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ON SUPERNILPOTENT RADICALS WITH THE AMITSUR PROPERTY

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Abstract

A radical α has the Amitsur property if $\alpha(A[x]) = (\alpha(A[x]) \cap A)[x]$ for all rings A. For rings $R \subseteq S$ with the same unity, we call S a finite centralizing extension of R if there exist $b_1, b_2, \ldots, b_t \in S$ such that $S = b_1R + b_2R + \cdots + b_tR$ and $b_ir = rb_i$ for all $r \in R$ and $i = 1, 2, \ldots, t$. A radical α is FCE-friendly if $\alpha(S) \cap R \subseteq \alpha(R)$ for any finite centralizing extension S of a ring R. We show that if α is a supernilpotent radical whose semisimple class contains the ring $\mathbb{Z}$ of all integers and α is FCE-friendly, then α has the Amitsur property. In this way the Amitsur property of many well-known radicals such as the prime radical, the Jacobson radical, the Brown–McCoy radical, the antisimple radical and the Behrens radical can be established. Moreover, applying this condition, we will show that the upper radical $\mathcal{U}(*_k)$ generated by the essential cover $*_k$ of the class $*$ of all $*$-rings has the Amitsur property and $\mathcal{U}(*_k)(A[x]) = \mathcal{U}(*_k)(A)[x]$, where a semiprime ring R is called a $*$-ring if the factor ring R/I is prime radical for every nonzero ideal I of R. The importance of $*$-rings stems from the fact that a $*$-ring A is Jacobson semisimple if and only if A is a primitive ring.

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1. Introduction

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. The fundamental definitions and properties of radicals can be found in [2] and [7]. A class μ of rings is called hereditary if μ is closed under ideals. If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. For any class μ of rings an ideal I of a ring A is called an μ-ideal if the factor ring A/I is in μ. As usual, for a radical α, the α radical of a ring A is denoted by $\alpha(A)$ and the class of all α-semisimple rings is denoted by $\mathcal{S}(\alpha)$. π denotes the class of all prime rings and $\beta = \mathcal{U}(\pi)$ denotes the prime radical.

The notation $I \lhd A$ means that I is a two-sided ideal of a ring A. Let

$$\mathrm{Spec}(A) = \{B \lhd A \mid A/B \in \pi\}.$$

An ideal I of a ring A is called essential in A if $I \cap J \neq 0$ for any nonzero two-sided ideal J of A. A ring A is called an essential extension of a ring I if I is an essential ideal of A. A class μ of rings is called essentially closed if $\mu = \mu_k$, where $\mu_k = \{A \mid A$ is an essential extension of some $I \in \mu\}$ is the essential cover of μ. A hereditary and essentially closed class of prime rings is called a special class and the upper radical generated by a special class is called a special radical. A hereditary radical containing the prime radical β is called a supernilpotent radical. Given a ring A, the polynomial ring over A in a commuting indeterminate x is denoted by $A[x]$.

An important issue in Radical Theory is to describe the radical of a polynomial ring $A[x]$. We say that a radical α has the Amitsur property if $\alpha(A[x]) = (\alpha(A[x]) \cap A)[x]$ for every ring A. Although many classical radicals have this important property, in several cases, it is not so easy to determine whether a given radical enjoys the Amitsur property. So it is desirable to have useful criteria for testing the Amistur property of radicals. In this paper we give such a criterion for supernilpotent radicals whose semisimple class contains the ring $\boldsymbol{Z}$ of all integers. In this way the Amitsur property of many important special radicals such as the prime radical, the Jacobson radical, the Brown–McCoy radical, the antisimple radical radical and the Behrens radical can be established.

A semiprime ring R is called a $*$-ring [5] if $R/I \in \beta$ for any nonzero ideal I of R. The class of all $*$-rings is denoted by $*$. For a nonzero ring R, the intersection PH_R (respectively, SH_R) of all the nonzero prime (respectively, semiprime) ideals of R is called the prime (respectively, semiprime) heart of R [5]. In [3] the prime heart of a ring R was called the pseudo radical of R and it was denoted by $ps(R)$. It was shown in [5] that both, the nonzero semiprime heart of a semiprime ring and the nonzero prime heart of a prime ring are $*$ -rings. We also have the following theorem.

