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THE TYPE SET OF A TORSION-FREE ABELIAN GROUP OF RANK TWO

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Abstract

In this paper we generalize a recent result of Freedman (1973) concerning the cardinality of the type set of a rank two torsion-free abelian group. We show that if A is such a group and A supports a non-trivial associative ring then the type set of A contains at most three elements.

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Throughout, the groups that we consider are abelian groups and the rings are associative rings. A ring on a group A is a ring whose additive group is (isomorphic to) A. We write $(A, \cdot)$ for a ring on A and say that A supports $(A, \cdot)$. A group is called non-nil if it supports a non-trivial ring. The type set of the torsion-free group A is denoted by $\mathscr{T}(A)$, and the type of $a \in A$ by $t(a)$. For the subset S of the torsion-free group A, $\langle S \rangle_*$ denotes the unique minimal pure subgroup of A containing S.

THEOREM 1. (Freedman (1973)). *Let A be a torsion-free group of rank two. If A supports a ring with identity then $\mathscr{T}(A)$ contains at most three elements.*

A partial generalization is contained in

THEOREM 2. (Feigelstock (1976)). *Suppose A is a torsion-free group of rank two, all of whose non-zero elements have non-idempotent type. Then either A is nil or* $|\mathscr{T}(A)| = 2$.

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The major part of Freedman's proof consists of showing that for a torsion-free group A of rank two, $\mathscr{T}(A)$ contains at most two maximal elements. More generally we can prove

PROPOSITION 3. *Let A be a torsion-free group of rank n with the property that every pure subgroup of A of rank greater than one is non-nil. Then $\mathscr{T}(A)$ contains at most n maximal elements.*

PROOF. We use an induction argument. Clearly the proposition is true for a rational group; so assume that every non-nil group of rank k $(k<n)$ satisfying the conditions of the proposition has the property that its type set contains at most k maximal elements. Suppose A is as stated in the proposition, and let $a_1, a_2, \ldots, a_{n+1}$ be $n+1$ distinct elements of A such that $t(a_i) \neq t(a_j)$ for $i \neq j$, and $t(a_i)$ is maximal in $\mathscr{T}(A)$ for each $i = 1, 2, \ldots, n+1$.

First we show that any subset of n distinct elements from $\{a_1, a_2, \ldots, a_{n+1}\}$ is a maximal independent set of elements of A. Clearly this amounts to showing that $\{a_1, a_2, \ldots, a_n\}$ is an independent set of elements of A. If $\{a_1, a_2, \ldots, a_n\}$ is not independent then there exists a $k \leqslant n$ for which $\{a_1, a_2, \ldots, a_{k-1}\}$ is independent but $\{a_1, a_2, \ldots, a_k\}$ is not. If $A_1 = \langle \oplus_{i=1}^{k-1} \langle a_i \rangle \rangle_*$ then $a_k \in A_1$ and, since A_1 is pure in A, $\mathscr{T}(A_1) \subseteq \mathscr{T}(A)$. But then A_1 is a rank $(k-1)$ torsion-free group satisfying the conditions of the proposition for which $\mathscr{T}(A_1)$ contains the k maximal elements $t(a_1), t(a_2), \ldots, t(a_k)$. Consequently $\{a_1, a_2, \ldots, a_n\}$ is a maximal independent set of elements of A.

We can now choose a non-zero integer m, and integers $m_1, m_2, \ldots, m_n$ such that

$$m a_{n+1} = m_1 a_1 + m_2 a_2 + \ldots + m_n a_n.$$

If $i \in \{1, 2, \ldots, n\}$ then the set $\{a_1, a_2, \ldots, a_{n+1}\} \setminus \{a_i\}$ is independent and so $m_i \neq 0$.

Consider now any ring $(A, \cdot)$ on A. For distinct i and j in $\{1, 2, \ldots, n+1\}$, the maximality of $t(a_i)$ and $t(a_j)$ in $\mathscr{T}(A)$ shows $a_i \cdot a_j = 0$. In particular for any $i \in \{1, 2, \ldots, n\}$

$$0 = m(a_{n+1} \cdot a_i) = m_i a_i^2.$$

Thus $m_i \neq 0$ yields $a_i^2 = 0$. Hence $(A, \cdot)$ must be the trivial ring on A. Since A is non-nil it now follows that $\mathscr{T}(A)$ contains at most n maximal elements.

