BULL. AUSTRAL. MATH. SOC. 47J20, 49J40
VOL. 68 (2003) [501–515]

SYSTEM OF GENERALISED SET-VALUED QUASI-VARIATIONAL-LIKE INEQUALITIES

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In this paper, we shall introduce a system of generalised set-valued quasi-variational-like inequalities, which generalises and unifies systems of generalised vector variational inequalities, systems of variational inequalities, generalised vector quasi-variational-like inequalities as well as various extensions of the classic variational inequalities in the literature. Some existence results for a solution of a system of generalised set-valued quasi-variational-like inequalities without any monotonity are obtained.

1. INTRODUCTION

The Vector Variational Inequality in a finite dimensional Euclidean space was introduced in [**24**] and applications were given. Chen and Cheng [**10**] studied the vector variational inequality in infinite dimensional space and applied it to vector optimisation problems. Since then, many authors [**9, 14, 11, 12, 37, 40, 41, 42, 44**] have intensively studied vector variational inequalities under different assumptions in infinite-dimensional spaces. Lee, Kim and Cho [**27**], Lee, Kim and Lee [**28**], Lin, Yang and Yao [**30**], Konnov and Yao [**26**], Daniilidis and Hadiisavvas [**16**], Yang and Yao [**43**], and Oettli and Schlager [**33**] studied the generalised vector variational inequality and obtained some existence results. Chen and Li [**13**] and Lee, Lee and Chang [**29**] introduced and studied generalised vector quasi-variational inequalities and established some existence theorems. Ansari [**2, 3**], Ding and Tarafdar [**21, 22**] and Luo [**31**] studied generalised vector variational-like inequalities. Ding [**19**] introduced and studied a class of generalised vector quasi-variational-like inequality problems. Pang [**34**], Cohen and Chaplais [**15**], Bianchi [**7**], and Ansari and Yao [**5**] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari, Schaible and Yao [**4**] considered a system of vector variational inequalities and obtained its existence results. Allevi,

Received 19th June, 2003
Supported by the National Natural Science Foundation of China and Applied Basic Research Foundation of Chongqing.

Gnudi and Konnov [1] considered a system of generalised vector variational inequalities and established some existence results with relative pseudomonoyonicity.

This paper introduces, a system of generalised set-valued quasi-variational-like inequalities, which generalises and unifies systems of generalised vector variational inequalities, systems of variational inequalities, generalised vector quasi-variational-like inequality as well as various extensions of the classic variational inequalities in the literature. Further some existence results of a solution for system of generalised set-valued quasi-variational-like inequalities without any monotonity are proved.

2. Problem statement and Preliminaries

Let intA denote the interior of a set A and I be an index set, for each $i \in I$, let Y_i be a Hausdorff topological vector space, E_i be a locally convex Hausdorff topological vector space. Consider a family of nonempty convex subsets $\{X_i\}_{i\in I}$ with $X_i \subset E_i$. Let $X = \prod_{i\in I} X_i$, and $E = \prod_{i\in I} E_i$. An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i, therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$. For each $i \in I$, let $\eta_i : X_i \times X_i \to E_i$ be a single-valued mapping and $C_i : X \to 2^{Y_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\mathrm{intC}_i(x) \neq \emptyset$ for each $x \in X$. Let $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{L(E_i,Y_i)}$ be two set-valued mappings, where $L(E_i, Y_i)$ denotes the space of all continuous linear operators from E_i into Y_i. Then, we consider a system of generalised set-valued quasi-variational-like inequalities, which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$, $\overline{x_i} \in D_i(\overline{x})$,

$$\forall y_i \in D_i(\overline{x}),\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, \eta_i(y_i, \overline{x_i})\rangle \notin -\mathrm{intC}_i(\overline{x}).$$

Then the point $\overline{x}$ is said to be a solution of the system of generalised set-valued quasi-variational-like inequalities.

It is easy to see that $\overline{x}$ is a solution of the system of generalised set-valued quasi-variational-like inequalities is equivalent to for each $i \in I$,

$$\overline{x_i} \in D_i(\overline{x}), \quad \forall y_i \in D_i(\overline{x}) : \ \langle T_i(\overline{x}), \eta_i(y_i, \overline{x_i})\rangle \not\subseteq -\mathrm{intC}_i(\overline{x}).$$

Where

$$\langle T_i(\overline{x}), \eta_i(y_i, \overline{x_i})\rangle = \bigcup_{v_i \in T_i(\overline{x})} \langle v_i, \eta_i(y_i, \overline{x_i})\rangle.$$

The following problems are the special cases of the system of generalised set-valued quasi-variational-like inequalities.