THEOREM 1 [5, Theorem 8]. *The following conditions are equivalent for a nonzero ring R.*

(1) $R \in *_k$.
(2) *R is a semiprime ring with nonzero semiprime heart.*
(3) *R is a prime ring with nonzero prime heart.*

The importance of the class $*_k$ is underlined by the two facts that follow.

THEOREM 2 [4, 8]. *If R is a nonzero $*$-ring, then the smallest special (respectively, supernilpotent) radical $\hat{l}_R$ (respectively, $\bar{l}_R$) containing R is an atom in the lattice of all special (respectively, supernilpotent) radicals.*

THEOREM 3 [5, Proposition 2]. *If $R \in *_k$ and μ is a special class of rings, then $R \in \mathcal{S}(\mathcal{U}(\mu))$ if and only if $R \in \mu$. Thus, in particular, a ring $R \in *_k$ is Jacobson semisimple if and only if R is primitive.*

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A long standing open question [6, Problem 1] asks whether $\beta = \mathcal{U}(*_k)$. In this context it is natural to ask whether $\mathcal{U}(*_k)$ behaves similarly to β when one takes polynomials. Using our criterion, we will answer this question in the positive by showing that $\mathcal{U}(*_k)$ has the Amitsur property and $\mathcal{U}(*_k)(A[x]) = \mathcal{U}(*_k)(A)[x]$.

2. Main results

For rings $R \subseteq S$ with the same unity, we say that S is a finite centralizing extension of R generated by $b_1, b_2, \ldots, b_t$ if there exist $b_1, b_2, \ldots, b_t \in S$ such that $S = b_1R + b_2R + \cdots + b_tR$ and $b_ir = rb_i$ for all $r \in R$ and $i = 1, 2, \ldots, t$. For example, the field $\mathbb{C}$ of all complex numbers is a centralizing extension of the field $\mathbb{R}$ of all real numbers generated by 1 and i.

It was proved in [11, Proposition 3.1] that if S is a finite centralizing extension of R and γ denotes the Behrens radical, that is, the upper radical generated by the class of all subdirectly irreducible rings each having a nonzero idempotent in its heart, then $\gamma(S) \cap R \subseteq \gamma(R)$. Moreover, it follows from [9, Corollary 10.2.10 and Corollary 10.4.15] that $\beta(S) \cap R \subseteq \beta(R)$ and $\mathcal{J}(S) \cap R \subseteq \mathcal{J}(R)$, where $\mathcal{J}$ denotes the Jacobson radical and S is a finite centralizing extension of R. Inspired by these results, we call a radical α FCE-friendly if $\alpha(S) \cap R \subseteq \alpha(R)$ for any finite centralizing extension S of R. Thus β and $\mathcal{J}$ are both FCE-friendly.

A radical α is strongly hereditary, if $A \in \alpha$ implies $L \in \alpha$ for every subring L of A.

It is a consequence of our next assertion (which is true for all rings, not necessarily with an identity element) that all strongly hereditary radicals are FCE-friendly. In particular, the prime radical, the locally nilpotent radical and the nil radical are all FCE-friendly. See also [1, Lemma 4 and the final paragraph].

LEMMA 4. *Let α be a strongly hereditary radical and let R be a subring of a ring S. Then $\alpha(S) \cap R \subseteq \alpha(R)$.*

PROOF. $\alpha(S) \cap R \subseteq \alpha(S) \in \alpha$, so $\alpha(S) \cap R \in \alpha$ since α is a strongly hereditary radical. But $\alpha(S) \cap R$ is an ideal of R. Thus $\alpha(S) \cap R \subseteq \alpha(R)$. □

Our next observation, contains a helpful criterion which can be used to identify special radicals which are FCE-friendly.

PROPOSITION 5. *Let μ be a special class of rings such that if S is a finite centralizing extension of R, then for every μ-ideal Q of R there exists a μ-ideal I of S such that $Q \supseteq I \cap R$. Then the special radical $\alpha = \mathcal{U}(\mu)$ is FCE-friendly.*

PROOF. Since α is a special radical, $\alpha(R)$ is the intersection of all ideals Q of R such that $R/Q \in \mu$. Then, by the assumption, for each such Q, there exists a μ-ideal I of S such that $Q \supseteq I \cap R$. Now, each such I contains $\alpha(S)$ and so $\alpha(R)$ contains $\alpha(S) \cap R$ which shows that α is FCE-friendly. □

To show some applications of Proposition 5, we need the following result which follows from [9, Theorem 10.2.4 and Theorem 10.2.9 and its proof].

