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ON SOME CLASSES OF UNIVALENT POLYNOMIALS

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1. Introduction. It was in the year 1931 that Dieudonné [**4**] proved the following necessary and sufficient condition for a polynomial to be univalent in the unit disk.

THEOREM A (Dieudonné criterion). *The polynomial*

(1) $P_n(z) = z + a_2z^2 + \ldots + a_nz^n$

is univalent in $|z| < 1$ *if and only if for every* θ *in* $[0, \pi/2]$ *the associated polynomial*

$$(2) \qquad \phi(z, \theta) = 1 + \frac{\sin 2\theta}{\sin \theta} a_2 z + \ldots + \frac{\sin n\theta}{\sin \theta} a_n z^{n-1}$$

does not vanish in $|z| < 1$. *For* $\theta = 0$, $\phi(z, \theta)$ *is to be interpreted as* $P_n'(z)$.

Since then very little was done about univalent polynomials until Brannan ([**1**], also see [**2**]) in the year 1967 used the above criterion in conjunction with the well-known Cohn rule [**7**] to get some interesting results. Subsequently, Suffridge (see for example [**10**; **11**]) made notable contributions to the theory of univalent polynomials. Amongst other things Brannan proved the following

THEOREM B. *Suppose* $P_3(z) = z + a_2z^2 + tz^3$, *where* t *is real and positive. Then for* $0 \leqq t \leqq 1/5$, $P_3(z)$ *is univalent in* $|z| < 1$ *if and only if* a_2 *lies in the ellipse*

$$\mathscr{E}_{3,t} : \left\{ x + iy \in \mathbf{C} \,\middle|\, \left(\frac{x}{1+3t}\right)^2 + \left(\frac{y}{1-3t}\right)^2 \leqq \frac{1}{4} \right\},$$

whereas, for $1/5 \leqq t \leqq 1/3$, $P_3(z)$ *is univalent in* $|z| < 1$ *if and only if* a_2 *lies in the intersection*

$$\bigcap_{(1-2t)/t \leqq d \leqq 3} \mathscr{E}_{d,t}$$

of the family of ellipses

$$\mathscr{E}_{d,t} : \left\{ x + iy \in \mathbf{C} \,\middle|\, \left(\frac{x}{1+td}\right)^2 + \left(\frac{y}{1-td}\right)^2 \leqq \frac{1}{1+d} \right\}.$$

The preceding result was also obtained by Cowling and Royster [**3**] by a completely different method.

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Let us now introduce some notations. The family of all polynomials of the form

(3) $P(z) = z + \alpha_p z^p + \beta_p z^{2p-1}, \quad p(\text{integer}) \geqq 2, \alpha_p \in \mathbf{C}, \beta_p \in \mathbf{C}$

which are univalent in $|z| < 1$ will be denoted by $\mathscr{S}_p$. We will denote the sub-families of $\mathscr{S}_p$ consisting of polynomials of the form (3) which are starlike and convex by $\mathscr{S}_p{}^*$ and $\mathscr{S}_p{}^c$, respectively.

Ruscheweyh and Wirths [**8**] considered the problem of determining the coefficient regions

$$\mathscr{B}_p \;: \{(\alpha_p, \beta_p)|z + \alpha_p z^p + \beta_p z^{2p-1} \in \mathscr{S}_p\},$$
$$\mathscr{B}_p{}^* : \{(\alpha_p, \beta_p)|z + \alpha_p z^p + \beta_p z^{2p-1} \in \mathscr{S}_p{}^*\}.$$

In order to extend Theorem B to the case $p \geqq 3$, they used the Dieudonné criterion and the Cohn rule like Brannan but they had to restrict themselves to the sub-family S_p consisting of those polynomials in $\mathscr{S}_p$ whose coefficients are real. They proved:

THEOREM C. *Given β_p in $[-1/(2p-1), 1/(2p-1)]$ let $v(\beta_p)$ be defined by the requirement that*

$$z + \alpha_p z^p + \beta_p z^{2p-1} \in S_p$$

for $|\alpha_p| \leqq v(\beta_p)$. Then the function $v(\beta_p)$ increases monotonically for $\beta_p \in [-1/(2p-1), 1/(2p-1)]$;

$$v(\beta_p) = \frac{1 + (2p-1)\beta_p}{p}, \quad -\frac{1}{2p-1} \leqq \beta_p \leqq \frac{p+1}{(2p-1)(3p-1)};$$

$$v(\beta_p) < \frac{1 + (2p-1)\beta_p}{p}, \quad \frac{p+1}{(2p-1)(3p-1)} < \beta_p \leqq \frac{1}{2p-1};$$

$$v\left(\frac{1}{2p-1}\right) = \frac{2p}{2p-1} \sin \frac{\pi}{2p}.$$

They also proved:

THEOREM D. *If $S_p{}^*$ denotes the sub-family of $\mathscr{S}_p{}^*$ consisting of those polynomials in $\mathscr{S}_p{}^*$ whose coefficients are real, then $z + \alpha_p z^p + \beta_p z^{2p-1}$ belongs to $S_p{}^*$ if and only if*

$$(4) \qquad |\alpha_p| \leqq \begin{cases} \dfrac{1 + (2p-1)\beta_p}{p}, & -\dfrac{1}{2p-1} \leqq \beta_p \leqq \dfrac{p+1}{(2p-1)(3p-1)}, \\[2ex] 4\left[\dfrac{\{1 - (2p-1)\beta_p\}p\beta_p}{(p+1)^2 - (3p-1)^2\beta_p}\right]^{1/2}, & \dfrac{p+1}{(2p-1)(3p-1)} \leqq \beta_p \leqq \dfrac{1}{2p-1}. \end{cases}$$

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Here we will characterize the regions $\mathscr{B}_p$, $\mathscr{B}_p^*$ as well as the coefficient region

$$\mathscr{B}_p^c : \{(\alpha_p, \beta_p) | z + \alpha_p z^p + \beta_p z^{2p-1} \in \mathscr{S}_p^c\}.$$

For this we will use the Dieudonné criterion like Brannan [**1**; **2**], Ruscheweyh and Wirths [**8**], Michel [**7**], etc. but instead of the Cohn rule we will use an elementary fact presented in Lemma 1.

We will also determine the radius of convexity of S_p^* as well as the radii of convexity and starlikeness of the families S_2 and S_3. Here our main tool will be Lemma 3 which is a result of independent interest.

Besides, we will determine the so-called Koebe constants for several of the above mentioned families.

2. Statement of results. 2.1. For the kind of problems under consideration, there is clearly no loss of generality in supposing that in (3), β_p is real and positive. Further, for sake of simplicity we will write t instead of β_p.

The region $\mathscr{B}_p$ is given by the following.

THEOREM 1. *Suppose* $P(z) = z + \alpha_p z^p + tz^{2p-1}$ *where* t *is real and positive, and* $\alpha_p \in \mathbf{C}$. *If*

$$(5) \qquad A_p(u) = \frac{1 + t\, U_{2p-2}(u)}{U_{p-1}(u)}, \quad B_p(u) = \frac{1 - t\, U_{2p-2}(u)}{U_{p-1}(u)},$$

where $U_k(u)$ *is the Chebyshev polynomial of the second kind of degree* k, *then* $P(z) \in \mathscr{S}_p$ *if and only if* α_p *lies in the intersection* $D_{p,t} = \bigcap_u E_{p,u,t}$ *of the ellipses*

$$(6) \qquad E_{p,u,t} : \left\{x + iy \,\middle|\, \frac{x^2}{(A_p(u))^2} + \frac{y^2}{(B_p(u))^2} \leqq 1\right\}, \quad 0 \leqq u \leqq 1.$$

The case $p = 2$ of this theorem is equivalent to Theorem B of Brannan.