(i) For each $i \in I$, $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the system of generalised set-valued quasi-variational-like inequalities reduces to the system of generalised set-valued quasi-variational inequalities which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$, $\overline{x_i} \in D_i(\overline{x})$,

$$\forall y_i \in D_i(\overline{x}),\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, y_i - \overline{x_i}\rangle \notin -\mathrm{intC}_i(\overline{x}).$$

(ii) For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, then the system of generalised set-valued quasi-variational-like inequalities reduces to the system of generalised set-valued variational-like inequalities which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$,

$$\forall y_i \in X_i,\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, \eta_i(y_i, x_i) \rangle \notin -\operatorname{intC}_i(\overline{x}).$$

(iii) For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, and $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the system of generalised set-valued quasi-variational-like inequalities reduces to the system of generalised set-valued variational inequalities which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that

$$\forall y_i \in X_i,\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, y_i - \overline{x_i} \rangle \notin -\operatorname{intC}_i(\overline{x}).$$

It is worth noting that the system of generalised set-valued quasi-variational-like inequalities, the system of generalised set-valued quasi-variational inequalities, the system of generalised set-valued variational-like inequalities and the system of generalised set-valued variational inequalities are new models of mathematics.

For each $i \in I$, for all $x \in X$, if $Y_i \equiv Y$ and $C_i(x) \equiv C$, where C is a convex closed and pointed cone in Y with $\operatorname{intC} \neq \emptyset$, then the system of generalised set-valued variational inequalities reduces to a system of set-valued variational inequalities which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that

$$\forall y_i \in X_i,\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, y_i - \overline{x_i} \rangle \notin -\operatorname{intC}.$$

This was studied by Allevi, Gnudi and Konnov [**1**].

If T_i is single-valued function, then the system of set-valued variational inequalities reduces to the system of vector variational inequalities, which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that

$$\langle T_i(\overline{x}), y_i - \overline{x_i} \rangle \notin -\operatorname{intC}, \quad \forall y_i \in X_i.$$

This was considered by Ansari, Schaible and Yao [**4**].

(iv) For each $i \in I$, for all $x \in X \subseteq R^n$, let $Y_i \equiv R$ and $C_i(x) \equiv R^+ = \{r \in R : r \geqslant 0\}$, let T_i be replaced by $f_i : X \to R$, then the system of vector variational inequalities reduces to the system of scalar variational inequalities which is to find $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that

$$\langle f_i(\overline{x}), y_i - \overline{x_i} \rangle \geqslant 0, \quad \forall y_i \in X_i.$$

This was considered in [**34**] and [**15, 7, 5**].

(v) If $I = \{1\}$, then the system of generalised set-valued quasi-variational-like inequalities reduces to the generalised set-valued quasi-variational-like inequality as finding $\overline{x}$ in X such that $\overline{x} \in D(\overline{x})$, and

$$\forall y \in D(\overline{x}),\ \exists \widehat{v} \in T(\overline{x}) : \langle \widehat{v}, \eta(y, \overline{x}) \rangle \notin -\operatorname{intC}(\overline{x}).$$

This was introduced and studied by Ding [**19**] with C^+-η-monotone and weakly C^+-η-monotone condions.

For all $x \in X$, if $D(x) \equiv X$, then the generalised set-valued quasi-variational-like inequality reduces to the generalised set-valued variational-like inequality problem (in short, GSVLI) which is to find $\overline{x}$ in X such that

$$\forall y \in X, \;\; \exists \widehat{v} \in T(\overline{x}) : \big\langle \widehat{v}, \eta(y, \overline{x}) \big\rangle \notin -\operatorname{intC}(\overline{x}).$$

This was studied in [**2, 3, 21, 22, 31**].

If T is a single-valued mapping and $\eta(y, x) = y - g(x), \quad \forall x, y \in X$, and $D(x) \equiv X$ for all $x \in X$, where $g : X \to E$ is a single-valued mapping, then the generalised set-valued quasi-variational-like inequality reduces to find $\overline{x}$ in X such that

$$\big\langle T(\overline{x}), y - g(\overline{x}) \big\rangle \notin -\operatorname{intC}(\overline{x}), \quad \forall y \in X.$$

This was considered by Siddiqi, Ansari and Khaliq in [**37**].

If $\eta(y, x) = y - x$, for all $x, y \in X$, then the generalised set-valued quasi-variational-like inequality reduces to finding $\overline{x}$ in X such that $\overline{x} \in D(\overline{x})$, and

$$\forall y \in D(\overline{x}), \;\; \exists \widehat{v} \in T(\overline{x}) : \langle \widehat{v}, y - x \rangle \notin -\operatorname{intC}(\overline{x}).$$

This problem was called the generalised set-valued quasi-variational inequality problem, which is new. When $C(x) = C$, for all $x \in X$ is a constant cone, the generalised set-valued quasi-variational problems reduces to the set-valued quasi-variational inequality problem which was studied by Chen and Li [**13**] and Lee, Lee and Chang [**29**].

