3
A closer look at the world-sheet

The careful reader has patiently suspended disbelief for a while now, allowing us to race through a somewhat rough presentation of some of the highlights of the construction of consistent relativistic strings. This enabled us, by essentially stringing lots of oscillators together, to go quite far in developing our intuition for how things work, and for key aspects of the language.

Without promising to suddenly become rigourous, it seems a good idea to revisit some of the things we went over quickly, in order to unpack some more details of the operation of the theory. This will allow us to develop more tools and language for later use, and to see a bit further into the structure of the theory.

3.1 Conformal invariance

We saw in section 2.2.8 that the use of the symmetries of the action to fix a gauge left over an infinite dimensional group of transformations which we could still perform and remain in that gauge. These are conformal transformations, and the world-sheet theory is in fact conformally invariant. It is worth digressing a little and discussing conformal invariance in arbitrary dimensions first, before specialising to the case of two dimensions. We will find a surprising reason to come back to conformal invariance in higher dimensions much later, so there is a point to this.

3.1.1 Diverse dimensions

Imagine[275] that we do a change of variables $x \to x'$. Such a change, if invertible, is a 'conformal transformation' if the metric is invariant up to

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an overall scale $\Omega(x)$, which can depend on position:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{3.1}$$

The name comes from the fact that angles between vectors are unchanged.

If we consider the infinitessimal change

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{3.2}$$

then from equation (1.1), we get:

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu), \tag{3.3}$$

and so we see that in order for this to be a conformal transformation,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = F(x) g_{\mu\nu}, \tag{3.4}$$

where, by taking the trace of both sides, it is clear that:

$$F(x) = \frac{2}{D} g^{\mu\nu} \partial_\mu \epsilon_\nu .$$

It is enough to consider our metric to be Minkowski space, in Cartesian coordinates, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. We can take one more derivative ∂_κ of the expression (3.4), and then do the permutation of indices $\kappa \to \mu, \mu \to \nu, \nu \to \kappa$ twice, generating two more expressions. Adding together any two of those and subtracting the third gives:

$$2\partial_\mu \partial_\nu \epsilon_\kappa = \partial_\mu F \eta_{\nu\kappa} + \partial_\nu F \eta_{\kappa\mu} - \partial_\kappa F \eta_{\mu\nu}, \tag{3.5}$$

which yields

$$2\Box \epsilon_\kappa = (2-D)\partial_\kappa F. \tag{3.6}$$

We can take another derivative this expression to get $2\partial_\mu \Box \epsilon_\kappa = (2-D)\partial_\mu \partial_\kappa F$, which should be compared to the result of acting with $\Box$ on equation (3.4) to eliminate ϵ leaving:

$$\eta_{\mu\nu} \Box F = (2-D)\partial_\mu \partial_\nu F \quad \Longrightarrow \quad (D-1)\Box F = 0, \tag{3.7}$$

where we have obtained the last result by contraction.

For general D we see that the last equations above ask that $\partial_\mu \partial_\nu F = 0$, and so F is linear in x. This means that ϵ is quadratic in the coordinates, and of the form:

$$\epsilon_\mu = A_\mu + B_{\mu\nu} x^\nu + C_{\mu\nu\kappa} x^\nu x^\kappa, \tag{3.8}$$

where C is symmetric in its last two indices.

Table 3.1. *The finite form of the conformal transformations and their infinitesimal generators*

Operation	Action	Generator
translations	$x'^{\mu} = x^{\mu} + A^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
rotations	$x'^{\mu} = M^{\mu}{}_{\nu}x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
dilations	$x'^{\mu} = \lambda x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
special conformal transformations	$x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}x^2}{1 - 2(\mathbf{x}\cdot\mathbf{b}) - b^{\mu}x^2}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

The parameter A_{μ} is obviously a translation. Placing the B term in equation (3.8) back into equation (3.4) yields that $B_{\mu\nu}$ is the sum of an antisymmetric part $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and a trace part λ:

$$B_{\mu\nu} = \omega_{\mu\nu} + \lambda\eta_{\mu\nu}. \tag{3.9}$$

This represents a scale transformation by $1 + \lambda$ and an infinitessimal rotation. Finally, direct substitution shows that

$$C_{\mu\nu\kappa} = \eta_{\mu\kappa}b_{\nu} + \eta_{\mu\nu}b_{\kappa} - \eta_{\nu\kappa}b_{\mu}, \tag{3.10}$$

and so the infinitesimal transformation which results is of the form

$$x'^{\mu} = x^{\mu} + 2(\mathbf{x}\cdot\mathbf{b})x^{\mu} - b^{\mu}x^2, \tag{3.11}$$

which is called a 'special conformal transformation'. Its finite form can be written as:

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}, \tag{3.12}$$

and so it looks like an inversion, then a translation, and then an inversion. We gather together all the transformations, in their finite form, in table 3.1.

Poincaré and dilatations together form a subgroup of the full conformal group, and it is indeed a special theory that has the full conformal invariance given by enlargement by the special conformal transformations.

It is interesting to examine the commutation relations of the generators, and to do so, we rewrite them as

$$\begin{aligned} &J_{-1,\mu} = \tfrac{1}{2}(P_{\mu} - K_{\mu}), \qquad J_{0,\mu} = \tfrac{1}{2}(P_{\mu} + K_{\mu}), \\ &J_{-1,0} = D, \quad J_{\mu\nu} = L_{\mu\nu}, \end{aligned} \tag{3.13}$$

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with $J_{ab} = -J_{ba}$, $a, b = -1, 0, \ldots, D$, and the commutators are:

$$[J_{ab}, J_{cd}] = -i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{db}J_{ac}). \tag{3.14}$$

Note that we have defined an extra value for our indices, and η is now $\mathrm{diag}(-1, -1, +1, \ldots)$. This is the algebra of the group $SO(D, 2)$ with $\frac{1}{2}(D + 2)(D + 1)$ parameters.

3.1.2 The special case of two dimensions

As we have already seen in section 2.2.8, the conformal transformations are equivalent to conformal mappings of the plane to itself, which is an infinite dimensional group. This might seem puzzling, since from what we saw just above, one might have expected $SO(2, 2)$, or in the case where we have Euclideanised the world-sheet, $SO(3, 1)$, a group with six parameters. Actually, this group is a very special subgroup of the infinite family, which is distinguished by the fact that the mappings are invertible. These are the *global* conformal transformations. Imagine that $w(z)$ takes the plane into itself. It can at worst have zeros and poles, (the map is not unique at a branch point, and is not invertible if there is an essential singularity) and so can be written as a ratio of polynomials in z. However, for the map to be invertible, it can only have a single zero, otherwise there would be an ambiguity determining the pre-image of zero in the inverse map. By working with the coordinate $\tilde{z} = 1/z$, in order to study the neighbourhood of infinity, we can conclude that it can only have a single simple pole also. Therefore, up to a trivial overall scaling, we have

$$z \to w(z) = \frac{az + b}{cz + d}, \tag{3.15}$$

where a, b, c, d are complex numbers, with for invertability, the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

should be non-zero, and after a scaling we can choose $ad - bc = 1$. This is the group $SL(2, \mathbb{C})$ which is indeed isomorphic to $SO(3, 1)$. In fact, since a, b, c, d is indistinguishable from $-a, -b, -c, -d$, the correct statement is that we have invariance under $SL(2, \mathbb{C})/\mathbb{Z}_2$.

