}_{\mathbb{Z}_l[G(L/K)]}\left(H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{L,S(L)}), \mathbb{Z}_l(1-r)) \right).$$

If $z_{L/K} \in \mathbb{Z}_l[G(E/K)]$ *satisfies*

$$\pi(z_{L/K}) \in \mathcal{J}^r_{L/K} \cdot \mathrm{ann}_{\mathbb{Z}_l[G(L/K)]}\left(\mathrm{Tors}\, H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{L,S(L)}), \mathbb{Z}_l(1-r)) \right)$$

then

$$\Big(\sum_{h \in G(E/L)} h \Big) \cdot z_{L/K} \in \mathrm{ann}_{\mathbb{Z}_l[G(E/F)]}\left(H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S(E)}), \mathbb{Z}_l(1-r)) \right).$$

Proof Let j denote the homomorphism on étale cohomology induced by the inclusion of L into E. Take $w \in H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{E,S(E)}), \mathbb{Z}_l(1-r))$ so that

$$\Big(\sum_{h \in G(E/L)} h \Big) \cdot z_{L/K} \cdot w = j(\mathrm{Tr}_{E/L}(z_{L/K} w)) = j(\pi(z_{L/K})\, \mathrm{Tr}_{E/L}(w)) = 0$$

by the induction hypothesis. ■

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