If $D(x) \equiv X$, for all $x \in X$ and $\eta(y, x) = y - x$, for all $x, y \in X$, then the generalised set-valued quasi-variational-like inequality reduces to find $\overline{x}$ in X such that

$$\forall y \in X, \;\; \exists \widehat{v} \in T(\overline{x}) : \big\langle \widehat{v}, \eta(y, \overline{x}) \big\rangle \notin -\operatorname{intC}(\overline{x}).$$

This Problem and its special cases are called the generalised vector variational inequality which was introduced and studied in [**27, 28, 30, 26, 16, 43, 33**].

If T is single-valued function and $D(x) \equiv X$, for all $x \in X$, then the generalised set-valued quasi-variational-like inequality reduces to find $\overline{x}$ in X such that

$$\big\langle T(\overline{x}), \eta(y, \overline{x}) \big\rangle \notin -\operatorname{intC}(\overline{x}), \quad \forall y \in X.$$

This problem and its special cases were studied by many authors, see [**27, 9, 14, 11, 12, 37, 40, 41, 42, 44**].

If $Y = R$ and $C(x) = [0, \infty)$, for all $x \in X$, then $L(E, Y) = E^*$, where E^* is the dual space of E, and the generalised set-valued quasi-variational-like inequality reduces to find $\overline{x}$ in X such that $\overline{x} \in D(\overline{x})$, and

$$\forall y \in D(\overline{x}), \;\; \exists \widehat{v} \in T(\overline{x}) : \big\langle \widehat{v}, \eta(y, \overline{x}) \big\rangle \geqslant 0.$$

This problem includes many classes of scalar type generalised quasi-variational inequality and generalised quasi-variational-like inequality problems as special cases, see [**36, 46, 8, 20, 17, 18, 45**].

In order to prove the main results, we need the following definitions and lemmas.

DEFINITION 2.1: For each $i \in I$, let E_i, Y_i be two real topological vector space, X_i be a nonempty and convex subset of E_i, $C_i : X \to 2^{Y_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone for each $x \in X$. Let $\eta_i : X_i \times X_i \to E_i$ be a single-valued mapping. $T_i : X \to 2^{L(E_i,Y_i)}$ is said to satisfy the generalised partial L-η_i-condition if and only if for any finite set $\{y_{i_1}, y_{i_2}, \dots, y_{i_n}\}$ in X_i, for all $\overline{x} = (\overline{x^i}, \overline{x_i})$ with $\overline{x_i} = \sum_{j=1}^{n} \alpha_j y_{i_j}$, where $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there exists $\overline{v_i} \in T_i(\overline{x})$ such that

$$\left\langle \overline{v_i}, \sum_{j=1}^{n} \alpha_j \eta_i(y_{i_j}, \overline{x_i}) \right\rangle \notin -\mathrm{intC}_i(\overline{x}).$$

REMARK 2.1. If $I = \{1\}$, then Definition 2.1 reduces to the generalised L-η-condition in [**21**].

REMARK 2.2. If $\eta_i(y_i, x_i)$ is affine in the first argument and for all $x = (x^i, x_i) \in X$, $\exists \overline{v_i} \in T_i(x)$, such that

$$\langle \overline{v_i}, \eta_i(x_i, x_i) \rangle \notin -\mathrm{intC}_i(x).$$

Then T_i satisfy the generalised partial L-η_i-condition.

REMARK 2.3. If $\eta_i(y_i, x_i) = y_i - x_i$, for all $x_i, y_i \in X_i$, then for any finite set $\{y_{i_1}, y_{i_2}, \dots, y_{i_n}\}$ in X_i, for all $\overline{x} = (\overline{x^i}, \overline{x_i})$ with $\overline{x_i} = \sum_{j=1}^{n} \alpha_j y_{i_j}$, where $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, we have that

$$\left\langle \overline{v}, \sum_{j=1}^{n} \alpha_j (y_{i_j} - \overline{x_i}) \right\rangle = \langle \overline{v}, \overline{x_i} - \overline{x_i}) \rangle = 0 \notin -\mathrm{intC}_i(\overline{x}), \quad \forall \overline{v} \in T_i(\overline{x}).$$

And hence T_i satisfy the generalised partial L-η_i-condition trivially.

DEFINITION 2.2: ([**6**].) Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then

(1) T is said to be upper semicontinuous if, for any $x_0 \in X$ and for each open set U in Y containing $T(x_0)$, there is a nerghborhood V of x_0 in X such that $T(x) \subseteq U$, for all $x \in V$.