For the open string we have the upper half-plane, and so we are restricted to considering maps which preserve (say) the real axis of the complex plane. The result is that a, b, c, d must be real numbers, and the resulting group of invertible transformations is $SL(2, \mathbb{R})/\mathbb{Z}_2$. Correspondingly, the infinite part of the algebra is also reduced in size by half, as the holomorphic and antiholomorphic parts are no longer independent.

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N.B. Notice that the dimension of the group $SL(2,\mathbb{C})$ is six, equivalent to three complex parameters. Often, in computations involving a number of operators located at points, z_i, a conventional gauge fixing of this invariance is to set three of the points to three values: $z_1 = 0, z_2 = 1, z_3 = \infty$. Similarly, the dimension of $SL(2,\mathbb{R})$ is three, and the convention used there is to set three (real) points on the boundary to $z_1 = 0, z_2 = 1, z_3 = \infty$.

3.1.3 States and operators

A very important class of fields in the theory are those which transform under the $SO(2,D)$ conformal group as follows:

$$\phi(x^\mu) \longrightarrow \phi(x'^\mu) = \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta}{D}} \phi(x^\mu) = \Omega^{\frac{\Delta}{2}}\phi(x^\mu). \tag{3.16}$$

Here, $\left|\frac{\partial x}{\partial x'}\right|$ is the Jacobian of the change of variables. (Δ is the dimension of the field, as mentioned earlier.) Such fields are called 'quasi-primary', and the correlation functions of some number of the fields will inherit such transformation properties:

$$\langle\!\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle\!\rangle = \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta_1}{D}}_{x=x_1} \cdots \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta_n}{D}}_{x=x_n} \langle\!\langle \phi_1(x'_1)\dots\phi_n(x'_n)\rangle\!\rangle. \tag{3.17}$$

In two dimensions, the relation is

$$\phi(z,\bar{z}) \longrightarrow \phi(z',\bar{z}') = \left(\frac{\partial z}{\partial z'}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{\bar{h}} \phi(z,\bar{z}), \tag{3.18}$$

where $\Delta = h + \bar{h}$, and we see the familiar holomorphic factorisation. This mimics the transformation properties of the metric under $z \to z'(z)$:

$$g'_{z\bar{z}} = \left(\frac{\partial z}{\partial z'}\right)\left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right) g_{z\bar{z}},$$

the conformal mappings of the plane. This is an infinite dimensional family, extending the expected six of $SO(2,2)$, which is the subset which is globally well-defined. The transformations (3.18) define what is called a 'primary field', and the quasi-primaries defined earlier are those restricted to $SO(2,2)$. So a primary is automatically a quasi-primary, but not vice versa.

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In any dimension, we can use the definition (3.16) to construct a definition of a conformal field theory (CFT). First, we have a notion of a vacuum $|0\rangle$ that is $SO(2, D)$ invariant, in which all the fields act. In such a theory, all of the fields can be divided into two categories: a field is either quasi-primary, or it is a linear combination of quasi-primaries and their derivatives. Conformal invariance imposes remarkably strong constraints on how the two- and three-point functions of the quasi-primary fields must behave. Obviously, for fields placed at positions x_i, translation invariance means that they can only depend on the differences $x_i - x_j$.

3.1.4 The operator product expansion

In principle, we ought to be imagining the possibility of constructing a new field at the point x^μ by colliding together two fields at the same point. Let us label the fields as ϕ_k, then we might expect something of the form:

$$\lim_{x \to y} \phi_i(x)\phi_j(y) = \sum_k c_{ij}{}^k(x-y)\phi_k(y), \tag{3.19}$$

where the coefficients $c_{ij}{}^k(x-y)$ depend only on which operators (labelled by i, j) enter on the left. Given the scaling dimensions Δ_i for ϕ_i, we see that the coordinate behaviour of the coefficient should be:

$$c_{ij}{}^k(x-y) \sim \frac{1}{(x-y)^{\Delta_i+\Delta_j-\Delta_k}}.$$

This 'operator product expansion' (OPE) in conformal field theory is actually a convergent series, as opposed to the case of the OPE in ordinary field theory where it is merely an asymptotic series. An asymptotic series has a family of exponential contributions of the form $\exp(-L/|x-y|)$, where L is a length scale appropriate to the problem. Here, conformal invariance means that there is no length scale in the theory to play the role of L in an asymptotic expansion, and so the convergence properties of the OPE are stronger. In fact, the radius of convergence of the OPE is essentially the distance to the next operator insertion.

The OPE only really has sensible meaning if we define the operators as acting with a specific time ordering, and so we should specify that $x^0 > y^0$ in the above. In two dimensions, after we have continued to Euclidean time and work on the plane, the equivalent of time ordering is radial ordering (see figure 2.4). All OPE expressions written later will be taken to be appropriately time ordered.

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Actually, the OPE is a useful way of giving us a definition of a normal ordering prescription in this operator language*. It follows from Wick's theorem, which says that the time ordered expression of a product of operators is equal to the normal ordered expression plus the sum of all contractions of pairs of operators in the expressions. The contraction is a number, which is computed by the correlator of the contracted operators.

$$\phi_i(x)\phi_j(y) = :\phi_i(x)\phi_j(y): + \langle\!\langle \phi_i(x)\phi_j(y)\rangle\!\rangle. \tag{3.20}$$

Actually, we can compute the OPE between objects made out of products of operators with this sort of way of thinking about it. We'll compute some examples later (for example in equations (3.37) and (3.39)) so that it will be clear that it is quite straightforward.

3.1.5 The stress tensor and the Virasoro algebra

The stress-energy-momentum tensor's properties can be seen directly from conformal invariance in many ways, because of its definition as a conjugate to the metric via equation (1.10) which we reproduce here:

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{3.21}$$

A change of variables of the form (3.2) gives, using equation (3.3):

$$S \longrightarrow S - \frac{1}{2}\int d^Dx\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu} = S + \frac{1}{2}\int d^Dx\sqrt{-g}\,T^{\mu\nu}\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right).$$

In view of equation (3.4), this is:

$$S \longrightarrow S + \frac{1}{D}\int d^Dx\sqrt{-g}\,T^\mu{}_\mu\partial_\nu\epsilon^\nu$$

for a conformal transformation. So if the action is conformally invariant, then the stress tensor must be traceless, $T^\mu{}_\mu = 0$.

We can formulate this more carefully using Noether's theorem, and also extract some useful information. Since the change in the action is

$$\delta S = \int d^Dx\,\sqrt{-g}\,\partial_\mu\epsilon_\nu T^{\mu\nu},$$

given that the stress tensor is conserved, we can integrate by parts to write this as

$$\delta S = \int_\partial \epsilon_\nu T^{\mu\nu}dS_\mu.$$

* For free fields, this definition of normal ordering is equivalent to the definition in terms of modes, where the annihilators are placed to the right.