Following Beaumont and Wisner (1959) we make the following definitions for the torsion-free group A of rank two. If $a \neq 0$ is an element of A then let

$$Q'_a = \{\alpha \in Q \mid \alpha a \in A\},$$

where Q is the group of rational numbers. Now define the nucleus D of A to be the subgroup $D = \bigcap_{a \in A} Q'_a$ of Q.

With the aid of Beaumont and Wisner (1959) the major result of Freedman (1973) can now be generalized.

THEOREM 4. *Suppose A is a torsion-free group of rank two that supports a non-trivial ring $(A, \cdot)$. Then $\mathcal{T}(A)$ contains at most three elements.*

PROOF. We consider two cases separately.

Case (i). $(A, \cdot)$ is non-commutative. Theorem 2 of Beaumont and Wisner (1959) now gives the structure of $(A, \cdot)$; suppose $a_1 \cdot a_2 = \phi(a_1) a_2$ for all a_1, a_2 in A, where $0 \neq \phi \in \mathrm{Hom}(A, D)$. It is clear that $D = \langle p^{-\infty} | pA = A \rangle$ and also that $\mathrm{Im}\,\phi$ is a rank one torsion-free group with the same type as D. Thus $\mathrm{Im}\,\phi \cong D$. Hence there is a non-zero $\theta \in \mathrm{Hom}(A, D)$ such that θ maps A onto D. We can now define a non-commutative ring $(A, \times)$ on A by letting $a_1 \times a_2 = \theta(a_1) a_2$ for all a_1, a_2 in A. Since $1 \in D$ there is an element $a \in A$ for which $\theta(a) = 1$. But then the element, a, will be a left identity of $(A, \times)$ and so for every $a' \in A$, $t(a) \leqslant t(a')$. (Notice that if $(A, \cdot)$ has the alternate description in Theorem 2 of Beaumont and Wisner (1959) then we can argue as above to again obtain $t(a) \leqslant t(a')$.)

Case (ii). $(A, \cdot)$ is commutative. It is readily checked that $(A, \cdot)$ non-trivial and commutative implies the existence of an element $a \in A$ such that $a^2 \neq 0$. Thus Lemma 1 of Beaumont and Wisner (1959) shows that we can choose an element $a_1 \in A$ such that a_1 and a_1^2 are independent. If a_2 is a non-zero element of A then there are integers $m \neq 0$, m_1 and m_2 such that $ma_2 = m_1 a_1 + m_2 a_1^2$. Consequently,

$$t(a_1) = t(a_1) \cap t(a_1^2) \leqslant t(a_2).$$

In either case $\mathcal{T}(A)$ contains a smallest element. We now argue as in Freedman (1973). Since A has rank two, each chain in $\mathcal{T}(A)$ is of length at most two. Proposition 3 shows $\mathcal{T}(A)$ contains at most two maximal elements. Therefore $|\mathcal{T}(A)| \leqslant 3$.

A consequence of the proof of Case (i) above is the following observation.

PROPOSITION 5. *Suppose $(A, \cdot)$ is a non-commutative ring on a torsion-free group A of rank two. Then A is completely decomposable.*

PROOF. It is clear that D can be made into a rank one module over itself, that is D is a projective D-module. As in the proof of Theorem 4 there is a non-zero $\theta \in \mathrm{Hom}(A, D)$ such that θ maps A onto D. It is readily checked that A is a D-module and $\theta \in \mathrm{Hom}_D(A, D)$. Consequently, A will contain a D-direct summand isomorphic to D. Thus A is completely decomposable.

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References

R. A. Beaumont and R. J. Wisner (1959), 'Rings with additive group which is a torsion-free group of rank two', *Acta Sci. Math.* (*Szeged*) **20**, 105–116.

S. Feigelstock (1976), 'On the type set of groups and nilpotence', *Comment. Math. Univ. St. Paul* **25**, 159–165.

H. Freedman (1973), 'On the additive group of a torsion-free ring of rank two', *Publ. Math. Debrecen* **20**, 85–87.

L. Fuchs (1973), *Infinite abelian groups*, Vol. II (Academic Press, New York).

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