(2) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

(3) T is said to be closed, if the set $\{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.1. ([**39**].) *Let X be a paracompact Hausdorff space and Y be a linear topological space. Suppose $T : X \to 2^Y$ is a set-valued mapping such that*

(i) *for each $x \in X$, $T(x)$ is nonempty,*

(ii) *for each $x \in X$, $T(x)$ is convex, and*

(iii) *T has open lower sections. Then there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$.*

Lemma 2.2. ([**6**].) *Let X and Y be topological spaces. If $T : X \to 2^Y$ is an upper semicontinuous set-valued mapping with closed values, then T is closed.*

Lemma 2.3. ([**38**].) *Let X and Y be topological spaces and $T : X \to 2^Y$ is an upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 2.4. ([**39**].) *Let X and Y be two topological spaces. Suppose $T : X \to 2^Y$ and $K : X \to 2^Y$ are set-valued mappings having open lower sections, then*

(i) *The set-valued mapping $F : X \to 2^Y$ defined by, for each $x \in X$, $F(x) = \mathrm{Co}\big(T(x)\big)$ has open lower sections.*

(ii) *The set-valued mapping $\theta : X \to 2^Y$ defined by, for each $x \in X$, $\theta(x) = T(x) \cap K(x)$ has open lower sections.*

Lemma 2.5. ([**23**].) *Let E be a locally convex topological linear space and X be a compact convex subset in E. Suppose $T : X \to 2^X$ is a set-valued mapping such that*

(i) *for each $x \in X$, $T(x)$ is nonempty,*

(ii) *for each $x \in X$, $T(x)$ is convex and closed,*

(iii) *T is upper semicontinuous.*

Then there exists a $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$.

Let Y be a real Hausdorff topological vector space and X be a nonempty convex subsets in a real locally convex Hausdorff topological vector space E. We denote by $L(E,Y)$ the space of all continuous linear operators from E into Y and by $\langle u, y\rangle$ the evaluation of $u \in L(E,Y)$ at $y \in E$. Let σ be the family of all bounded subsets of X whose union is total in E, that is, the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let β be a neighbourhood base of 0 in Y. When S runs through σ, V through β, the family

$$M(S,V) = \Big\{l \in L(E,Y) : \bigcup_{x \in S} \langle l, x\rangle \subset V\Big\}$$

is a neighbourhood base of 0 in $L(E,Y)$ at $x \in E$ (see [**35**, pp. 79–80]). By the Corollary of Schaefer [**35**, pp. 80], $L(E,Y)$ becomes a locally convex topological vector space under the σ-topology, where Y is assumed a locally convex topological space.

LEMMA 2.6. ([21, 19].) *Let E and Y be real Hausdorff topological vector spaces and $L(E,Y)$ be the topological vector space under the σ-topology. Then, the bilinear mapping*

$$\langle .,. \rangle : L(E,Y) \times E \to Y$$

is continuous on $L(E,Y)$, where $\langle l, x\rangle$ denotes the evaluation of the linear operator $l \in L(X,Y)$ at $x \in X$.

3. EXISTENCE RESULTS

In this section, we shall present some existence results for a solution to the system of generalised set-valued quasi-variational-like inequalities without any monotone conditions.

THEOREM 3.1. *Let I be an index set and I be countable. For each $i \in I$, let Y_i be a real Hausdorff topological vector space, X_i be a nonempty, compact, convex and metrisable set in a real locally convex Hausdorff topological vector space E_i, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, and let $L(E_i, Y_i)$ be equipped with the σ-topology. Suppose that*

(i) *for each $i \in I$, $C_i : X \to 2^{Y_i}$ is a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\mathrm{intC}_i(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M_i = Y_i \backslash (-\mathrm{intC}_i) : X \to 2^{Y_i}$ be upper semicontinuous;*

(ii) *for each $i \in I$, $T_i : X \to 2^{L(E_i,Y_i)}$ is an upper semicontinous set-valued mapping with nonempty compact values, $\eta_i : X_i \times X_i \longrightarrow E_i$ be continuous with respect to the second argument, such that T_i satisfies the generalised partial L-η_i-condition.*

Then, there exists $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$, $\overline{x_i} \in D_i(\overline{x})$ and for all $y_i \in D_i(\overline{x})$, $\exists \overline{v_i} \in T_i(\overline{x})$: $\langle \overline{v_i}, \eta_i(y_i, \overline{x_i})\rangle \notin -\mathrm{intC}_i(\overline{x})$. that is, the system of generalised set-valued quasi-variational-like inequalities has a solution $\overline{x} \in X$.

PROOF: For each $i \in I$, define a set-valued mapping $P_i : X \to 2^{X_i}$ by

$$\begin{aligned} P_i(x) &= \Big\{ y_i \in X_i : \big\langle T_i(x), \eta_i(y_i, x_i)\big\rangle \subseteq -\mathrm{intC}_i(x)\Big\} \\ &= \Big\{ y_i \in X_i : \big\langle v_i, \eta_i(y_i, x_i)\big\rangle \in -\mathrm{intC}_i(x), \quad \forall v_i \in T_i(x)\Big\}, \quad \forall x \in X. \end{aligned}$$

Thus, proving the theorem is equivalent to showing that there exists $\overline{x} \in X$ such that, for each $i \in I$, $\overline{x_i} \in D_i(\overline{x})$ and $D_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