Now let $p = 3$. We have

$$A_3(u) = \frac{1 + t(16u^4 - 12u^2 + 1)}{4u^2 - 1}, \quad B_3(u) = \frac{1 - t(16u^4 - 12u^2 + 1)}{4u^2 - 1},$$

$$(0 \leqq u \leqq 1).$$

The minor axis $B_3(u)$ decreases for $0 \leqq u \leqq 1$ whereas

$$\frac{\partial}{\partial u} A_3(u) = 0 \quad \text{for } u = \frac{1}{2}\sqrt{1 + \left(\frac{1-t}{t}\right)^{1/2}}$$

which lies in the range $0 \leqq u \leqq 1$ only when $1/10 \leqq t \leqq 1/5$. For $u < (1/2)\sqrt{1 + ((1-t)/t)^{1/2}}$, $A_3(u)$ decreases and for $u > (1/2)\sqrt{1 + ((1-t)/t)^{1/2}}$ it $(A_3(u))$ increases. Hence the following analogue of Theorem B holds in the case $p = 3$.

THEOREM 1′. *Let* $P_5(z) = z + \alpha_3 z^3 + tz^5$ *where* t *is real and positive. Then for* $0 \leqq t \leqq 1/10$, $P_5(z)$ *is univalent in* $|z| < 1$ *if and only if* α_3 *lies in the ellipse*

$$E_{3,1,t} : \left\{ x + iy \in \mathbf{C} \,\middle|\, \left(\frac{x}{1+5t}\right)^2 + \left(\frac{y}{1-5t}\right)^2 \leqq \frac{1}{9} \right\}$$

whereas, for $1/10 \leqq t \leqq 1/5$, $P_5(z)$ *is univalent in* $|z| < 1$ *if and only if* α_3 *lies in the intersection*

$$\bigcap_{(1/2)\sqrt{1+((1-t)/t)^{1/2}} \leqq u \leqq 1} E_{3,u,t}$$

of the ellipses $E_{3,u,t}$ *defined in* (6).

From Theorem 1′ we readily obtain the following result of Ruscheweyh and Wirths [**8**, p. 350].

COROLLARY 1. *The polynomial* $P(z) = z + \alpha_3 z^3 + tz^5$ *where* α_3 *is real, is univalent in* $|z| < 1$ *if and only if*

$$(7) \qquad |\alpha_3| \leqq \begin{cases} \dfrac{1+5t}{7}, & 0 \leqq t \leqq 1/10 \\ 2\sqrt{t(1-t)} - t, & 1/10 \leqq t \leqq 1/5. \end{cases}$$

We believe that for all $p \geqq 2$ and $t \in [0, (p+1)/((2p-1)(3p-1))]$, $P(z) = z + \alpha_p z^p + tz^{2p-1} \in \mathscr{S}_p$ if and only if α_3 lies in the ellipse

$$E_{p,1,t} : \left\{ x + iy \in \mathbf{C} \,\middle|\, \frac{x^2}{\left(\dfrac{1+(2p-1)t}{p}\right)^2} + \frac{y^2}{\left(\dfrac{1-(2p-1)t}{p}\right)^2} \leqq 1 \right\}$$

but we are unable to prove it for $p > 3$.

The following theorem gives the region $\mathscr{B}_p^*$.

THEOREM 2. *The polynomial* $P(z) = z + \alpha_p z^p + tz^{2p-1}$, $p \geqq 2$ *belongs to the class* $\mathscr{S}_p^*$ *if and only if* α_p *lies in the region* $D_{p,t}^*$ *which is symmetrical with respect to the coordinate axes and the portion of* $\partial D_{p,t}^*$ *lying in the first quadrant has the parametric equation*

$$(8) \qquad \begin{cases} x(\varphi) = \dfrac{\begin{aligned}&\{(p+1)+(3p-1)t\}(1+t)\\ &\quad\times\{(p+1)+(3p-1)(2p-1)t\}\\ &\quad - 4pt\{(p+1)^2+(3p-1)^2t\}\cos^2\varphi\end{aligned}}{p[\{(p+1)+(3p-1)t\}^2 - 4(p+1)(3p-1)t\cos^2\varphi]}\cos\varphi \\[2ex] y(\varphi) = \dfrac{\begin{aligned}&\{(p+1)+(3p-1)t\}^2\{1-(2p-1)t\}\\ &\quad - 4pt\{(p+1)^2-(3p-1)^2t\}\cos^2\varphi\end{aligned}}{p[\{(p+1)+(3p-1)t\}^2 - 4(p+1)(3p-1)t\cos^2\varphi]}\sin\varphi \end{cases}$$

where φ *is to vary from* 0 *to* $\pi/2$ *or from*

$$\varphi_0 = \arccos \frac{\{1-(2p-1)t\}^{1/2}\{(p+1)+(3p-1)t\}}{[4pt\{(p+1)^2-(3p-1)^2t\}]^{1/2}} \quad \textit{to } \pi/2$$

according as

$$0 \leqq t \leqq \frac{p+1}{(3p-1)(2p-1)} \quad \text{or} \quad \frac{p+1}{(3p-1)(2p-1)} \leqq t \leqq \frac{1}{2p-1},$$

respectively.

The preceding result extends Theorem D to the case of complex coefficients.

Remark. It may be noted that if $t = 1/(2p-1)$ then $P(z) = z + \alpha_p z^p + tz^{2p-1}$ can belong to $\mathcal{S}_p$ only if α_p is real, whereas it belongs to $\mathcal{S}_p{}^*$ if and only if α_p is zero.

The next result follows from Theorem 2 on using the fact that $zP'(z) \in \mathcal{S}_p{}^*$ if and only if $P(z) \in \mathcal{S}_p{}^c$.

THEOREM 3. *The polynomial*

$$P(z) = z + \alpha_p z^p + tz^{2p-1}, \quad p \geqq 2$$

belongs to the class $\mathcal{S}_p{}^c$ *if and only if* α_p *lies in the region* $D_{p,t}{}^c$ *which is symmetrical with respect to the coordinate axes and the portion of* $\partial D_{p,t}{}^c$ *lying in the first quadrant has the parametric equation*

$$(9) \quad \begin{cases} x(\varphi) = \dfrac{\begin{array}{l}\{(p+1)+(2p-1)(3p-1)t\}\{1+(2p-1)t\} \\ \quad \times \{(p+1)+(2p-1)^2(3p-1)t\} \\ \quad - 4p(2p-1)t\{(p+1)^2+(3p-1)^2(2p-1)t\}\cos^2\varphi\end{array}}{\begin{array}{l}p^2[\{(p+1)+(3p-1)(2p-1)t\}^2 \\ \qquad - 4(p+1)(3p-1)(2p-1)t\cos^2\varphi]\end{array}} \cos\varphi \\[2ex] y(\varphi) = \dfrac{\begin{array}{l}\{(p+1)+(2p-1)(3p-1)t\}^2\{1-(2p-1)^2t\} \\ \quad - 4p(2p-1)t\{(p+1)^2-(3p-1)^2(2p-1)t\}\cos^2\varphi\end{array}}{\begin{array}{l}p^2[\{(p+1)+(3p-1)(2p-1)t\}^2 \\ \qquad - 4(p+1)(3p-1)(2p-1)t\cos^2\varphi]\end{array}} \sin\varphi \end{cases}$$