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LEMMA 6. *Let S be a finite centralizing extension of R. If $I \in \mathrm{Spec}(S)$, then $I \cap R \in \mathrm{Spec}\ (R)$ and if $Q \in \mathrm{Spec}(R)$, then there exists $I \in \mathrm{Spec}(S)$ such that $Q = I \cap R$.*

COROLLARY 7. *Let μ denote any of the following special classes: the class of all subdirectly irreducible rings with idempotent hearts; the class of all simple rings with identity or the class $*_k$. Then the radical $\mathcal{U}(\mu)$ is FCE-friendly.*

PROOF. Let S be a finite centralizing extension of R.

First we show that the antisimple radical β_φ, that is, the upper radical generated by the class of all subdirectly irreducible rings with idempotent hearts is FCE-friendly. Let $Q \lhd R$ be such that R/Q is a subdirectly irreducible ring with an idempotent heart H/Q, where H is an ideal of R strictly containing Q. Then clearly $Q \in \mathrm{Spec}(R)$ and, it follows from Lemma 6, that there exists a prime ideal I of S such that $Q = I \cap R$. Then, by Zorn's lemma, there exists an ideal P of S that is maximal with respect to having $Q = P \cap R$. It is easy to check that this P is a prime ideal of S. Moreover, it was proved in [11, proof of Proposition 3.1] that S/P is a subdirectly irreducible ring with heart containing $(H + P)/P \simeq H/Q$ which is nonzero. Thus S/P is a subdirectly irreducible ring with an idempotent heart which shows that the class of all subdirectly irreducible rings with idempotent hearts satisfies the condition of Proposition 5. Consequently β_φ is FCE-friendly.

To show that the Brown–McCoy radical $\mathcal{G} = \mathcal{U}(\text{all simple rings with identity})$ is FCE-friendly, it suffices to show that the class of all simple rings with identity satisfies the condition of Proposition 5. Let $Q \lhd R$ be such that R/Q is a simple ring with identity. Then $Q \in \mathrm{Spec}(R)$ and, arguing as in the first part of the proof, we can find an ideal P of S that is maximal with respect to having $Q = P \cap R$ and $P \in \mathrm{Spec}(S)$. Then $Q \supseteq P \cap R$. We will show that S/P is a simple ring with identity. Let J/P be a nonzero ideal of S/P, where J is an ideal of S strictly containing P. Then, by the maximality of P, $J \cap R$ strictly contains Q, or equivalently, $(J \cap R)/Q$ is a nonzero ideal of the simple ring R/Q. Consequently $(J \cap R)/Q = R/Q$ which implies $R \subseteq J$ and, since R and S have the same identity and J is an ideal of S, it follows that $S = J$. Thus R/P is a simple ring. with identity which ends the proof.

We will now show that the class $*_k$ satisfies the condition of Proposition 5. Let $Q \lhd R$ be such that $R/Q \in *_k$. If $Q = R$, then $I = S$ satisfies the conditions of Proposition 5. So assume that $Q \subsetneq R$. This, in view of Theorem 1, means that R/Q is a prime ring with $0 \neq ps(R/Q) = H/Q$, where H is an ideal of R containing Q. Since such Q is a prime ideal of R, arguing as in the first part of the proof again, we can find an ideal P of S that is maximal with respect to having $Q = P \cap R$ and $P \in \mathrm{Spec}(S)$. We will show that $ps(S/P) \neq 0$. Let J/P be a nonzero prime ideal of S/P. Then J is a prime ideal of S strictly containing P and so, by the maximality of P, $J \cap R$ strictly contains Q. Then $J \cap R$ is a prime ideal of R and it follows that $(J \cap R)/Q$ is a nonzero prime ideal of R/Q. This implies that $H/Q \subseteq (J \cap R)/Q$, so J contains H and hence J/P contains $(H + P)/P$ which is nonzero. Thus $ps(S/P) \supseteq (H + P)/P \neq 0$. But, since S/P is a prime ring, this, in view of Theorem 1, shows that $S/P \in *_k$. Thus the class $*_k$ satisfies the condition of Proposition 5 which implies that the radical $\mathcal{U}(*_k)$ is FCE-friendly and ends the proof. $\square$

Our next result shows a connection between the FCE-friendliness and the Amitsur property of radicals and gives a useful criterion for identifying supernilpotent radicals with the Amitsur property. In what follows, the usual extension of a ring A obtained by adjoining unity is denoted by A^1.

