TWO ISOMORPHS OF THE FOUR-COLOUR PROBLEM

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To Professor H. S. M. Coxeter on his sixtieth birthday

It is known that the four-colour problem for the faces of a map on a sphere is isomorphic with the four-colour problem for the vertices of its dual, and the problem is here discussed in the latter form. The isomorphs described below are concerned with codes for the four colours and for a change in colour as we pass along an edge from vertex to vertex. In the first (algebraic) isomorph, the coding involves the four fourth roots of unity, and leads to a graphical representation in the complex plane. In the second (arithmetical) isomorph, the coding involves the integers mod 4, and also the face-edge incidence matrix of the dual. The two isomorphs may be said to be logarithmically related. In each of them the problem is to assign colouring instructions on the edges to satisfy consistency conditions so that, after completing a circuit of edges, we restore its original colour to the vertex from which we started.

1. The map M and its dual D. Let M be a map on the surface of a sphere. The number of faces is finite, and three faces meet at each vertex. To form the dual D, we choose a point in each face of M and join the points in adjacent faces by curves which meet only at those points. D is the graph so obtained. The four-colour problem for M in the usual sense (colour the faces) is isomorphic with the four-colour problem for D in the sense that the vertices of D are to be coloured with not more than four colours, two vertices joined by an edge receiving different colours.

Let us then henceforth think only of D, and use the words *face*, *edge*, and *vertex* in reference to D, not M. Let F, E, and V denote the numbers of faces, edges, and vertices. The faces of D are triangular, and so $3F = 2E$. Combining this with Euler's equation $F - E + V = 2$, we see that there exists an integer n (we may call it the *order* of D) such that

$$F = 2n, \qquad E = 3n, \qquad V = n + 2. \tag{1.1}$$

We orient the faces by assigning a positive sense of circulation to the boundary of each face, and we orient the edges by assigning a positive sense to each edge. Then any face and any edge together define an incidence index I defined as $I = 0$ if the edge does not belong to the face, and $I = \pm 1$ if the edge belongs to the face and the senses agree or disagree, respectively.

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There is no simple canonical plan for orienting the edges, but we can and shall orient the faces by using the plane projection of the dual, and choosing the counterclockwise sense as positive for all faces except the one that goes to infinity; we orient it by taking the clockwise sense on its (inner) boundary to be positive. By using this canonical orientation we ensure that if two faces share a common edge, the product of the two incidence indices is -1.

2. The algebraic isomorph. Let K_1, K_2, K_3, K_4 be four colours. We code them by association with the four fourth roots of unity:

$$K_1 \leftrightarrow i, \qquad K_2 \leftrightarrow i^2, \qquad K_3 \leftrightarrow i^3, \qquad K_4 \leftrightarrow i^4 = 1, \tag{2.1}$$

or any permutation of this association.

We now assign *colouring instructions* to the oriented edges by assigning to each edge a number j belonging to the set (i, i^2, i^3). These instructions are to be used as follows.

Let v, v' be two vertices joined by an edge with the instruction j. Let a colour c (in the code (2.1)) be assigned to v. Then we are to give to v' the colour $c' = jc$ if the sense $v \rightarrow v'$ is the positive sense of the edge, and the colour $c' = j^{-1}c$ if $v \rightarrow v'$ is the negative sense of the edge.

This ensures that the colours of v and v' are different. Moreover, the instructions are consistent in the sense that if we go from v to v' and then back to v with the same instruction, then the final colour of v is the same as its initial colour.

Suppose now that we have assigned instructions j arbitrarily (in the range i, i^2, i^3) to all the edges. Choose any vertex v and assign a colour to it. Take any open path formed by edges, without loops. If we follow the instructions along this path, we give to all vertices on it colours which are uniquely determined, and the colours of consecutive vertices differ.

But if we modify the above by taking a loop of edges, beginning and ending at v, in general the final colour of v will differ from the initial colour. In brief, arbitrary instructions are in general inconsistent. In order that a set of instructions assigned to the edges may lead to an unambiguous colouring of all the vertices in four (or less) colours, it is necessary and sufficient that certain *consistency conditions* be satisfied.

To examine these conditions, consider any face. Denote by I_1, I_2, I_3 its incidence indices with the edges which bound it. Let j_1, j_2, j_3 be the colour-instructions assigned to those edges. Then, if we start from any vertex of the face with colour c, and go round the boundary in the positive sense, on completion of the circuit we arrive back at that vertex with colour

$$c' = wc, \tag{2.2}$$

where

$$w = j_1^{I_1} j_2^{I_2} j_3^{I_3}. \tag{2.3}$$

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Let us call w the *weight* of the face. It can take only the four values ± 1, $\pm i$. Obviously the consistency condition for the face is $w = 1$.