where φ *is to vary from* 0 *to* $\pi/2$ *or from*

$$\varphi_1 = \operatorname{arc\,cos} \frac{\{1-(2p-1)^2t\}^{1/2}\{(p+1)+(3p-1)(2p-1)t\}}{[4p(2p-1)t\{(p+1)^2-(3p-1)^2(2p-1)t\}]^{1/2}}$$

to $\pi/2$ *according as*

$$0 \leqq t \leqq \frac{p+1}{(2p-1)^2(3p-1)} \quad \text{or} \quad \frac{p+1}{(2p-1)^2(3p-1)} \leqq t \leqq \frac{1}{(2p-1)^2},$$

respectively.

As a special case of the preceding result, we have

COROLLARY 2. *If* $S_p{}^c$ *denotes the sub-family of* $\mathcal{S}_p{}^c$ *consisting of those polynomials in* $\mathcal{S}_p{}^c$ *whose coefficients are real, then* $z + \alpha_p z^p + tz^{2p-1}$ *belongs to* $S_p{}^c$

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if and only if

$$(10)\quad |\alpha_p| \leqq \begin{cases} \dfrac{1+(2p-1)^2 t}{p^2}, & 0 \leqq t \leqq \dfrac{p+1}{(2p-1)^2(3p-1)}, \\ \dfrac{4}{p}\left[\dfrac{p(2p-1)t\{1-(2p-1)^2 t\}}{(p+1)^2-(3p-1)^2(2p-1)t}\right]^{1/2}, & \dfrac{p+1}{(2p-1)^2(3p-1)} \leqq t \leqq \dfrac{1}{(2p-1)^2}. \end{cases}$$

No doubt, Corollary 2 can be deduced from Theorem D as well.

2.2. Once the regions $D_{p,t}$, $D_{p,t}^*$, $D_{p,t}^c$ have been characterized, we may argue as follows in order to determine the radius of starlikeness r_p^* of the family $\mathscr{S}_p$ and the radii of convexity r_p^c, $r_{p,*}^c$ of the families $\mathscr{S}_p$, $\mathscr{S}_p^*$. If $P(z) = z + \alpha_p z^p + tz^{2p-1}$ belongs to $\mathscr{S}_p$ then $(1/\rho)P(\rho z) = z + \rho^{p-1}\alpha_p z^p + \rho^{2p-2}tz^{2p-1}$ is starlike in $|z| < 1$ for $0 < \rho \leqq r_p^*$ and so

$$(11)\quad \rho^{p-1}\alpha_p \in D_{p,\rho^{2p-2}t}^*.$$

The largest value of ρ for which (11) holds for all $t \in [0, 1/(2p-1)]$ is r_p^*. We may argue the same way for r_p^c, $r_{p,*}^c$. Since the regions $D_{p,t}$, $D_{p,t}^*$, $D_{p,t}^c$ are very complicated it is not really easy to carry out the details. We therefore restrict ourselves to the case of real coefficients.

For a fixed t in $[0, 1/(2p-1)]$ let $S_{p,t}^*$ denote the class of all polynomials of the form $z + \alpha_p z^p + tz^{2p-1}$, $\alpha_p \in \mathbf{R}$ which are starlike and univalent in $|z| < 1$. In order to determine the radius of convexity $\rho_{p,*}^c$ of the family S_p^* we prove:

THEOREM 4. *Let*

$$(12)\quad \begin{cases} t_0 = -\dfrac{1}{2p-1}, \quad t_1 = \dfrac{p+1}{(2p-1)(3p-1)}, \\ t_2 = \dfrac{(p+1)(6p^5-11p^4+5p^2-6p+2)}{(2p-1)(3p-1)(2p^5+3p^4-18p^3+17p^2-10p+2)}, \\ t_3 = \dfrac{2p^4-p^3-4p^2-3p+2}{(2p-1)(2p^4+3p^3-18p^2+11p-2)}, \quad t_4 = \dfrac{1}{2p-1}, \end{cases}$$

$$(13)\quad A(t) = \left[\frac{pt\{1-(2p-1)t\}}{(p+1)^2-(3p-1)^2 t}\right]^{1/2}.$$

Then the radius of convexity ρ_t of the class $S_{p,t}^*$ is given by the formula

$$(14)\quad \rho_t = \omega_i(t) \quad \textit{for } t \in [t_{i-1}, t_i],\ i = 1, 2, 3, 4,$$

where

$$\omega_1(t) = \left[\frac{p\{1+(2p-1)t\} + \sqrt{p^2\{1+(2p-1)t\}^2 - 4(2p-1)^2 t}}{2}\right]^{-1/(p-1)}$$

$$\omega_2(t) = \{2p^2 A(t) + \sqrt{4p^4 A^2(t) - (2p-1)^2 t}\}^{-1/(p-1)}$$

$$\omega_3(t) = \left[\frac{(p-1)^2\{(2p-1)(p^2+4p-1)t - (p+1)^2\}}{(2p-1)t\{(2p-1)(3p-1)^2 t - (5p^2-1)\}}\right]^{1/2(p-1)}$$

$$\omega_4(t) = \{(2p-1)^2 t\}^{-1/2(p-1)}.$$

The extremal polynomials have the form

$$P(z) = z \pm \alpha_p z^p + t z^{2p-1},$$

where

$$\alpha_p = \begin{cases} \dfrac{1+(2p-1)t}{p} & \text{if } t \in [t_0, t_1] \\ 4\left[\dfrac{p\{1-(2p-1)t\}}{(p+1)^2 - (3p-1)^2 t}\right]^{1/2} & \text{if } t \in [t_1, t_4]. \end{cases}$$

COROLLARY 3. *In the notations of Theorem 4, every polynomial $P(z) \in S_p^*$ is convex in the disk $|z| < \omega_2(\rho^*)$, where ρ^* is the unique root of the equation*

$$(15) \quad A(t)(p-1)^2\{(p+1)^2(4p^3+4p^2-1) - 2(p+1)^2(2p-1) \times (4p^3+8p^2-6p+1)t + (2p-1)(3p-1)^2 \times (4p^3+8p^2-6p+1)t^2\} + 2p^3\{(2p-1)(3p-1)^2 t^2 - 2(2p-1)(p+1)^2 t + (p+1)^2\}\sqrt{4p^4A^2(t) - (2p-1)^2 t} = 0$$

lying in the interval

$$\left(\frac{p+1}{(2p-1)(3p-1)}, \frac{(2p-1)(p+1)^2 - (p+1)(p-1)\sqrt{2(2p-1)(p-1)}}{(2p-1)(3p-1)^2}\right)$$

Remark. Since the polynomials

$$P(z) = z \pm 4A(\rho^*)z^p + \rho^* z^{2p-1} \quad \in S_p^*$$

are convex in $|z| < \omega_2(\rho^*)$ and in no larger disk, $\omega_2(\rho^*)$ is, in fact, the radius of convexity of the family S_p^*.