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We see that the current $j^\mu = T^{\mu\nu}\epsilon_\mu$, with ϵ_ν given by equation (3.4) is associated to the conformal transformations. The charge constructed by integrating over an equal time slice

$$Q = \int d^{D-1}x J^0,$$

is conserved, and it is responsible for infinitessimal conformal transformations of the fields in the theory, defined in the standard way:

$$\delta_\epsilon \phi(x) = \epsilon[Q, \phi]. \tag{3.22}$$

In two dimensions, infinitesimally, a coordinate transformation can be written as

$$z \to z' = z + \epsilon(z), \qquad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}).$$

As we saw in the previous chapter, or can be verified using the above discussion, the tracelessness condition yields $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and the conservation of the stress tensor is

$$\partial_z T_{zz}(z) = 0 = \partial_{\bar{z}} T_{\bar{z}\bar{z}}(\bar{z}).$$

For simplicity, we shall often use the shorthand: $T(z) \equiv T_{zz}(z)$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$. On the plane, an equal time slice is over a circle of constant radius, and so we can define

$$Q = \frac{1}{2\pi i} \oint \left(T(y)\epsilon(y)dy + \bar{T}(\bar{y})\bar{\epsilon}(\bar{y})d\bar{y}\right).$$

Infinitesimal transformations can then be constructed by an appropriate definition of the commutator $[Q, \phi(z)]$ of Q with a field ϕ.

Notice that this commutator requires a definition of two operators at a point, and so our previous discussion of the OPE comes into play here. We also have the added complication that we are performing a y-contour integration around one of the operators, inserted at z or $\bar{z}$. Under the integral sign, the OPE requires that $|z| < |y|$, when we have $Q\phi(y)$, or that $|z| > |y|$ if we have $\phi(y)Q$. The commutator requires the difference between these two, and after consulting figure 3.1, can be seen in the limit $y \to z$ to simply result in the y contour integral around the point z of the OPE $T(z)\phi(y)$ (with a similar discussion for the antiholomorphic case):

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \frac{1}{2\pi i} \oint \left(\{T(y)\phi(z,\bar{z})\}\epsilon(y)dy + \{\bar{T}(\bar{y})\phi(z,\bar{z})\}\bar{\epsilon}(\bar{y})d\bar{y}\right). \tag{3.23}$$

The result should simply be the infinitesimal version of the defining equation (3.18), which the reader should check is:

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \left(h\frac{\partial\epsilon}{\partial z}\phi + \epsilon\frac{\partial\phi}{\partial z}\right) + \left(\bar{h}\frac{\partial\bar{\epsilon}}{\partial\bar{z}}\phi + \bar{\epsilon}\frac{\partial\phi}{\partial\bar{z}}\right). \tag{3.24}$$

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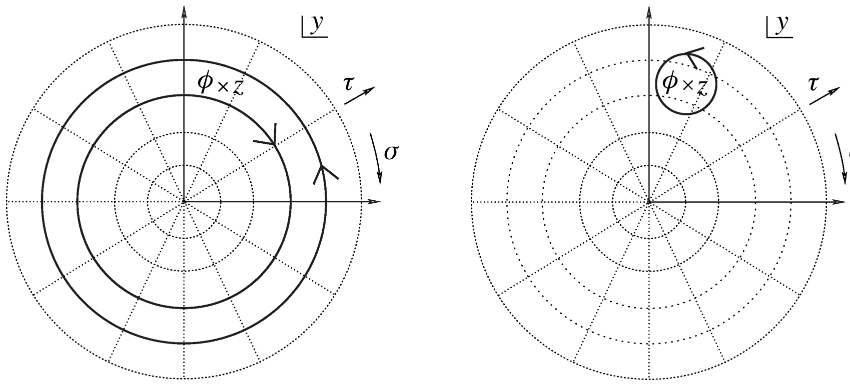

Fig. 3.1. Computing the commutator between the generator Q, defined as a contour in the y-plane, and the operator ϕ, inserted at z. The result in the limit $y \to z$ is on the right.

This defines the operator product expansions $T(z)\phi(z,\bar{z})$ and $\bar{T}(\bar{z})\phi(z,\bar{z})$ for us as:

$$\begin{aligned} T(y)\phi(z,\bar{z}) &= \frac{h}{(y-z)^2}\phi(z,\bar{z}) + \frac{1}{(y-z)}\partial_z\phi(z,\bar{z}) + \cdots \\ \bar{T}(\bar{y})\phi(z,\bar{z}) &= \frac{\bar{h}}{(\bar{y}-\bar{z})^2}\phi(z,\bar{z}) + \frac{1}{(\bar{y}-\bar{z})}\partial_z\phi(z,\bar{z}) + \cdots, \end{aligned} \tag{3.25}$$

where the ellipsis indicates that we have ignored parts which are regular (analytic). These OPEs constitute an alternative definition of a primary field with holomorphic and antiholomorphic weights $h, \bar{h}$, often referred to simply as an $(h, \bar{h})$ primary.

We are at liberty to Laurent expand the infinitesimal transformation around $(z, \bar{z}) = 0$:

$$\epsilon(z) = -\sum_{n=-\infty}^{\infty} a_n z^{n+1}, \qquad \bar{\epsilon}(\bar{z}) = -\sum_{n=-\infty}^{\infty} \bar{a}_n \bar{z}^{n+1},$$

where the $a_n, \bar{a}_n$ are coefficients. The quantities which appear as generators, $\ell_n = z^{n+1}\partial_z, \bar{\ell}_n = \bar{z}^{n+1}\partial_{\bar{z}}$, satisfy the commutation relations

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m}, \\ [\ell_n, \bar{\ell}_m] &= 0, \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m}, \end{aligned} \tag{3.26}$$

which is the classical version of the Virasoro algebra we saw previously in equation (2.63), or the quantum case in equation (2.71) with the central extension, $c = \bar{c} = 0$.

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Now we can compare with what we learned here. It should be clear after some thought that $\ell_{-1}, \ell_0, \ell_1$ and their antiholomorphic counterparts form the six generators of the global conformal transformations generating $SL(2,\mathbb{C}) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. In fact, $\ell_{-1} = \partial_z$ and $\bar{\ell}_{-1} = \partial_{\bar{z}}$ generate translations, $\ell_0 + \bar{\ell}_0$ generates dilations, $i(\ell_0 - \bar{\ell}_0)$ generates rotations, while $\ell_1 = z^2\partial_z$ and $\bar{\ell}_1 = \bar{z}^2\partial_{\bar{z}}$ generate the special conformal transformations.