We first prove that $x_i \notin \mathrm{Co}\big(P_i(x)\big)$ for all $x = (x^i, x_i) \in X$. To see this, suppose, by way of contradiction, that there exists some $i \in I$ and some point $\overline{x} = (\overline{x^i}, \overline{x_i}) \in X$ such that $\overline{x_i} \in \mathrm{Co}\big(P_i(\overline{x})\big)$. Then there exists a finite number of points $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ in X_i, and $\alpha_j \geqslant 0$ with $\sum\limits_{j=1}^{n} \alpha_j = 1$ such that $\overline{x} = \sum\limits_{j=1}^{n} \alpha_j y_{i_j}$ and $y_{i_j} \in P(\overline{x})$ for all $j = 1, 2, \ldots, n$.

https://doi.org/10.1017/S0004972700037904 Published online by Cambridge University Press

That is,

$$\langle v_i, \eta_i(y_{i_j}, \overline{x_i})\rangle \in -\operatorname{intC}_i(\overline{x}), \quad \forall v_i \in T_i(\overline{x})$$

and $j = 1, 2, \ldots, n$. Since $\operatorname{int}C_i(\overline{x})$ is convex, we obtain

$$\left\langle v_i, \sum_{j=1}^{n} \alpha_j \eta_i(y_{i_j}, \overline{x_i})\right\rangle \in -\operatorname{intC}_i(\overline{x}), \quad \forall v_i \in T_i(x),$$

which contradicts the fact that T_i satisfies the generalised partial L-η_i-condition. Therefore $x_i \notin \operatorname{Co}(P_i(x))$ for all $x \in X$.

Now we prove that the set

$$\begin{aligned} P_i^{-1}(y_i) &= \left\{x \in X : \langle T_i(x), \eta_i(y_i, x_i)\rangle \subseteq -\operatorname{intC}_i(x)\right\} \\ &= \left\{x \in X : \langle v_i, \eta_i(y_i, x_i)\rangle \in -\operatorname{intC}_i(x), \quad \forall v_i \in T_i(x)\right\} \end{aligned}$$

is open for each $i \in I$ and for each $y_i \in X_i$. That is, P_i has open lower sections in X. We only need to prove that

$$\begin{aligned} S_i(y_i) &= \left\{x \in X : \langle T_i(x), \eta_i(y_i, x_i)\rangle \not\subseteq -\operatorname{intC}_i(x)\right\} \\ &= X \setminus P_i^{-1}(y_i) = \left\{x \in X : \exists\, v_i \in T_i(x) \quad \text{such that} \quad \langle v_i, \eta_i(y_i, x_i)\rangle \notin -\operatorname{intC}_i(x)\right\}. \end{aligned}$$

is closed for all $y_i \in X_i$.

In fact, consider a net $x_t \in S_i(y_i)$ such that $x_t \to x \in X$. Since $x_t \in S_i(y_i)$, there exists $s_t \in T_i(x_t)$ such that

$$\langle s_t, \eta_i(y_i, x_{i_t})\rangle \notin -\operatorname{intC}_i(x_t).$$

From the upper semicontinuous and compact values of T_i and Lemma 2.3, it suffices to find a subset $\{s_{t_j}\}$ which converges to some $s \in T_i(x)$. By Lemma 2.6, we know that $\langle . \rangle$ is continuous, and hence

$$\langle s_{t_j}, \eta_i(y_i, x_{i_{t_j}})\rangle \to \langle s, \eta_i(y_i, x_i)\rangle.$$

By Lemma 2.2 and upper semicontinuity of M_i, we have $\langle s, \eta_i(y_i, x_i)\rangle \notin -\operatorname{intC}_i(x)$, and hence $x \in S_i(y_i)$, $S_i(y_i)$ is closed. For each $i \in I$, also define another set-valued mapping, $G_i : X \to 2^{X_i}$ by $G_i(x) = D_i(x) \cap \operatorname{Co}(P_i(x))$, for all $x \in X$. Let the set $W_i = \{x \in X : G_i(x) \neq \emptyset\}$. Since D_i and P_i has open lower sections in X, and by Lemma 2.4, we know that $\operatorname{Co}(P_i)$ and G_i also has open lower sections in X. Hence, $W_i = \bigcup_{y_i \in X_i} G_i^{-1}(y_i)$ is an open set in X. Then, the set-valued mapping $G_i|_{W_i}: W_i \to 2^{X_i}$ has open lower sections in W_i, and for all $x \in W_i$, $G_i(x)$ is nonempty and convex. Also, since X is a metrisable space [**25**, p. 50], W_i is paracompact [**32**, p .831]. Hence, by

Lemma 2.1, there is a continuous function $f_i : W_i \to X_i$ such that $f_i(x) \in G_i(x) \subset D_i(x)$ for all $x \in W_i$. Define $T_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} f_i(x) & \text{if } x \in W_i, \\ D_i(x) & \text{if } x \notin W_i. \end{cases}$$