By a reasoning different from the one explained at the beginning of this section we will determine the radii of convexity R_2^c, R_3^c and the radii of starlikeness R_2^*, R_3^* of the families S_2, S_3 respectively.

THEOREM 5. $R_2^c = 1/\sqrt{7}$, $R_3^c = \{(9+\sqrt{305})/112\}^{1/2}$.

THEOREM 6. $R_2^* = 3/\sqrt{11}$, $R_3^* = (10/13)^{1/4}$.

2.3. As usual, we define the Koebe constant $K(\mathscr{F})$ of a family $\mathscr{F}$ of normalized functions $z + a_2z^2 + \ldots$ holomorphic in the unit disk ($|z| < 1$) as the radius of the largest disk centred at the origin which is always contained in the image of the unit disk by an arbitrary function belonging to the family $\mathscr{F}$. We prove

THEOREM 7. *Let S_p, S_p^*, S_p^c be as above. Then*

$$K(S_2) = \frac{3-\sqrt{5}}{2}, \quad K(S_3) = 2-\sqrt{2},$$

$$K(S_p^*) = \frac{6(p-1)^2}{(2p-1)(3p-1)},$$

$$K(S_p^c) = \frac{(p^2-1)(12p^2-16p+3)}{p(2p-1)^2(3p-1)},$$

and the extremal functions have the form

$$z \pm \frac{2}{5}\sqrt{5}\, z^2 + \frac{5-\sqrt{5}}{10} z^3, \quad z \pm \frac{3\sqrt{2}-2}{4} z^3 + \frac{2-\sqrt{2}}{4} z^5,$$

$$z \pm \frac{4}{3p-1} z^p + \frac{(p+1)}{(2p-1)(3p-1)} z^{2p-1},$$

$$z \pm \frac{4}{p(3p-1)} z^p + \frac{(p+1)}{(2p-1)^2(3p-1)} z^{2p-1}.$$

3. Lemmas. For the determination of the regions $\mathscr{B}_p$, $\mathscr{B}_p^*$ and $\mathscr{B}_p^c$ we use the Dieudonné criterion in conjunction with the following lemma.

LEMMA 1. *If*

$$f(z) = 1 + az + bz^2, \quad b \text{ real}, \; a \in \mathbf{C}$$

does not vanish in $|z| < 1$, then a lies in the ellipse

$$E : \left\{ x + iy \,\middle|\, \left(\frac{x}{1+b}\right)^2 + \left(\frac{y}{1-b}\right)^2 \leqq 1 \right\}$$

if $-1 < b < 1$. If $b = 1$, then $a \in [-2, 2]$, whereas $a \in [-2i, 2i]$ if $b = -1$.

Proof. Since the transformation

$$w(z) = -\left(\frac{1}{z} + bz\right), \quad b \neq \pm 1$$

maps the unit disk $|z| < 1$ onto the exterior of the ellipse E, a cannot lie outside E or else $1/z + bz + a$ would vanish in $|z| < 1$ and so would $1 + az + bz^2$. We similarly see that $a \in [-2, 2]$ if $b = 1$ and that $a \in [-2i, 2i]$ if $b = -1$.

LEMMA 2. *Let $a > b > d > c > 0$, and let*

$$(16) \quad \Delta(\varphi) = a^2d^2 - \{d^2(a^2-b^2) + (d^2-c^2)(a^2+d^2-c^2)\}\cos^2\varphi + (d^2-c^2)(a^2-b^2+d^2-c^2)\cos^4\varphi, \quad 0 \leqq \varphi \leqq \pi/2.$$

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Then the envelope of the family of circles

(17) $(x - a\cos\varphi)^2 + (y - b\sin\varphi)^2 = c^2\cos^2\varphi + d^2\sin^2\varphi, \quad 0 \leqq \varphi \leqq \pi/2$

has the parametric equation

$$(18)\quad \left.\begin{aligned} x(\varphi) &= \frac{a\{a^2 + (d^2 - c^2)\} - a\{(a^2 - b^2) + (d^2 - c^2)\} \times \cos^2\varphi \pm b\sqrt{\Delta(\psi)}}{a^2 - (a^2 - b^2)\cos^2\varphi}\cos\varphi \\ y(\varphi) &= \frac{a^2 b - b\{(a^2 - b^2) + (d^2 - c^2)\} \times \cos^2\varphi \pm a\sqrt{\Delta(\varphi)}}{a^2 - (a^2 - b^2)\cos^2\varphi}\sin\varphi \end{aligned}\right\}, \quad 0 \leqq \varphi \leqq \frac{\pi}{2}$$

where the plus sign before $a\sqrt{\Delta(\varphi)}$ *goes with the plus sign before* $b\sqrt{\Delta(\varphi)}$ *and the minus sign before* $a\sqrt{\Delta(\varphi)}$ *goes with the minus sign before* $b\sqrt{\Delta(\varphi)}$.

Proof. The envelope is given by the system of equations

$$(19)\quad \left.\begin{aligned} &(x - \cos\varphi)^2 + (y - b\sin\varphi)^2 = c^2\cos^2\varphi + d^2\sin^2\varphi \\ &(x - a\cos\varphi)a\sin\varphi - (y - b\sin\varphi)b\cos\varphi = (d^2 - c^2)\sin\varphi\cos\varphi \end{aligned}\right\}, \quad 0 \leqq \varphi \leqq \pi/2.$$

On eliminating x between these two equations we obtain

$$(20)\quad \{a^2 - (a^2 - b^2)\cos^2\varphi\}(y - b\sin\varphi)^2 + 2b(d^2 - c^2) \times (y - b\sin\varphi)\sin\varphi\cos^2\varphi - [a^2d^2 - (d^2 - c^2) \times \{a^2 + (d^2 - c^2)\}\cos^2\varphi]\sin^2\varphi = 0,$$

which gives us

$$(21)\quad y - b\sin\varphi = \frac{-b(d^2 - c^2)\cos^2\varphi \pm a\sqrt{\Delta(\varphi)}}{a^2 - (a^2 - b^2)\cos^2\varphi}\sin\varphi$$

where $\Delta(\varphi)$ is defined in (16). This readily leads us to the desired result.

The next lemma gives us some useful information about the location of the zeros of a polynomial $P(z) \in \mathscr{S}_p$.

LEMMA 3. *If* $P(z) = z + \alpha_p z^p + \beta_p z^{2p-1}$ *is univalent in* $|z| < 1$, *then* $z^{-1}P(z) \neq 0$ *in* $|z| < (2p - 1)^{1/2(p-1)}$.

Proof. For the proof we will not need univalence of $P(z)$ but the weaker requirement that $P'(z) \neq 0$ in $|z| < 1$. The lemma will be proved if we show that the polynomial

$$h(z) = 1 + \alpha_p z + \beta_p z^2$$

does not vanish in $|z| < (2p - 1)^{1/2}$. For this we observe that $h(z)$ is the composition (in the sense of G. Szegö [**6**]) of the polynomials

$$f(z) = 1 + p\,\alpha_p\,z + (2p - 1)\beta_p\,z^2,$$

$$g(z) = 1 + \frac{2}{p}z + \frac{1}{2p - 1}z^2.$$

Since $f(z) = P'(z^{1/(p-1)}) \neq 0$ in $|z| < 1$ and the zeros of $g(z)$ lie on $|z| = (2p-1)^{1/2}$, it follows [**6**, pp. 65–66] that $h(z)$ does not vanish in $|z| < (2p-1)^{1/2}$.