Let us note some useful pieces of terminology and physics here. Recall that we had defined physical states to be those annihilated by the $\ell_n, \bar{\ell}_n$ with $n > 0$. Then ℓ_0 and $\bar{\ell}_0$ will measure properties of these physical states. Considering them as operators, we can find a basis of ℓ_0 and $\bar{\ell}_0$ eigenstates, with eigenvalues h and $\bar{h}$ (two independent numbers), which are the 'conformal weights' of the state: $\ell_0|h\rangle = h|h\rangle$, $\bar{\ell}_0|\bar{h}\rangle = \bar{h}|\bar{h}\rangle$. Since the sum and difference of these operators are the dilations and the rotations, we can characterise the scaling dimension and the spin of a state or field as $\Delta = h + \bar{h}$, $s = h - \bar{h}$.

It is worth noting here that the stress-tensor itself is *not* in general a primary field of weight $(2,2)$, despite the suggestive fact that it has two indices. There can be an anomalous term, allowed by the symmetries of the theory:

$$\begin{aligned} T(z)T(y) &= \frac{c}{2}\frac{1}{(z-y)^4} + \frac{2}{(z-y)^2}T(y) + \frac{1}{z-y}\partial_y T(y), \\ \bar{T}(\bar{z})\bar{T}(\bar{y}) &= \frac{\bar{c}}{2}\frac{1}{(\bar{z}-\bar{y})^4} + \frac{2}{(\bar{z}-\bar{y})^2}\bar{T}(\bar{y}) + \frac{1}{\bar{z}-\bar{y}}\partial_{\bar{y}} \bar{T}(\bar{y}). \end{aligned} \tag{3.27}$$

The holomorphic conformal anomaly c and its antiholomorphic counterpart $\bar{c}$, can in general be non-zero. We shall see this occur below.

It is worthwhile turning some of the above facts into statements about commutation relation between the modes of $T(z), \bar{T}(\bar{z})$, which we remind the reader are defined as:

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} L_n z^{-n-2}, \qquad L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1}T(z), \\ \bar{T}(\bar{z}) &= \sum_{n=-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2}, \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{z}^{n+1}\bar{T}(\bar{z}). \end{aligned} \tag{3.28}$$

In these terms, the resulting commutator between the modes is that displayed in equation (2.71), with D replaced by $\bar{c}$ and c on the right and left.

The definition (3.24) of the primary fields ϕ translates into

$$[L_n, \phi(y)] = \frac{1}{2\pi i}\oint dz z^{n+1}T(z)\phi(y) = h(n+1)y^n\phi(y) + y^{n+1}\partial_y\phi(y). \tag{3.29}$$

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It is useful to decompose the primary into its modes:

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-h}, \qquad \phi_n = \frac{1}{2\pi i}\oint dz\; z^{h+n-1}\phi(z). \tag{3.30}$$

In terms of these, the commutator between a mode of a primary and of the stress tensor is:

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m}, \tag{3.31}$$

with a similar antiholomorphic expression. In particular this means that our correspondence between states and operators can be made precise with these expressions. $L_0|h\rangle = h|h\rangle$ matches with the fact that $\phi_{-h}|0\rangle = |h\rangle$ would be used to make a state, or more generally $|h,\bar{h}\rangle$, if we include both holomorphic and antiholomorphic parts. The result $[L_0, \phi_{-h}] = h\phi_{-h}$ guarantees this.

In terms of the finite transformation of the stress tensor under $z \to z'$, the result (3.27) is

$$T(z) = \left(\frac{\partial z'}{\partial z}\right)^2 T(z') + \frac{c}{12}\left(\frac{\partial z'}{\partial z}\right)^{-2}\left[\frac{\partial z'}{\partial z}\frac{\partial^3 z'}{\partial z^3} - \frac{3}{2}\left(\frac{\partial^2 z'}{\partial z^2}\right)^2\right], \tag{3.32}$$

where the quantity multiplying $c/12$ is called the 'Schwarzian derivative', $S(z,z')$. It is interesting to note (and the reader should check) that for the $SL(2,\mathbb{C})$ subgroup, the proper global transformations, $S(z,z') = 0$. This means that the stress tensor is in fact a quasi-primary field, but not a primary field.

3.2 Revisiting the relativistic string

Now we see the full role of the energy-momentum tensor which we first encountered in the previous chapter. Its Laurent coefficients there, L_n and $\bar{L}_n$, realised there in terms of oscillators, satisfied the Virasoro algebra, and so its role is to generate the conformal transformations. We can use it to study the properties of various operators in the theory of interest to us.

First, we translate our result of equation (2.44) into the appropriate coordinates here:

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'} :\partial_z X^\mu(z)\partial_z X_\mu(z):, \\ \bar{T}(\bar{z}) &= -\frac{1}{\alpha'} :\partial_{\bar{z}} X^\mu(\bar{z})\partial_{\bar{z}} X_\mu(\bar{z}):. \end{aligned} \tag{3.33}$$

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We can use here our definition (3.20) of the normal ordering at the operator level here, which we construct with the OPE. To do this, we need to know the result for the OPE of ∂X^μ with itself. This we can get by observing that the propagator of the field $X^\mu(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$ is

$$\begin{aligned}\langle X(z)^\mu X^\nu(y)\rangle &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-y),\\ \langle \bar{X}(\bar{z})^\mu \bar{X}^\nu(\bar{y})\rangle &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log(\bar{z}-\bar{y}).\end{aligned} \tag{3.34}$$

By taking a couple of derivatives, we can deduce the OPE of $\partial_z X^\mu(z)$ or $\partial_{\bar{z}}\bar{X}^\mu(\bar{z})$:

$$\begin{aligned}\partial_z X^\mu(z)\partial_y X^\nu(y) &= -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-y)^2} + \cdots\\ \partial_{\bar{z}}\bar{X}^\nu(\bar{z})\partial_{\bar{y}}\bar{X}^\mu(\bar{y}) &= -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(\bar{z}-\bar{y})^2} + \cdots.\end{aligned} \tag{3.35}$$

So in the above, we have, using our definition of the normal ordered expression using the OPE (see discussion below equation (3.20)):

$$T(z) = -\frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) := -\frac{1}{\alpha'}\lim_{y\to z}\left[\partial_z X^\mu(z)\partial_z X_\mu(y) - \frac{D}{(\bar{z}-\bar{y})^2}\right], \tag{3.36}$$

with a similar expression for the antiholomorphic part. It is now straightforward to evaluate the OPE of $T(z)$ and $\partial_z X^\nu(y)$. We simply extract the singular part of the following:

$$\begin{aligned}T(z)\partial_y X^\nu(y) &= \frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) : \partial_y X^\nu(y)\\ &= 2\cdot\frac{1}{\alpha'}\partial_z X^\mu(z)\langle\!\langle\partial_z X_\mu(z)\partial_z X^\nu(y)\rangle\!\rangle + \cdots\\ &= \partial_z X^\nu(z)\frac{1}{(z-y)^2} + \cdots.\end{aligned} \tag{3.37}$$

In the above, we were instructed by Wick to perform the two possible contractions to make the correlator. The next step is to Taylor expand for small $(z-y)$: $X^\nu(z) = X^\nu(y) + (z-y)\partial_y X^\nu(y) + \cdots$, substitute into our result, to give:

$$T(z)\partial_y X^\nu(y) = \frac{\partial_y X^\nu(y)}{(z-y)^2} + \frac{\partial_y^2 X^\nu(y)}{z-y} + \cdots, \tag{3.38}$$

and so we see from our definition in equation (3.25) that that the field $\partial_z X^\nu(z)$ is a primary field of weight $h = 1$, or a $(1,0)$ primary

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field, since from the OPEs (3.35), its OPE with $\bar{T}$ obviously vanishes. Similarly, the antiholomorphic part is a $(0,1)$ primary. Notice that we should have suspected this to be true given the OPE we deduced in (3.35).