Now, we prove that T_i is upper semicontinuous. In fact, for each open set V_i in X_i, the set

$$\begin{aligned} \{x \in X : T_i(x) \subset V_i\} &= \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X \setminus W_i : D_i(x) \subset V_i\} \\ &\subset \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : D_i(x) \subset V_i\}. \end{aligned}$$

On the other hand, when $x \in W_i$, and $f_i(x) \in V_i$, we have $T_i(x) = f_i(x) \in V_i$. when $x \in X$ and $D_i(x) \subset V_i$, since $f_i(x) \in D_i(x)$ if $x \in W_i$, we know that $T_i(x) \subset V_i$ and so

$$\{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : D_i(x) \subset V_i\} \subset \{x \in X : T_i(x) \subset V_i\}.$$

Therefore,

$$\{x \in X : T_i(x) \subset V_i\} = \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : D_i(x) \subset V_i\}$$

Since f_i is continuous and D_i is upper semicontinuous, the sets $\{x \in W_i : f_i(x) \in V_i\}$ and $\{x \in X : D_i(x) \subset V_i\}$ are open. It follows that $\{x \in X : T_i(x) \subset V_i\}$ is open and so the mapping $T_i : X \to 2^{X_i}$ is upper semicontinuous. Now define $T : X \to 2^X$ by $T(x) = \prod_{i \in I} T_i(x)$, for each $x \in X$. By Lemma 3 [**23**, p .124], T is upper semicontinuous. Since for each $x \in X$, $T(x)$ is convex, closed, and nonempty, by Lemma 2.5, there is $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$. Note that for each $i \in I$, $\overline{x} \notin W_i$. Otherwise, there is some $i \in I$ such that $\overline{x} \in W_i$. then $\overline{x_i} = f_i(\overline{x}) \in \mathrm{Co}(P_i(\overline{x}))$, which contradicts $x_i \in \mathrm{Co}(P_i(x))$ for all $x = (x^i, x_i) \in X$. Thus $\overline{x_i} \in D_i(\overline{x})$ and $G_i(\overline{x}) = \emptyset$ for each $i \in I$. That is, $\overline{x_i} \in D_i(\overline{x})$ and $D_i(\overline{x}) \cap \mathrm{Co}(P_i(\overline{x})) = \emptyset$ for each $i \in I$, which implies $\overline{x_i} \in D_i(\overline{x})$ and $D_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$ for each $i \in I$. Consequently, there exists $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$,

$$\overline{x_i} \in D_i(\overline{x}) \quad \text{and} \quad \forall y_i \in D_i(\overline{x}), \quad \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, \eta_i(y_i, \overline{x_i}) \rangle \notin -\mathrm{intC}_i(\overline{x}).$$

Hence, the solution set of system of generalised set-valued quasi-variational-like inequalities is nonempty. □

By Theorem 3.1 and Remark 2.2, we have

COROLLARY 3.2. *Let I be an index set and I be countable. For each $i \in I$, let Y_i be a real Hausdorff topological vector space, X_i be a nonempty, compact, convex and metrisable set in a real locally convex Hausdorff topological vector space E_i, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex*

https://doi.org/10.1017/S0004972700037904 Published online by Cambridge University Press

closed values and open lower sections, and $L(E_i, Y_i)$ be equipped with the σ-topology. Suppose that

- (i) *for each $i \in I$, $C_i : X \to 2^{Y_i}$ is a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with* $\mathrm{intC}_i(x) \neq \emptyset$ *for each $x \in X$, and the set-valued mapping $M_i = Y_i \backslash (-\mathrm{intC}_i) : X \to 2^{Y_i}$ be upper semicontinuous;*
- (ii) *for each $i \in I$, $T_i : X \to 2^{L(E_i, Y_i)}$ is an upper semicontinuous set-valued mapping with nonempty compact values, $\eta_i : X_i \times X_i \longrightarrow E_i$ be continuous with respect to the second argument and affine with respect to the first argument and for all $x = (x^i, x_i) \in X$, $\exists \overline{v_i} \in T_i(x)$, such that*

$$\langle \overline{v_i}, \eta_i(x_i, x_i) \rangle \notin -\mathrm{intC}_i(x).$$

Then, there exists $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$,

$$\overline{x_i} \in D_i(\overline{x}) \quad \text{and} \quad \forall y_i \in D_i(\overline{x}),\ \exists \overline{v_i} \in T_i(\overline{x}): \quad \langle \overline{v_i}, \eta_i(y_i, \overline{x_i}) \rangle \notin -\mathrm{intC}_i(\overline{x}).$$

that is, the system of generalised set-valued quasi-variational-like inequalities has a solution $\overline{x} \in X$.