By considering the univalent polynomials

$$z + \lambda \frac{2p}{2p-1}\left(\sin \frac{\pi}{2p}\right) z^p + \frac{1}{2p-1} z^{2p-1}$$

where λ is any number such that $-1 \leqq \lambda \leqq 1$ [**9**], we see that the result is sharp.

4. Proofs of theorems.

Proof of Theorem 1. According to the Dieudonné criterion, $P(z) = z + \alpha_p z^p + tz^{2p-1}$ is univalent in $|z| < 1$ if and only if for all $\theta \in [0, \pi/2]$ the polynomial

$$f_\theta(z) = 1 + \alpha_p \frac{\sin p\theta}{\sin \theta} z^{p-1} + t \frac{\sin (2p-1)\theta}{\sin \theta} z^{2p-2}, \quad (f_0(z) = P'(z))$$

has no zeros in $|z| < 1$.

Replacing z^{p-1} by ζ and putting $\cos \theta = u$ we conclude that $P(z) \in \mathcal{S}_p$ if and only if the function

$$1 + \alpha_p U_{p-1}(u)\zeta + tU_{2p-2}(u)\zeta^2 \quad \text{where } U_k(u) = \frac{\sin \{(k+1)(\text{arc} \cos u)\}}{\sin (\text{arc} \cos u)}$$

is the Chebyshev polynomial of the second kind of degree k, does not vanish in $|\zeta| < 1$ for all $u \in [0, 1]$. Now the desired result follows on applying Lemma 1.

Proof of Theorem 2. The polynomial $P(z) = z + \alpha_p z^p + tz^{2p-1}$ belongs to $\mathcal{S}_p{}^*$ if and only if

(i) $P(z)/z \neq 0$ in $|z| \leqq 1$,

(ii) $\text{Re } zP'(z)/P(z) \geqq 0$ in $|z| \leqq 1$.

Since $\text{Re } zP'(z)/P(z)$ is harmonic in $|z| \leqq 1$, (ii) may be replaced by the condition that $\text{Re } zP'(z)/P(z) \geqq 0$ on $|z| = 1$, or equivalently

$$|zP'(z)/P(z) + p| \geqq |zP'(z)/P(z) - p| \quad \text{on } |z| = 1.$$

This leads us to the requirement that

$$|(p+1) + 2p\alpha_p\zeta + (3p-1)t\zeta^2| \geqq |(p-1) - (p-1)t\zeta^2|$$

for all ζ on the unit circle. Writing this inequality in the form

$$(22) \quad \left| \frac{(p+1)}{2p} e^{-i\varphi} + \alpha_p + \frac{(3p-1)}{2p} te^{i\varphi} \right| \geqq \left| \frac{(p-1)}{2p} e^{-i\varphi} - \frac{(p-1)t}{2p} e^{i\varphi} \right|, \quad (0 \leqq \varphi < 2\pi)$$

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we see that α_p must lie outside the ring shaped region G generated by a disk D_φ of varying radius

$$r(\varphi) = \frac{(p-1)}{2p}\sqrt{(1+t)^2 - 2t\cos 2\varphi}$$

and centre

$$\left(\frac{(p+1)+(3p-1)t}{2p}\cos\varphi, \frac{(p+1)-(3p-1)t}{2p}\sin\varphi\right)$$

moving along the ellipse

$$\frac{x^2}{\left\{\frac{(p+1)+(3p-1)t}{2p}\right\}^2} + \frac{y^2}{\left\{\frac{(p+1)-(3p-1)t}{2p}\right\}^2} = 1.$$

But for $P(z)$ to belong to $\mathscr{S}_p{}^*$ we must also have $P(z)/z \neq 0$ in $|z| \leqq 1$. So in view of Lemma 1, $P(z) \in \mathscr{S}_p{}^*$ if and only if α_p belongs to the maximal connected set $D_{p,t}{}^*$ containing the origin and lying in the complement of G. In order to determine the boundary of $D_{p,t}{}^*$ we look at the envelope of the family of disks D_φ $(0 \leqq \varphi < 2\pi)$. Since $D_{p,t}{}^*$ is clearly symmetrical with respect to the coordinate axes we may focus our attention on the sub-family D_φ $(0 \leqq \varphi \leqq \pi/2)$. We apply Lemma 2 to get the envelope and see that the portion which is relevant for our purpose has the parametric equation

$$x(\varphi) = \frac{\begin{array}{l}\{(p+1)+(3p-1)t\}(1+t) \\ \qquad \times\{(p+1)+(3p-1)(2p-1)t\} \\ \qquad\qquad - 4pt\{(p+1)^2+(3p-1)^2t\}\cos^2\varphi\end{array}}{p[\{(p+1)+(3p-1)t\}^2 - 4(p+1)(3p-1)t\cos^2\varphi]}\cos\varphi$$

$$y(\varphi) = \frac{\begin{array}{l}\{(p+1)+(3p-1)t\}^2\{1-(2p-1)t\} \\ \qquad\qquad - 4pt\{(p+1)^2-(3p-1)^2t\}\cos^2\varphi\end{array}}{p[\{(p+1)+(3p-1)t\}^2 - 4(p+1)(3p-1)t\cos^2\varphi]}\sin\varphi$$

where φ is to vary from 0 to $\pi/2$. If $0 \leqq t \leqq (p+1)/((2p-1)(3p-1))$, then $y(\varphi)$ vanishes only once in the interval $[0, \pi/2]$, namely at $\varphi = 0$. For other values of φ it is positive. But if $(p+1)/((2p-1)(3p-1)) < t \leqq 1/(2p-1)$ then

$$y(\varphi) \leqq 0$$

$$\text{for } 0 \leqq \varphi \leqq \varphi_0 = \arccos\frac{\{1-(2p-1)t\}^{1/2}\{(p+1)+(3p-1)t\}}{[4pt\{(p+1)^2-(3p-1)^2t\}]^{1/2}}.$$

For $\varphi_0 < \varphi \leqq \pi/2$, $y(\varphi) > 0$. Hence, if $(p+1)/((2p-1)(3p-1)) < t \leqq 1/(2p-1)$, the portion of $\partial D_{p,t}{}^*$ lying in the first quadrant is given by the above parametric equation where the parameter φ varies from φ_0 to $\pi/2$. This completes the proof of Theorem 2.

Proof of Theorem 4. Observe that ρ_t is the largest number with the property that for every polynomial $P(z) = z + \alpha_p z^p + tz^{2p-1} \in S_{p,t}^*$ and all $\rho \in (0, \rho_t]$ the corresponding polynomial

$$f(z, \rho) = P(z\rho)/\rho = z + \alpha_p \rho^{p-1} z^p + t\rho^{2p-2} z^{2p-1}$$

is convex in $|z| < 1$. Thus, clearly $|t|(\rho_t)^{2p-2} \leqq 1/(2p - 1)^2$.