THEOREM 8. *Let α be a supernilpotent radical. If α is an FCE-friendly and $\mathbb{Z} \in \mathcal{S}(\alpha)$, then α has the Amitsur property.*

PROOF. We adapt the proof of [11, Theorem 3.2]. First observe that, for every ring $T \neq T^1$, $T[x]$ is an ideal of $T^1[x]$ and so $\alpha(T[x]) \subseteq \alpha(T^1[x])$. On the other hand, $T^1[x]/T[x] \simeq \mathbb{Z}[x]$. Since $\mathbb{Z}$ is a prime ring with identity and the centre of $\mathbb{Z}$ is infinite, it follows from [2, Lemma 5, p. 243] that $\mathbb{Z}[x]$ is a subdirect sum of copies of $\mathbb{Z} \in \mathcal{S}(\alpha)$. Hence $\mathbb{Z}[x] \in \mathcal{S}(\alpha)$. Thus $\alpha(T[x]) = \alpha(T^1[x])$.

Now, since α is a supernilpotent radical, it follows from [1] that to complete the proof, it suffices to show that if A is a ring of prime characteristic p, then $\alpha(A[x]) \cap A[x^p - x] \subseteq \alpha(A[x^p - x])$. Now, if A has the identity 1, the result follows because α is FCE friendly and $S = A[x]$ is a finite centralizing extension of $R = A[x^p - x]$ generated by $b_1 = 1$, $b_2 = x, \ldots, b_p = x^{p-1}$. This is because

$$x^p = (x^p - x) + x \in 1R + xR + \cdots + x^{p-1}R,$$
$$x^{p+1} = x(x^p - x) + x^2 \in 1R + xR + \cdots + x^{p-1}R$$

and, it follows by the mathematical induction that $x^{p+n} \in 1R + xR + \cdots + x^{p-1}R$ for any natural number n. If A has no identity, then

$$\alpha(A[x]) \cap A[x^p - x] \subseteq \alpha(A^1[x]) \cap A^1[x^p - x] \subseteq \alpha(A^1[x^p - x]) = \alpha(A[x^p - x])$$

which ends the proof. $\square$

In a private communication, E. R. Puczylowski gave the following example which shows that the Amitsur property of a radical α does not imply that α is FCE-friendly.

EXAMPLE 9. Consider the special radical $\alpha = \mathcal{U}(\mathbb{R})$. According to [7, Theorem 4.9.22], in order to show that α has the Amitsur property, it suffices to show that $\alpha(A[x]) \cap A = 0$ implies $\alpha(A[x]) = 0$ for all rings A. So let A be a ring such that $\alpha(A[x]) \cap A = 0$ and suppose that $\alpha(A[x]) \neq 0$. Then

$$A \simeq A/(A \cap \alpha(A[x])) \simeq (A + \alpha(A[x]))/\alpha(A[x]) \subseteq A[x]/\alpha(A[x]).$$

But, as $A[x]/\alpha(A[x]) \in \mathcal{S}(\alpha)$, it follows that $A[x]/\alpha(A[x])$ is a subdirect sum of copies of $\mathbb{R}$ and hence $A[x]/\alpha(A[x])$ is torsion free. Then A is also torsion free and then it follows from [1] that $\alpha(A[x]) \neq 0$ implies $\alpha(A[x]) \cap A \neq 0$ which gives a contradiction. Thus α has the Amitsur property. However, α is not FCE-friendly because $\alpha(\mathbb{C}) \cap \mathbb{R} = \mathbb{C} \cap \mathbb{R} = \mathbb{R} \nsubseteq \{0\} = \alpha(\mathbb{R})$.