Another operator we used last chapter was the normal ordered exponentiation $V(z) =: \exp(i\mathbf{k}\cdot\mathbf{X}(z)) :$, which allowed us to represent the momentum of a string state. Here, the normal ordering means that we should not contract the various Xs which appear in the expansion of the exponential with each other. We can extract the singular part to define the OPE with $T(z)$ by following our noses and applying the Wick procedure as before:

$$
\begin{aligned}
T(z)V(y) &= \frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) :: e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\
&= \frac{1}{\alpha'}(\langle \partial_z X^\mu(z) i\mathbf{k}\cdot\mathbf{X}(y)\rangle)^2 : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\
&\qquad\qquad + 2\cdot\frac{1}{\alpha'}\partial_z X^\mu(z)\langle \partial_z X_\mu(z) i\mathbf{k}\cdot\mathbf{X}(y)\rangle : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\
&= \frac{\alpha' k^2}{4}\frac{1}{(z-y)^2} : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : + \frac{i\mathbf{k}\cdot\partial_z\mathbf{X}(z)}{(z-y)} : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\
&= \frac{\alpha' k^2}{4}\frac{V(y)}{(z-y)^2} + \frac{\partial_y V(y)}{(z-y)}.
\end{aligned}
\tag{3.39}
$$

We have Taylor expanded in the last line, and throughout we only displayed explicitly the singular parts. The expressions tidy up themselves quite nicely if one realises that the worst singularity comes from when there are two contractions with products of fields using up both pieces of $T(z)$. Everything else is either non-singular, or sums to reassemble the exponential after combinatorial factors have been taken into account. This gives the first term of the second line. The second term of that line comes from single contractions. The factor of two comes from making two choices to contract with one or other of the two identical pieces of $T(z)$, while there are other factors coming from the n ways of choosing a field from the term of order n from the expansion of the exponential. After dropping the non-singular term, the remaining terms (with the n) reassemble the exponential again. (The reader is advised to check this explicitly to see how it works.) The final result (when combined with the antiholomorphic counterpart) shows that $V(y)$ is a primary field of weight $(\alpha' k^2/4, \alpha' k^2/4)$.

Now we can pause to see what this all means. Recall from section 2.4.1 that the insertion of states is equivalent to the insertion of operators into the theory, so that:

$$S \to S' = S + \lambda \int d^2z \mathcal{O}(z, \bar{z}). \tag{3.40}$$

In general, we may consider such an operator insertion for a general theory. For the theory to remain conformally invariant, the operator must be a *marginal* operator, which is to say that $\mathcal{O}(z, \bar{z})$ must at least have dimension $(1, 1)$ do that the integrated operator is dimensionless. In principle, the dimension of the operator after the deformation (i.e. in the new theory defined by S') can change, and so the full condition for the operator is that it must remain $(1, 1)$ after the insertion (see insert 3.1). It in fact defines a direction in the space of couplings, and λ can be thought of as an infinitessimal motion in that direction. The statement of the existence of a marginal operator is then referred to the existence of a 'flat direction'.

In the first instance, we recall that the use of the tachyon vertex operator $V(z, \bar{z})$ corresponds to the addition of $\int d^2z\, V(z, \bar{z})$ to the action. We wish the theory to remain conformal (preserving the relativistic string's symmetries, as stressed in chapter 1), and so $V(z, \bar{z})$ must be (1,1). In fact, since our conformal field theory is actually free, we need do no more to check that the tachyon vertex is marginal. So we require that $(\alpha' k^2/4, \alpha' k^2/4) = (1, 1)$. Therefore we get the result that $M^2 \equiv -k^2 = -4/\alpha'$, the result that we obtained previously for the tachyon.

Another example is the level one closed string vertex operator:

$$: \partial_z X^\mu \partial_{\bar{z}} X^\nu \exp(i\mathbf{k} \cdot \mathbf{X}) : .$$

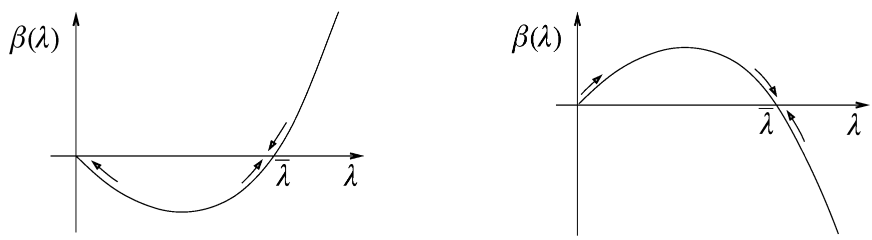
It turns out that there are no further singularities in contracting this with the stress tensor, and so the weight of this operator is $(1 + \alpha' k^2/4, 1 + \alpha' k^2/4)$. So, marginality requires that $M^2 \equiv -k^2 = 0$, which is the massless result that we encountered earlier.

Another computation that the reader should consider doing is to work out explicitly the $T(z)T(y)$ OPE, and show that it is of the form (3.27) with $c = D$, as each of the D bosons produces a conformal anomaly of unity. This same is true from the antiholomorphic sector, giving $\bar{c} = D$. Also, for open strings, we get the same amount for the anomaly. This result was alluded to in chapter 2. This is problematic, since this conformal anomaly prevents the full operation of the string theory. In particular, the anomaly means that the stress tensor's trace does not in fact vanish quantum mechanically.

This is all repaired in the next section, since there is another sector which we have not yet considered.

Insert 3.1. Deformations, RG flows, and CFTs

A useful picture to have in mind for later use is of a conformal field theory as a 'fixed point' in the space of theories coordinatised by the coefficients of possible operators such as in equation (3.40). (There is an infinite set of such perturbations and so the space is infinite dimensional.) In the usual reasoning using the renomalisation group (RG), once the operator is added with some value of the coupling, the theory (i.e. the value of the coupling) flows along an RG trajectory as the energy scale μ is changed. The 'β-function', $\beta(\lambda) \equiv \mu\partial\lambda/\partial\mu$ characterises the behaviour of the coupling. One can imagine the existence of *'fixed points'* of such flows, where $\beta(\lambda) = 0$ and the coupling tends to a specific value, as shown in the diagram.