The detailed calculations for determining ρ_t depend on whether

(i) $t \in \left[-\frac{1}{2p-1}, \frac{p+1}{(2p-1)(3p-1)}\right]$ or

(ii) $t \in \left(\frac{p+1}{(2p-1)(3p-1)}, \frac{1}{2p-1}\right]$.

First, let $t \in [t_0, t_1]$. We show that ρ_t cannot be larger than

$$\{(p + 1)/((2p - 1)^2(3p - 1)t)\}^{1/2(p-1)}.$$

For this let $\hat{\rho}_t$ denote the largest number in

$$[0, \{(p + 1)/((2p - 1)^2(3p - 1)t)\}^{1/2(p-1)}]$$

with the property that $f(z, \rho) = P(z\rho)/\rho$ is convex in $|z| < 1$ for all $\rho \in (0, \hat{\rho}_t]$. According to (10)

$$|\alpha_p \rho^{p-1}| \leqq (1 + (2p - 1)^2 t\rho^{2p-2})/p^2$$

for $0 \leqq \rho \leqq \hat{\rho}_t$ as long as $z + \alpha_p z^p + tz^{2p-1} \in S_p^*$, i.e. $|\alpha_p| \leqq (1 + (2p - 1)t)/p$ Hence

$$\hat{\rho}_t = [\{p(1 + (2p-1)t) + \sqrt{p^2(1 + (2p-1)t)^2 - 4(2p-1)^2 t}\}/2]^{-1/(p-1)} < \left\{\frac{p+1}{(2p-1)^2(3p-1)t}\right\}^{1/2(p-1)}.$$

Since $\hat{\rho}_t$ turns out to be strictly less than $\{(p + 1)/((2p - 1)^2(3p - 1)t)\}^{1/2(p-1)}$ it follows that

$$\rho_t = \hat{\rho}_t \leqq [(p\{1 + (2p - 1)t\} + \sqrt{p^2\{1 + (2p - 1)t\}^2 - 4(2p - 1)^2 t})/2]^{-1/(p-1)} = \omega_1(t).$$

Secondly, let $t \in (t_1, t_2)$. Here again we see that ρ_t is smaller than

$$\{(p + 1)/((2p - 1)^2(3p - 1)t)\}^{1/2(p-1)}.$$

In fact, in this case, the number $\hat{\rho}_t$ defined above turns out to be equal to

$$\omega_2(t) = \{2p^2 A(t) + \sqrt{4p^4 A^2(t) - (2p - 1)^2 t}\}^{-1/(p-1)}$$

which happens to be strictly less than $\{(p + 1)/((2p - 1)^2(3p - 1)t)\}^{1/2(p-1)}$

for $t \in (t_1, t_2)$. Hence

$$\rho_t = \hat{\rho}_t \leqq \omega_2(t).$$

Now let $t \in [t_2, t_4]$. It is easily verified that in this case

$$\rho_t \geqq \hat{\rho}_t \geqq \{(p+1)/((2p-1)^2(3p-1)t)\}^{1/2(p-1)}.$$

Hence in view of (10)

$$|\alpha_p \rho^{p-1}| \leqq \frac{4}{p}\left[\frac{p(2p-1)t\{1-(2p-1)^2 t\}}{(p+1)^2-(3p-1)^2(2p-1)t}\right]^{1/2}$$

for $0 \leqq \rho \leqq \rho_t$ as long as $z + \alpha_p z^p + tz^{2p-1} \in S_p^*$. Since according to (4), $|\alpha_p|$ can be as large as $4[\{1-(2p-1)t\}pt/((p+1)^2-(3p-1)^2t)]^{1/2}$, we conclude that $\rho_t \leqq \omega_3(t)$. But, as remarked earlier ρ_t cannot be larger than $\omega_4(t) = \{(2p-1)^2 t\}^{-1/2(p-1)}$. Hence

$$\rho_t \leqq \min\{\omega_3(t), \omega_4(t)\} = \begin{cases} \omega_3(t) & \text{if } t_2 \leqq t \leqq t_3 \\ \omega_4(t) & \text{if } t_3 \leqq t \leqq t_4. \end{cases}$$

Thus we have shown that

$$\rho_t \leqq \omega_i(t) \quad \text{for } t \in [t_{i-1}, t_i],\ i = 1, 2, 3, 4.$$

By considering the polynomials $P(z) = z \pm \alpha_p z^p + tz^{2p-1}$ where

$$\alpha_p = \begin{cases} \dfrac{1+(2p-1)t}{p} & \text{if } t \in [\mathrm{t}_0, \mathrm{t}_1] \\ 4\left[\dfrac{p\{1-(2p-1)t\}}{(p+1)^2-(3p-1)^2 t}\right]^{1/2} & \text{if } t \in [t_1, t_4] \end{cases}$$

we see that, in fact

$$\rho_t = \omega_i(t) \quad \text{for } t \in [t_{i-1}, t_i],\ i = 1, 2, 3, 4.$$

Remark. The proof of Theorem 4 and the result contained in Theorem C show that every polynomial

$$P(z) = z + \alpha_p z^p + tz^{2p-1} \in S_p, \quad \left(-\frac{1}{2p-1} \leqq t \leqq \frac{p+1}{(2p+1)(3p-1)}\right),$$

is convex in

$$|z| < \left(\frac{2p^2-(p-1)\sqrt{4p^2+2p-1}}{(2p-1)(p+1)}\right)^{1/(p-1)}.$$

Proof of Theorem 5. Let $P(z) = z + a_2 z^2 + tz^3 \in S_2$. Then $P'(z)$ does not vanish in $|z| < 1$ and

$$|a_2| \leqq \begin{cases} (1+3t)/2 & \text{for } -1/3 \leqq t \leqq 1/5 \\ 2\sqrt{t(1-t)} & \text{for } 1/5 \leqq t \leqq 1/3. \end{cases}$$

Hence for $0 < t \leqq 1/3$ we may write

$$P'(z) = (1 - z/\mu e^{i\alpha})(1 - z/\mu e^{-i\alpha}),$$

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where $\mu \geqq 1$,

$$(23)\quad |\cos\alpha| \leqq \begin{cases} (\mu^2+1)/(2\mu) & \text{for } \sqrt{5/3} \leqq \mu < \infty \\ \dfrac{2}{3\mu}\sqrt{3\mu^2-1} & \text{for } 1 \leqq \mu \leqq \sqrt{5/3}. \end{cases}$$

It is clear that $\text{Re}\,\{1 + zP''(z)/P'(z)\} > 0$ if and only if

$$(24)\quad \left|1 + z\frac{P''(z)}{P'(z)} + 2\right| > \left|1 + z\frac{P''(z)}{P'(z)} - 2\right|.$$