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COROLLARY 10. *If α is any of the following special radicals: β, $\mathcal{J}$, the Behrens radical γ, β_φ, $\mathcal{G}$ or $\mathcal{U}(*_k)$, then α has the Amitsur property.*

PROOF. Let α be any of the radicals listed in Corollary 10. Since any special radical is supernilpotent, each such α is supernilpotent and Corollary 7 and our previous remarks imply that each such α is FCE-friendly. Moreover, since $\mathbb{Z}$ is the subdirect sum of the finite fields $\mathbb{Z}_{p_i}$, where p_i ranges through the prime numbers and each $\mathbb{Z}_{p_i} \in \mathcal{S}(\alpha)$, it follows that $\mathbb{Z} \in \mathcal{S}(\alpha)$. Thus it follows from Theorem 8 that each such α has the Amitsur property. $\square$

Our remaining results show that $\mathcal{U}(*_k)$ behaves similarly to β when one takes polynomials.

THEOREM 11. *The radical $\mathcal{U}(*_k)$ is polynomially extensible, that is, if $A \in \mathcal{U}(*_k)$, then $A[x] \in \mathcal{U}(*_k)$ for any ring A.*

PROOF. Suppose $A[x] \notin \mathcal{U}(*_k)$. It suffices to show that $A \notin \mathcal{U}(*_k)$. Since $A[x] \notin \mathcal{U}(*_k)$, then $0 \neq A[x]/P \in *_k$ for some $P \lhd A[x]$. Then it follows from Theorem 1 that $A[x]/P \in \pi$ and $ps(A[x]/P) \neq 0$. But this together with [3, Lemma 4.2] implies that $ps(A/P \cap A) \neq 0$. But then, since $A/(P \cap A) \in \pi$, it follows from Theorem 1 that $0 \neq A/(P \cap A) \in *_k$. This shows that $A \notin \mathcal{U}(*_k)$ and ends the proof. $\square$

COROLLARY 12. *For any ring A, $\mathcal{U}(*_k)(A[x]) = \mathcal{U}(*_k)(A)[x]$.*

PROOF. Theorem 11 shows that $\mathcal{U}(*_k)$ is polynomially extensible. Since, being a special radical, $\mathcal{U}(*_k)$ is also hereditary, it follows from [7, Proposition 4.9.21] that $\mathcal{U}(*_k)(A) = \mathcal{U}(*_k)(A[x]) \cap A$. This and the fact that $\mathcal{U}(*_k)$ has the Amitsur property imply that $\mathcal{U}(*_k)(A[x]) = (\mathcal{U}(*_k)(A[x]) \cap A)[x] = \mathcal{U}(*_k)(A)[x]$ which ends the proof. $\square$

COROLLARY 13. *For any ring A, if $A \in *_k$, then $A[x] \in \mathcal{S}(\mathcal{U}(*_k))$.*

PROOF. Let $A \in *_k$. Then $\mathcal{U}(*_k)(A) = 0$. Then Corollary 12 implies that $\mathcal{U}(*_k)(A[x]) = 0$ which means that $A[x] \in \mathcal{S}(\mathcal{U}(*_k))$. $\square$

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References

[1] S. A. Amitsur, 'Radicals of polynomial rings', *Canad. J. Math.* **8** (1956), 355–361.
[2] V. A. Andrunakievich and Yu. M. Ryabukhin, *Radicals of Algebra and Structure Theory* (Nauka, Moscow, 1979), in Russian.
[3] M. Ferrero and R. Wisbauer, 'Unitary strongly prime rings and related radicals', *J. Pure Appl. Algebra* **181** (2003), 209–226.
[4] H. France-Jackson, 'On atoms of the lattice of supernilpotent radicals', *Quaest. Math.* **10** (1987), 251–256.

[5] ———, 'Rings related to special atoms', *Quaest. Math.* **24** (2001), 105–109.
[6] B. J. Gardner, 'Some recent results and open problems concerning special radicals', in: *Radical Theory, Proceedings of the 1988 Sendai Conference, Sendai, 24–30 July 1988* (ed. S. Kyono) (Uchida Rokakuho, Tokyo, 1989), pp. 25–56.
[7] B. J. Gardner and R. Wiegandt, *Radical Theory of Rings* (Marcel Dekker, New York, 2004).
[8] H. Korolczuk, 'A note on the lattice of special radicals', *Bull. Pol. Acad. Sci. Math.* **29** (1981), 103–104.
[9] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings* (John Wiley & Sons, Chichester, 1987).
[10] N. H. McCoy, *The Theory of Rings* (MacMillan, New York, 1964).
[11] P.-H. Lee and E. R. Puczylowski, 'On the Behrens radical of matrix rings and polynomial rings', *J. Pure Appl. Algebra* **212** (2008), 2163–2169.

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