On the left, $\bar{\lambda}$ is an 'infra-red (IR) fixed point', since the coupling is driven to it for decreasing μ, while on the right, $\bar{\lambda}$ is an 'ultra-violet (UV) fixed point', since the coupling is driven to it for increasing μ. The origins of each diagram of course define a fixed point of the opposite type to that at $\bar{\lambda}$. A conformal field theory is then clearly such a fixed point theory, where the scale dependence of all couplings exactly vanishes. A *'marginal operator'* is an operator which when added to the theory, does not take it away from the fixed point. A *'relevant operator'* deforms a theory increasingly as μ goes to the IR, while an *'irrelevant operator'* is increasingly less important in the IR. This behaviour is reversed on going to the UV. When applied to a fixed point, such non-marginal operators can be used to deform fixed point theories away from the conformal point, often allowing us to find other interesting theories, as we will do in later chapters. $D = 4$ Yang–Mills theories, for sufficiently few flavours of quark (like QCD), have negative β-function, and so behave roughly as the neighbourhood of the origin in the left diagram. *'Asymptotic freedom'* is the process of being driven to the origin (zero coupling) in the UV. Later, we will see examples of both type of fixed point theory.

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3.3 Fixing the conformal gauge

It must not be forgotten where all of the riches of the previous section – the conformal field theory – came from. We made a gauge choice in equation (2.41) from which many excellent results followed. However, despite everything, we saw that there is in fact a conformal anomaly equal to D (or a copy each on both the left and the right hand side, for the closed string). The problem is that we have not made sure that the gauge fixing was performed properly. This is because we are fixing a local symmetry, and it needs to be done dynamically in the path integral, just as in gauge theory. This is done with Faddeev–Popov ghosts in a very similar way to the methods used in field theory. Let us not go into the details of it here, but assume that the interested reader can look into the many presentations of the procedure in the literature. The key difference with field theory approach is that it introduces two ghosts, c^a and b_{ab} which are rank one and rank two tensors on the world sheet. The action for them is:

$$S^{\rm gh} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}c^c\nabla_a b_{bc}, \tag{3.41}$$

and so b_{ab} and c^a, which are anticommuting, are conjugates of each other.

3.3.1 Conformal ghosts

Once the conformal gauge has been chosen, (see equation (2.41)) picking the diagonal metric, we have

$$S^{\rm gh} = -\frac{1}{2\pi}\int d^2z\,\left(c(z)\partial_{\bar z}b(z) + \bar c(\bar z)\partial_z \bar b(\bar z)\right). \tag{3.42}$$

From equation (3.41), the stress tensor for the ghost sector is:

$$T^{\rm gh}(z) =: c(z)\partial_z b(z) : + : 2(\partial_z c(z))b(z) :, \tag{3.43}$$

with a similar expression for $\bar T_{\rm ghost}(\bar z)$. Just as before, as the ghosts are free fields, with equations of motion $\partial_z c = 0 = \partial_z b$, we can Laurent expand them as follows:

$$b(z) = \sum_{n=-\infty}^{\infty} b_n\, z^{-n-2}, \qquad c(z) = \sum_{n=-\infty}^{\infty} c_n\, z^{-n+1}, \tag{3.44}$$

which follows from the property that b is of weight 2 and c is of weight -1, a fact which might be guessed from the structure of the action (3.41). The quantisation yields

$$\{b_m, c_n\} = \delta_{m+n}. \tag{3.45}$$

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and the stress tensor is

$$L_n^{\rm gh} = \sum_{m=-\infty}^{\infty} (2n-m) : b_m c_{n-m} : -\delta_{n,0}, \tag{3.46}$$

where we have a normal ordering constant -1, as in the previous sector,

$$[L_m^{\rm gh}, b_n] = (m-n) b_{m+n}, \qquad [L_m^{\rm gh}, c_n] = -(2m+n) c_{m+n}. \tag{3.47}$$

The OPE for the ghosts is given by

$$\begin{aligned} b(z)c(y) &= \frac{1}{(z-y)} + \cdots, \qquad & c(z)b(y) &= \frac{1}{(z-y)} + \cdots, \\ b(z)b(y) &= O(z-y), & c(z)c(y) &= O(z-y), \end{aligned} \tag{3.48}$$

where the second expression is obtained from the first by the anticommuting property of the ghosts. The second line also follows from the anticommuting property. There can be no non-zero result for the singular parts there.

As with everything for the closed string, we must supplement the above expressions with very similar ones referring to $\bar{z}$, $\bar{c}(\bar{z})$ and $\bar{b}(\bar{z})$. For the open string, we carry out the same procedures as before, defining everything on the upper half-plane, reflecting the holomorphic into the anti-holomorpic parts, defining a single set of ghosts (see also insert 3.2).

3.3.2 The critical dimension

Now comes the fun part. We can evaluate the conformal anomaly of the ghost system, by using the techniques for computation of the OPE that we refined in the previous section. We can do it for the ghosts in as simple a way as for the ordinary fields, using the expression (3.43) above. In the following, we will focus on the most singular part, to isolate the conformal anomaly term. This will come from when there are two contractions in each term. The next level of singularity comes from one contraction, and so on:

$$\begin{aligned} &T^{\rm gh}(z)T^{\rm gh}(y) \\ &\quad = (: \partial_z b(z) c(z) : + : 2b(z)\partial_z c(z) :)(: \partial_y b(y) c(y) : + : 2b(y)\partial_y c(y) :) \\ &\quad = : \partial_z b(z) c(z) :: \partial_y b(y) c(y) : +2 : b(z)\partial_z c(z) :: \partial_y b(y) c(y) : \\ &\quad + 2 : \partial_z b(z) c(z) :: b(y)\partial_y c(y) : +4 : b(z)\partial_z c(z) :: b(y)\partial_y c(y) : \\ &\quad = \langle \partial_z b(z) c(y)\rangle\langle c(z)\partial_y b(y)\rangle + 2\langle b(z)c(y)\rangle\langle \partial_z c(z)\partial_y b(y)\rangle \\ &\quad + 2\langle \partial_z b(z)\partial_y c(y)\rangle\langle c(z) b(y)\rangle + 4\langle b(z)\partial_y c(y)\rangle\langle \partial_z c(z) b(y)\rangle \\ &\quad = -\frac{13}{(z-y)^4}, \end{aligned} \tag{3.49}$$

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Insert 3.2. Further aspects of conformal ghosts

Notice that the flat space expression (3.42) is also consistent with the stress tensor

$$T(z) =: \partial_z b(z)c(z) : -\kappa : \partial_z[b(z)c(z)] :, \tag{3.50}$$

for arbitrary κ, with a similar expression for the antiholomorphic sector. It is a useful exercise to use the OPEs of the ghosts given in equation (3.48) to verify that this gives b and c conformal weights $h = \kappa$ and $h = 1 - \kappa$, respectively. The case we studied above was $\kappa = 2$. Further computation (recommended) reveals that the conformal anomaly of this system is $c = 1 - 3(2\kappa - 1)^2$, with a similar expression for the antiholomorphic version of the above.