Thus our problem is to determine the largest disk $|z| < R_2{}^c$ in which (24) holds. We may write (24) in the form

$$|5z^2 - 8\mu z\cos\alpha + 3\mu^2| > |z^2 - \mu^2|.$$

This inequality clearly holds for $z = 0$ and in the punctured disk $0 < |z| < R_2{}^c$ it will hold if and only if

$$|5z - 8\mu\cos\alpha + 3\mu^2/z| > |z - \mu^2/z|$$

holds. Thus if $z = re^{i\theta}$ we wish to determine $R_2{}^c$ such that

$$(25)\quad w(\mu, r, \alpha, \theta) := 8\mu^2r^2\cos^2\theta - 2\mu r(3\mu^2 + 5r^2)(\cos\alpha)\cos\theta + (\mu^2 - r^2)(\mu^2 - 3r^2) + 8\mu^2r^2\cos^2\alpha > 0$$

for $0 < r < R_2{}^c$ and θ real. Without loss of generality, we may assume $0 \leqq \alpha \leqq \pi/2$. For given μ, r, α the minimum of $w(\mu, r, \alpha, \theta)$ can occur only if

$$\text{(i)}\ \cos\theta = \frac{(3\mu^2 + 5r^2)}{8\mu r}\cos\alpha$$

(which is admissible only if

$$(26)\quad \cos\alpha \leqq \frac{8\mu r}{3\mu^2 + 5r^2}\Bigg),$$

or

$$\text{(ii)}\ \cos\theta = 0$$

or

$$\text{(iii)}\ \cos\theta = 1.$$

If $\cos\theta = ((3\mu^2 + 5r^2)/8\mu r)\cos\alpha$ is admissible, then

$$w(\mu, r, \alpha, \theta) = (\mu^2 - r^2)\{(\mu^2 - 3r^2) - \tfrac{1}{8}(9\mu^2 - 25r^2)\cos^2\alpha\}$$

which is positive for $r < \mu/\sqrt{5}$ and so certainly for $r < 1/\sqrt{7}$, since (26) holds.

If $\cos\theta = 0$, then clearly, $w(\mu, r, \alpha, \theta) > 0$ for $r < \mu/\sqrt{3}$.

Finally, if $\cos\theta = 1$, then

$$\begin{aligned} w(\mu, r, \alpha, \theta) &= 8\mu^2r^2 - 2\mu r(3\mu^2 + 5r^2)\cos\alpha + (\mu^2 - r^2)(\mu^2 - 3r^2) \\ &\quad + 8\mu^2r^2\cos^2\alpha = (3r^2 - 4\mu r\cos\alpha + \mu^2)(r^2 - 2\mu r\cos\alpha + \mu^2) \end{aligned}$$

$$= \begin{cases} \{3r^2 - 2r(\mu^2+1) + \mu^2\}\{r^2 - r(\mu^2+1) + \mu^2\} & \text{if } \sqrt{5/3} \leqq \mu < \infty \\ \left(3r^2 - \frac{8r}{3}\sqrt{3\mu^2 - 1 + \mu^2}\right)\left(r^2 - \frac{4r}{3}\sqrt{3\mu^2 - 1 + \mu^2}\right) & \text{if } 1 \leqq \mu \leqq \sqrt{5/3}. \end{cases}$$

Hence in this case, $w(\mu, r, \alpha, \theta) > 0$ if $r < (\mu^2 + 1 - \sqrt{\mu^4 - \mu^2 + 1})/3$ or $r < (4\sqrt{3\mu^2 - 1} - \sqrt{21\mu^2 - 16})/9$ according as $\sqrt{5/3} \leqq \mu < \infty$ or $1 \leqq \mu \leqq \sqrt{5/3}$ respectively. From this it follows that for $0 < t \leqq 1/3$ the polynomial $P(z) = z + a_2z^2 + tz^3 \in S_2$ is convex in $|z| < 1/\sqrt{7}$. If $-1/3 \leqq t < 0$ then $|a_2| \leqq \frac{1}{2}(1 + 3t)$ and $P'(z)$ has two real zeros μ_1, μ_2 with $|\mu_1| \geqq 1, |\mu_2| \geqq 1$. By the same reasoning as above it can be shown that in this case $P(z) = z + a_2z^2 + tz^3 \in S_2$ is convex in $|z| < 1/\sqrt{3}$, so that $R_2^c = 1/\sqrt{7}$. Extremal polynomials are

$$P(z) = z \pm \frac{8}{23}\sqrt{7}z^2 + \frac{7}{23}z^3.$$

We can similarly prove that $R_3^c = \{(9 + \sqrt{305})/112\}^{1/2}$ where the extremal polynomials are

$$P(z) = z \pm 112\frac{(50 + 18\sqrt{305})}{(81 + 9\sqrt{305})^2 + 112^2}z^3 + \frac{112^2}{(81 + 9\sqrt{305})^2 + 112^2}z^5.$$

Proof of Theorem 6. Let $P(z) = z + \alpha_pz^p + tz^{2p-1} \in S_p$, $p \geqq 2$. For positive t we may, in view of Lemma 3 write:

$$P(z) = (1/\lambda^2)z(z^{p-1} - \lambda e^{i\alpha})(z^{p-1} - \lambda e^{-i\alpha}) = z - \frac{2\cos\alpha}{\lambda}z^p + \frac{1}{\lambda^2}z^{2p-1},$$

where $\lambda \geqq (2p-1)^{1/2}$. Besides, there is no loss of generality in supposing that $0 \leqq \alpha \leqq \pi/2$. Since $\operatorname{Re} zP'(z)/P(z) > 0$ if and only if

(27) $|zP'(z)/P(z) + p| > |zP'(z)/P(z) - p|$,

our problem is to determine the largest disk $|z| < r_p^*$ in which (27), i.e.

(28) $|(3p-1)z^{2(p-1)} - 4\lambda pz^{p-1}\cos\alpha + (p+1)\lambda^2| > (p-1)|z^{2(p-1)} - \lambda^2|$

holds. Inequality (28) holds for $z = 0$. On dividing the two sides of this inequality by $|z^{p-1}|$ and putting $z^{p-1} = Re^{i\varphi}$ it takes the form

$$\begin{aligned} 2\lambda R[2p\lambda R\cos^2\varphi - \{(3p-1)R^2 + (p+1)\lambda^2\}(\cos\alpha)(\cos\varphi) \\ + 2p\lambda R\cos^2\alpha] + \{(2p-1)R^2 - \lambda^2\}(R^2 - \lambda^2) > 0. \end{aligned}$$

It is clear that this latter inequality certainly holds as long as

$$\begin{aligned} W(p, \lambda, R, \alpha) := &\{4p(2p-1) - (3p-1)^2\cos^2\alpha\}R^4 \\ &- 2\lambda^2\{4p^2 - (5p^2 - 2p + 1)\cos^2\alpha\}R^2 \\ &+ \lambda^4\{4p - (p+1)^2\cos^2\alpha\} > 0. \end{aligned}$$

For fixed p, λ, R the function $W(p, \lambda, R, \alpha)$ is smallest when $\cos\alpha$ assumes its

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largest admissible value. Using the fact that, if $p = 2$, then

$$\cos\alpha \leqq \begin{cases} \dfrac{\lambda^2+3}{4\lambda} & \text{for } \lambda \in [\sqrt{5}, \infty) \\ \dfrac{\sqrt{\lambda^2-1}}{\lambda} & \text{for } \lambda \in [\sqrt{3}, \sqrt{5}], \end{cases}$$

whereas, if $p = 3$, then

$$\cos\alpha \leqq \begin{cases} \dfrac{\lambda^2+5}{6\lambda} & \text{for } \lambda \in [\sqrt{10}, \infty) \\ \dfrac{2\sqrt{\lambda^2-1}-1}{2\lambda} & \text{for } \lambda \in [\sqrt{5}, \sqrt{10}], \end{cases}$$

it can be shown that for positive t, $P(z) = z + \alpha_p z^p + tz^{2p-1} \in S_p$ is starlike in $|z| < 3/\sqrt{11}$ if $p = 2$ and in $|z| < (10/13)^{1/4}$ if $p = 3$. As in the case of Theorem 5 it turns out that the same holds *a fortiori* for negative t, so that $R_2^* = 3/\sqrt{11}$, $R_3 = (10/13)^{1/4}$.