If $\eta_i(y_i, x_i) = y_i - x_i$, for all $x_i, y_i \in X_i$, by Remark 2.3 and Theorem 3.1, it is easy to obtain the existence of a solution for the system of generalised set-valued quasi-variational inequalities as follows.

Corollary 3.3. *Let I be an index set and I be countable. For each $i \in I$, let Y_i be a real Hausdorff topological vector space, X_i be a nonempty, compact, convex and metrisable set in a real locally convex Hausdorff topological vector space E_i, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, and $L(E_i, Y_i)$ be equipped with the σ-topology. Suppose that*

- (i) *for each $i \in I$, $C_i : X \to 2^{Y_i}$ is a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with* $\mathrm{intC}_i(x) \neq \emptyset$ *for each $x \in X$, and the set-valued mapping $M_i = Y_i \backslash (-\mathrm{intC}_i) : X \to 2^{Y_i}$ be upper semicontinuous;*
- (ii) *for each $i \in I$, $T_i : X \to 2^{L(E_i, Y_i)}$ is an upper semicontinuous set-valued mapping with nonempty compact values.*

Then, there exists $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$,

$$\overline{x_i} \in D_i(\overline{x}) \quad \text{and} \quad \forall y_i \in D_i(\overline{x}),\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, y_i - \overline{x_i} \rangle \notin -\mathrm{intC}_i(\overline{x}).$$

that is, the system of generalised set-valued quasi-variational inequalities has a solution $\overline{x} \in X$.

Remark 3.1. If $D_i(x) = X_i$ and $C_i(x) = C$ for each $i \in I$ and for all $x \in X$, where C is a pointed convex cone with $\mathrm{intC} \neq \emptyset$, then by Corollary 3.3, we can obtain the existence of a solution for a system of set-valued variational inequalities which is different from those results in [**1**]. Moreover, let T_i be a single-valued mapping, then by Corollary 3.3, we can recover Theorem 3.1 in [**4**] with the additional condition of metrisablity of X_i. Hence, Theorem 3.1, Corollary 3.2 and Corollary 3.3 are generalisations of [**4**, Theorem 3.1].

THEOREM 3.4. *Let I be an index set and I be countable. For each $i \in I$, let Y_i be a real Hausdorff topological vector space, X_i be a nonempty, compact, convex and metrisable set in a real locally convex Hausdorff topological vector space E_i, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, and let $L(E_i, Y_i)$ be equipped with the σ-topology. Suppose that*

(i) *for each $i \in I$, $C_i : X \to 2^{Y_i}$ is a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\operatorname{intC}_i(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M_i = Y_i \backslash (-\operatorname{intC}_i) : X \to 2^{Y_i}$ be upper semicontinuous;*

(ii) *for each $i \in I$, $T_i : X \to 2^{L(E_i,Y_i)}$ is an upper semicontinuous set-valued mapping with nonempty compact values, $\eta_i : X_i \times X_i \longrightarrow E_i$ be continuous with respect to the second argument;*

(iii) *for each $i \in I$, there exists a mapping $h_i : X_i \times X_i \to Y_i$, such that:*

   (a) *For all $x = (x^i, x_i) \in X$, $\forall y_i \in X_i$, $\exists v_i \in T_i(x)$, such that*

$$h_i(x_i, y_i) - \langle v_i, \eta_i(y_i, x_i)\rangle \in -\operatorname{intC}_i(x);$$

   (b) *For any finite set $\{y_{i_1}, y_{i_2}, \dots, y_{i_n}\} \subseteq X_i$ and for all $x = (x^i, x_i) \in X$ with $x_i = \sum_{j=1}^{n} \alpha_j y_{i_j}$, where $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there is a $j \in \{1, 2, \dots, n\}$, such that $h_i(x_i, y_{i_j}) \notin -\operatorname{intC}_i(x)$.*

Then, there exists $\overline{x} = (\overline{x^i}, \overline{x_i})$ in X such that for each $i \in I$,

$$\overline{x_i} \in D_i(\overline{x}) \quad \text{and} \quad \forall y_i \in D_i(\overline{x}),\ \exists \overline{v_i} \in T_i(\overline{x}) : \langle \overline{v_i}, \eta_i(y_i, \overline{x_i})\rangle \notin -\operatorname{intC}_i(\overline{x}).$$

that is, the system of generalised set-valued quasi-variational-like inequalities has a solution $\overline{x} \in X$.