The case of fermionic ghosts will be of interest to us later. In that case, the action and stress tensor are just like before, but with $b \to \beta$ and $c \to \gamma$, where β and γ, are *fermionic*. Since they are fermionic, they have singular OPEs

$$\beta(z)\gamma(y) = -\frac{1}{(z-y)} + \cdots, \qquad \gamma(z)\beta(y) = \frac{1}{(z-y)} + \cdots. \tag{3.51}$$

A computation gives conformal anomaly $3(2\kappa - 1)^2 - 1$, which in the case $\kappa = 3/2$, gives an anomaly of 11. In this case, they are the 'superghosts', required by supersymmetry in the construction of superstrings later on.

and so comparing with equation (3.27), we see that the ghost sector has conformal anomaly $c = -26$. A similar computation gives $\bar{c} = -26$.

So recalling that the 'matter' sector, consisting of the D bosons, has $c = \bar{c} = D$, we have achieved the result that the conformal anomaly vanishes in the case $D = 26$. This also applies to the open string in the obvious way.

3.4 The closed string partition function

We have all of the ingredients we need to compute our first one-loop diagram[†]. It will be useful to do this as a warm up for more complicated

[†] Actually, we've had them for some time now, essentially since chapter 2.

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examples later, and in fact we will see structures in this simple case which will persist throughout.

Consider the closed string diagram of figure 3.2(a). This is a vacuum diagram, since there are no external strings. This torus is clearly a one loop diagram and in fact it is easily computed. It is distinguished topologically by having two completely independent one-cycles. To compute the path integral for this we are instructed, as we have seen, to sum over all possible metrics representing all possible surfaces, and hence all possible tori.

Well, the torus is completely specified by giving it a flat metric, and a complex structure, τ, with $\mathrm{Im}\tau \geq 0$. It can be described by the lattice given by quotienting the complex w-plane by the equivalence relations

$$w \sim w + 2\pi n; \quad w \sim w + 2\pi m\tau, \tag{3.52}$$

for any integers m and n, as shown in figure 3.2(b). The two one-cycles can be chosen to be horizontal and vertical. The complex number τ specifies the *shape* of a torus, which cannot be changed by infinitesimal diffeomorphisms of the metric, and so we must sum over all all of them. Actually, this naive reasoning will make us overcount by a lot, since in fact there are a lot of τs which define the same torus. For example, clearly for a torus with given value of τ, the torus with $\tau + 1$ is the same torus, by the equivalence relation (3.52). The full family of equivalent tori can be reached from any τ by the 'modular transformations':

$$\begin{aligned} T: &\quad \tau \to \tau + 1 \\ S: &\quad \tau \to -\frac{1}{\tau}, \end{aligned} \tag{3.53}$$

which generate the group $SL(2,\mathbb{Z})$, which is represented here as the group

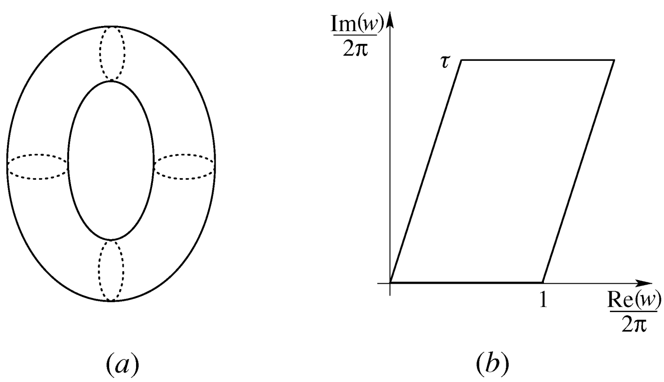

Fig. 3.2. (a) A closed string vacuum diagram. (b) The flat torus and its complex structure.

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of 2×2 unit determinant matrices with integer elements:

$$SL(2,\mathbb{Z}): \quad \tau \to \frac{a\tau + b}{c\tau + d}; \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \qquad (3.54)$$

(It is worth noting that the map between tori defined by S exchanges the two one-cycles, therefore exchanging space and (Euclidean) time.) The full family of inequivalent tori is given not by the upper half-plane $H_\perp$ (i.e. τ such that $\mathrm{Im}\tau \geq 0$) but the quotient of it by the equivalence relation generated by the group of modular transformations. This is $\mathcal{F} = H_\perp / PSL(2,\mathbb{Z})$, where the P reminds us that we divide by the extra $\mathbb{Z}_2$ which swaps the sign on the defining $SL(2,\mathbb{Z})$ matrix, which clearly does not give a new torus. The commonly used fundamental domain in the upper half-plane corresponding to the inequivalent tori is drawn in figure 3.3. Any point outside that can be mapped into it by a modular transformation.

The fundamental region $\mathcal{F}$ is properly defined as follows: Start with the region of the upper half-plane which is in the interval $(-\frac{1}{2}, +\frac{1}{2})$ and above the circle of unit radius. we must then identify the two vertical edges, and also the two halves of the remaining segment of the circle. This produces a space which is smooth everywhere except for two points about which there are conical singularities, described in insert 3.3.

The string propagation on our torus can be described as follows. Imagine that the string is of length 1, and lies horizontally. Mark a point on the string. Running time upwards, we see that the string propagates for a time $t = 2\pi \mathrm{Im}\tau \equiv 2\pi\tau_2$. Once it has got to the top of the diagram, we see that

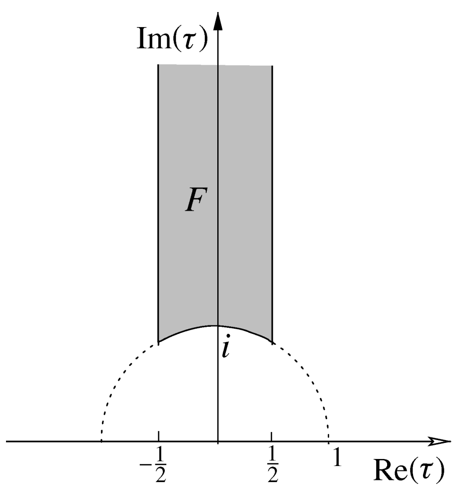

Fig. 3.3. The space of inequivalent tori.

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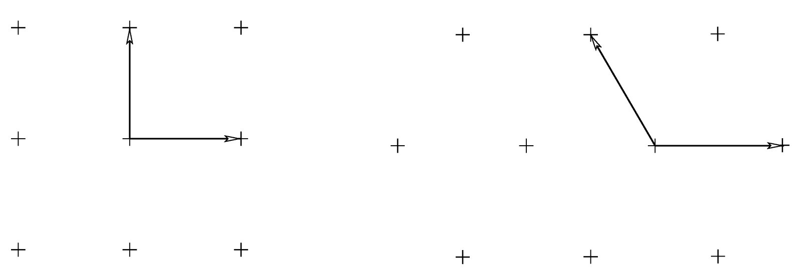
Insert 3.3. Special points in the moduli space of tori

Actually, there are two very special points of interest on $\mathcal{F}$, depicted in figure 3.3. They can be clearly seen in the figure. The point $\tau = i$ and the point $\tau = e^{\frac{2\pi i}{3}}$, which is one sharp corner (its mirror image is also visible). The significance of these points is that they are fixed points of certain elements of $SL(2,\mathbb{Z})$. The point $\tau = i$ is fixed by the element S, while the other point is fixed by the element ST.