In the case $p = 2$, the extremal polynomials are

$$P_2(z) = z \pm \frac{2\sqrt{2}}{3}z^2 + \frac{1}{3}z^3$$

and Re $\{zP'_2(z)/P_2(z)\}$ vanishes on $|z| = 3/\sqrt{11}$ at $z = \mp (3/\sqrt{11})e^{i\theta_0}$ where $\theta_0 = \arccos\sqrt{(8/11)}$.

In the case $p = 3$, the extremal polynomials are

$$P_3(z) = z \pm \frac{3}{5}z^3 + \frac{1}{5}z^5$$

and Re $\{zP_3'(z)/P_3(z)\}$ vanishes on $|z| = (10/13)^{1/4}$ at $z = \mp(10/13)^{1/4}e^{i\theta_1}$ where $\theta_1 = \arccos\,(17/(2\sqrt{130}))$.

Proof of Theorem 7. Let $\mathscr{F}_p$ denote any one of the families $\mathscr{S}_p$, S_p, S_p^*, S_p^c, and for an admissible t let $\mathscr{F}_{p,t}$ denote the class of all polynomials of the form

$$P(z) = z + \alpha_p z^p + tz^{2p-1}$$

belonging to $\mathscr{F}_p$. It is clear that $K(\mathscr{F}_p) = \min_t K(\mathscr{F}_{p,t})$. So we may fix our attention on the family $\mathscr{F}_{p,t}$. From Theorem 1 it follows that for $P(z) \in \mathscr{F}_{p,t}$ the region of variability of α_p is a set $\Delta_{p,t}$ contained in the ellipse

$$E_t: \left\{x + iy \,\middle|\, \left(\frac{x}{1+t}\right)^2 + \left(\frac{y}{1-t}\right)^2 \leqq 1\right\}.$$

Now our idea consists in observing that $K(\mathscr{F}_{p,t})$ is equal to the shortest distance between $\partial\Delta_{p,t}$ and ∂E_t. In fact

$$\begin{aligned} K(\mathscr{F}_{p,t}) &= \min_{P(z)\in\mathscr{F}_{p,t}} \min_{0\leqq\theta<2\pi} |P(e^{i\theta})| \\ &= \min_{\alpha_p\in\Delta_{p,t}} \min_{0\leqq\theta<2\pi} |e^{-i(p-1)\theta} + \alpha_p + te^{i(p-1)\theta}| \\ &= \min_{\alpha_p\in\Delta_{p,t}} \min_{0\leqq\theta<2\pi} |e^{-i\theta} + te^{i\theta} + \alpha_p| \end{aligned}$$

which obviously represents the shortest distance d_t between $\partial\Delta_{p,t}$ and ∂E_t.

If $\mathscr{F}_p$ is one of the families $\mathscr{S}_p$, S_p, S_p^*, S_p^c then the corresponding set $\Delta_{p,t}$ is known to be convex. In such a case

$$d_t = \min_{0 \leqq \varphi < 2\pi} \{k_1(\varphi) - k_2(\varphi)\}$$

where $k_1(\varphi)$, $k_2(\varphi)$ are the supporting functions of the sets E_t, $\Delta_{p,t}$ respectively.

In order to illustrate the method it is enough to carry out the details for the class S_p^*. In this case the set $\Delta_{p,t}$ is the interval $[-c_t, c_t]$ where

$$c_t = \begin{cases} \dfrac{1 + (2p - 1)t}{p}, & -\dfrac{1}{2p - 1} \leqq t \leqq \dfrac{p + 1}{(2p - 1)(3p - 1)} \\ \left[\dfrac{\{1 - (2p - 1)t\}pt}{(p + 1)^2 - (3p - 1)^2 t}\right]^{1/2}, & \dfrac{p + 1}{(2p - 1)(3p - 1)} \leqq t \leqq \dfrac{1}{2p - 1}. \end{cases}$$

The supporting functions $k_1(\varphi)$, $k_2(\varphi)$ are

$$k_1(\varphi) = \sqrt{(1 - t)^2 + 4t\cos^2\varphi}, \quad 0 \leqq \varphi < 2\pi$$

$$k_2(\varphi) = c_t|\cos\varphi|, \quad 0 \leqq \varphi < 2\pi,$$

respectively. It is easily seen that for

$$t \in [-1/(2p - 1), (p + 1)/((2p - 1)(3p - 1))]$$

$$(29) \quad d_t = k_1(0) - k_2(0) = 1 + t - \frac{1 + (2p - 1)t}{p}.$$

If $t \in [(p + 1)/((2p - 1)(3p - 1)), 1/(2p - 1)]$, then for $0 \leqq \varphi < \pi/2$

$$k_1'(\varphi) - k_2'(\varphi) = 4\sin\varphi\left[\left\{\frac{(1 - (2p - 1)t)pt}{(p + 1)^2 - (3p - 1)^2 t}\right\}^{1/2} - \frac{t\cos\varphi}{\sqrt{(1 - t)^2 + 4t\cos^2\varphi}}\right].$$

Now observe that if $t \in [(p + 1)/((2p - 1)(3p - 1)), t_0]$ where t_0 is the unique root of the equation

$$\left\{\frac{(1 - (2p - 1)t)pt}{(p + 1)^2 - (3p - 1)^2 t}\right\}^{1/2} = \frac{t}{1 + t}$$

in $[(p + 1)/((2p - 1)(3p - 1)), 1/(2p - 1)]$ then $k_1'(\varphi) - k_2'(\varphi)$ does not vanish at any point in $[0, \pi/2]$ other than $\varphi = 0$. This helps us to conclude that for $t \in [(p + 1)/((2p - 1)(3p - 1)), t_0]$

$$(30) \quad \begin{aligned} d_t &= \min_{\varphi} \{k_1(\varphi) - k_2(\varphi)\} = k_1(0) - k_2(0) \\ &= (1 + t) - 4\left\{\frac{(1 - (2p - 1))pt}{(p + 1)^2 - (3p - 1)^2 t}\right\}^{1/2}. \end{aligned}$$

For $t \in [t_0, 1/(2p - 1)]$, d_t turns out to be equal to $k_1(\varphi_0) - k_2(\varphi_0)$ where φ_0 is the unique root of the equation

$$\frac{t\cos\varphi}{\sqrt{(1 - t)^2 + 4t\cos^2\varphi}} = \left\{\frac{(1 - (2p - 1)t)pt}{(p + 1)^2 - (3p - 1)^2 t}\right\}^{1/2}$$

in $(0, \pi/2]$, i.e.

$$(31) \quad d_t = \frac{(p-1)(1-t)\sqrt{1-t}}{\sqrt{(p+1)^2 - (3p-1)^2 t}}.$$

From (29), (30), and (31) it can be deduced that

$$K(S_p^*) = \frac{6(p-1)^2}{(2p-1)(3p-1)}.$$

The proofs of the other assertions in Theorem 7 are omitted.

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