PROOF: For each $i \in I$, define two set-valued mappings $P_i : X \to 2^{X_i}$, $Q_i : X \to 2^{X_i}$ by

$$P_i(x) = \Big\{y_i \in X_i : \langle v_i, \eta_i(y_i, x_i)\rangle \in -\operatorname{intC}_i(x), \quad \forall v_i \in T_i(x)\Big\}, \quad \forall x \in X$$
$$Q_i(x) = \{y_i \in X_i : h_i(x_i, y_i) \in -\operatorname{intC}_i(x)\}, \quad \forall x \in X.$$

We first prove that $x_i \notin \operatorname{Co}(Q_i(x))$ for each $i \in I$ and for all $x = (x^i, x_i) \in X$. To see this, suppose, by way of contradiction, that there exists some point $\overline{x} = (\overline{x^i}, \overline{x_i}) \in X$

such that $\overline{x_i} \in \mathrm{Co}\big(Q_i(\overline{x})\big)$. Then there exists finite points $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ in X_i, and $\alpha_j \geqslant 0$ with $\sum_{j=1}^{n} \alpha_j = 1$ such that $\overline{x_i} = \sum_{j=1}^{n} \alpha_j y_{i_j}$ and $y_{i_j} \in Q_i(\overline{x})$ for all $j = 1, 2, \ldots, n$. That is, $h_i(\overline{x_i}, y_{i_j}) \in -\mathrm{intC}_i(\overline{x})$, $j = 1, 2, \ldots, n$. This contradicts the condition (b) of (iii). Therefore $x_i \notin \mathrm{Co}\big(Q_i(x)\big)$ for each $i \in I$ and for all $x = (x^i, x_i) \in X$. The condition (a) of (iii) implies that $Q_i(x) \supseteq P_i(x)$ for all $x \in X$. Hence, $x_i \notin \mathrm{Co}\big(P_i(x)\big)$, for all $x = (x^i, x_i) \in X$. The remainder of the proof is same as that in the proof of Theorem 3.1. ∎

COROLLARY 3.5. *Let I be an index set and I be countable. For each $i \in I$, let Y_i be a real Hausdorff topological vector space, X_i be a nonempty, compact, convex and metrisable set in a real locally convex Hausdorff topological vector space E_i, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, and let $L(E_i, Y_i)$ be equipped with the σ-topology. Suppose that*

(i) *for each $i \in I$, $C_i : X \to 2^{Y_i}$ is a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\mathrm{intC}_i(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M_i = Y_i \backslash (-\mathrm{intC}_i) : X \to 2^{Y_i}$ be upper semicontinuous;*

(ii) *for each $i \in I$, $T_i : X \to 2^{L(E_i, Y_i)}$ is an upper semicontinuous set-valued mapping with nonempty compact values, $\eta_i : X_i \times X_i \longrightarrow E_i$ be continuous with respect to the second argument;*

(iii) *for each $i \in I$, there exists a mapping $h_i : X_i \times X_i \to Y_i$, such that:*

  (a) *For all $x = (x^i, x_i) \in X$, $\forall y_i \in X_i$, $\exists v_i \in T_i(x)$, such that $h_i(x_i, y_i) - \big\langle v_i, \eta_i(y_i, x_i)\big\rangle \in -\mathrm{intC}_i(x)$;*

  (b) *For all $x = (x^i, x_i) \in X$, the set $\big\{y_i \in X_i : h_i(x_i, y_i) \in -\mathrm{intC}_i(x)\big\}$ is convex;*

  (c) *For all $x = (x^i, x_i) \in X$, $h_i(x_i, x_i) \notin -\mathrm{intC}_i(x)$.*

PROOF: It is only needed to show that (b) of (iii) in Theorem 3.4 holds. If the condition (b) of (iii) in Theorem 3.4 does not hold, then there exists a finite set $\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\} \subseteq X_i$ and some point $x = (x^i, x_i) \in X$ with $x_i = \sum_{j=1}^{n} \alpha_j y_{i_j}$, where $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, satisfying $h_i(x_i, y_{i_j}) \in -\mathrm{intC}_i(x)$ for all $j \in \{1, 2, \ldots, n\}$. that is, $y_{i_j} \in \big\{y_i \in X_i : h_i(x_i, y_i) \in -\mathrm{intC}_i(x)\big\}$ for all $j \in \{1, 2, \ldots, n\}$. By the convexity of the set $\big\{y_i \in X_i : h_i(x_i, y_i) \in -\mathrm{intC}_i(x)\big\}$, we have $x_i \in \big\{y_i \in X_i : h_i(x_i, y_i) \in -\mathrm{intC}_i(x)\big\}$. Hence, $h_i(x_i, x_i) \in -\mathrm{intC}_i(x)$, which contradicts to the condition (c) of (iii). Then, by Theorem 3.4, we know that the conclusion holds. ∎

REMARK 3.2. By the results in section 3, it is easy to obtain the existence results for all of the special models of the system of generalised set-valued quasi-variational-like inequalities mentioned in the section 2. For example, let $I = \{1\}$, by Theorem 3.1, Corollary 3.2, Theorem 3.4 and Corollary 3.5, respectively, we obtain the existence results of a solution for generalised set-valued quasi-variational-like inequalities which are generalisations of the main results in **[21]** and Theorem 1 in **[31]** from the cases of generalised set-valued variational-like inequalities to the cases of generalised set-valued quasi-variational-like inequalities.

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