These points are 'orbifold' singularities, a term that will become more widely used here after chapter 4. For our purposes here, this means that they have a conical deficit angle. For example, the point $\tau = i$, because it is at the tip of a region formed by folding the plane in half (remember we identified the two halves of the circle segment), has a deficit angle of π. In other words, because of the folding, one only needs to go half way around a circle in order to return to where one started. Similalry, the other orbifold point has a deficit angle of $4\pi/3$: one only needs to go a third of the way around a circle in order to return to where one started.

One may visualise the significance of these points, recalling that we make the tori from lattices in the plane. The lattices for these two points have special, and familiar, symmetry. The $\tau = i$ point is simply a *square* lattice, and S is in fact just a $\pi/2$ rotation. Notice that $S^4 = 1$, which fits with this fact nicely. The $\tau = e^{\frac{2\pi i}{3}}$ point is an *hexagonal* lattice, and ST is a rotation by $\pi/3$, which dovetails nicely with the relation $(ST)^6 = 1$. We draw the lattice below, with appropriate basis vectors. It might be worth studying the action of S and ST, and considering the tori to which they correspond.



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our marked point has shifted rightwards by an amount $x = 2\pi\text{Re}\tau \equiv 2\pi\tau_1$. We actually already have studied the operators that perform these two operations. The operator for time translations is the Hamiltonian (2.64), $H = L_0 + \bar{L}_0 - (c+\bar{c})/24$ while the operator for translations along the string is the momentum $P = L_0 - \bar{L}_0$ discussed above equation (2.73). Recall that $c = \bar{c} = D-2 = 24$. So our vacuum path integral is

$$Z = \text{Tr}\left\{e^{-2\pi\tau_2 H}e^{2\pi i\tau_1 P}\right\} = \text{Tr}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}. \tag{3.55}$$

Here, $q \equiv e^{2\pi i\tau}$, and the trace means a sum over everything which is discrete and an integral over everything which is continuous, which in this case, is simply τ. This is easily evaluated, as the expressions for L_0 and $\bar{L}_0$ give a family of simple geometric sums (see insert 3.4 (p. 92)), and the result can be written as:

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(q), \quad \text{where} \tag{3.56}$$

$$Z(q) = |\tau_2|^{-12}(q\bar{q})^{-1}\left|\prod_{n=1}^{\infty}(1-q^n)^{-24}\right|^2 = (\sqrt{\tau_2}\eta\bar{\eta})^{-24}, \tag{3.57}$$

is the 'partition function', with Dedekind's function

$$\eta(q) \equiv q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,; \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{3.58}$$

This is a pleasingly simple result. One very interesting property it has is that it is actually 'modular invariant'. It is invariant under the T transformation in equation (3.52), since under $\tau \to \tau + 1$, we get that $Z(q)$ picks up a factor $\exp(2\pi i(L_0 - \bar{L}_0))$. This factor is precisely unity, as follows from the level matching formula (2.73). Invariance of $Z(q)$ under the S transformation $\tau \to -1/\tau$ follows from the property mentioned in equation(3.58), after a few steps of algebra, and using the result $S: \ \tau_2 \to \tau_2/|\tau|^2$.

Modular invariance of the partition function is a crucial property. It means that we are correctly integrating over all inequivalent tori, which is required of us by diffeomorphism invariance of the original construction. Furthermore, we are counting each torus only once, which is of course important.

Note that $Z(q)$ really deserves the name 'partition function' since if it is expanded in powers of q and $\bar{q}$, the powers in the expansion – after multiplication by $4/\alpha'$ – refer to the $(\text{mass})^2$ level of excitations on the left

Insert 3.4. Partition functions

It is not hard to do the sums. Let us look at one dimension, and so one family of oscillators α_n. We need to consider

$$\mathrm{Tr}\, q^{L_0} = \mathrm{Tr}\, q^{\sum_{n=0}^{\infty} \alpha_{-n}\alpha_n}.$$

We can see what the operator $q^{\sum_{n=0}^{\infty} \alpha_{-n}\alpha_n}$ means if we write it explicitly in a basis of all possible multiparticle states of the form $\alpha_{-n}|0\rangle$, $(\alpha_{-n})^2|0\rangle$, etc.:

$$q^{\alpha_{-n}\alpha_n} = \begin{pmatrix} 1 & & & & \\ & q^n & & & \\ & & q^{2n} & & \\ & & & q^{3n} & \\ & & & & \ddots \end{pmatrix},$$

and so clearly $\mathrm{Tr} q^{\alpha_{-n}\alpha_n} = \sum_{i=1}^{\infty} (q^n)^i = (1-q^n)^{-1}$, which is remarkably simple! The final sum over all modes is trivial, since

$$\mathrm{Tr}\, q^{\sum_{n=0}^{\infty} \alpha_{-n}\alpha_n} = \prod_{n=0}^{\infty} \mathrm{Tr}\, q^{\alpha_{-n}\alpha_n} = \prod_{n=0}^{\infty} (1-q^n)^{-1}.$$

We get a factor like this for all 24 dimensions, and we also get contributions from both the left and right to give the result.
Notice that if our modes were fermions, ψ_n, things would be even simpler. We would not be able to make multiparticle states $(\psi_{-n})^2|0\rangle$, (Pauli), and so we only have a 2×2 matrix of states to trace in this case, and so we simply get

$$\mathrm{Tr}\, q^{\psi_{-n}\psi_n} = (1+q^n).$$

Therefore the partition function is

$$\mathrm{Tr}\, q^{\sum_{n=0}^{\infty} \psi_{-n}\psi_n} = \prod_{n=0}^{\infty} \mathrm{Tr} q^{\psi_{-n}\psi_n} = \prod_{n=0}^{\infty} (1+q^n).$$

We will encounter such fermionic cases later.

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and right, while the coefficient in the expansion gives the degeneracy at that level. The degeneracy is the number of partitions of the level number into positive integers. For example, at level three this is three, since we have $\alpha_{-3}, \alpha_{-1}\alpha_{-2}$, and $\alpha_{-1}\alpha_{-1}\alpha_{-1}$.

The overall factor of $(q\bar{q})^{-1}$ sets the bottom of the tower of masses. Note for example that at level zero we have the tachyon, which appears only once, as it should, with $M^2 = -4/\alpha'$. At level one, we have the massless states, with multiplicity 24^2, which is appropriate, since there are 24^2 physical states in the graviton multiplet $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$. Introducing a common piece of terminology, a term $q^{w_1}\bar{q}^{w_2}$, represents the appearance of a 'weight' (w_1, w_2) field in the 1+1 dimensional conformal field theory, denoting its left-moving and right-moving weights or 'conformal dimensions'.

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