We may then state the following theorem.

THEOREM I. *The four-colour problem is isomorphic to the problem of assigning colour-instructions j from the set i, i^2, i^3 to the edges so as to make the weight of every face unity.*

It is easy to see that the condition $w = 1$ for every face is equivalent to the single condition

$$\sum w = 2n, \tag{2.4}$$

where $\sum$ denotes summation over all the $2n$ faces. For suppose that there are x faces with $w = 1$, y faces with $w = -1$, z faces with $w = i$, and t faces with $w = -i$. Then

$$\sum w = x - y + i(z - t), \qquad x + y + z + t = 2n. \tag{2.5}$$

Thus (2.4) is true if and only if $x = 2n, y = z = t = 0$.

For any choice of colour-instructions, we have a complex number $\sum w$. To explore its behaviour, write

$$\sum w = \xi + i\eta, \qquad \xi = x - y, \eta = z - t, \tag{2.6}$$

x, y, z, t being non-negative integers satisfying the second of (2.5). Hence,

$$|\pm\xi \pm \eta| \leqslant 2n. \tag{2.7}$$

This tells us that, if we plot the point $\sum w$ in the complex plane, it cannot lie outside the square with vertices $(\pm 2n, \pm 2ni)$.

If, for any given map, the four-colour conjecture is true, then there exists a set of colour-instructions, say $\{j\}$, which puts the point $\sum w$ in the complex plane at the point $2n$, viz. the right-hand extremity of the square (2.7). If, for that same map, we started with a different set of instructions, say $\{j'\}$, yielding a point elsewhere in the square, then we could carry this point to the desired position by changing the instructions one at a time. To treat the matter systematically, it seems desirable to make some canonical choice of $\{j'\}$, and the choice we shall make is $j' = i^2 = -1$ for all edges. This choice is invariant under changes in the orientations of the faces and edges, because $(-1)^{-1} = -1$. It gives $\sum w = -2n$, which means that we start from that point of the square (2.7) most distant from the goal $\sum w = 2n$. We may call it the *worst choice.*

We now have a systematic plan to test the four-colour conjecture for any given map. Start with the worst choice, $j' = i^2$ for every edge, and $\sum w = -2n$. Number the edges $1, 2, \ldots, 3n$. There are 3^{3n} possible choices of instructions in the range (i, i^2, i^3), and each such choice gives us a point in the complex plane corresponding to the value of $\sum w$. If the conjecture is true for the map in question, this set of points includes $\sum w = 2n$; if false, the set does not contain that point.

We may think of the matter in terms of a game played on the integer lattice points of the complex plane. We start in the worst position, $\sum w = -2n$ with all instructions i^2, and alter the instructions one by one, moving a counter from the worst position to the successive positions indicated by the complex value of $\sum w$. The player seeks to attain the point $\sum w = 2n$. If the four-colour conjecture is true for the map in question, then that goal is attainable in a finite number of moves, and the prize would go to the player who minimizes that number.

If the four-colour conjecture should happen to be false for the map in question, the game becomes more interesting. The aim of the player is not to attain $\sum w = 2n$ (we suppose that to be impossible), but to carry his counter as far as possible to the right in the complex plane. For there exists a limit line L

$$\operatorname{Re} \sum w = M < 2n, \tag{2.8}$$

where M is an integer, such that at least one lattice point on L is attainable but no lattice point to the right of L is attainable. The player has to get his counter on to L, and we might require him to minimize the number of moves required.

In fact, every map may be said to possess a non-negative *index of uncolourability*

$$U = 2n - M, \tag{2.9}$$

such that $U = 0$ is necessary and sufficient for colourability in four (or less) colours.

3. Sidelights on the algebraic isomorph. The reduction of the four-colour problem to the game described above might raise a wild hope that we have a solution near at hand. For, to prove the four-colour conjecture, all we have to do is this: Show that, when the counter is in any position (other than $\sum w = 2n$), it can be moved to the right by a change in the colour-instructions. Such hopes, however, are not fulfilled. It may be interesting, however, to examine this game a little further.

THEOREM II. *The product of all the weights is unity*:

$$\Pi w = 1. \tag{3.1}$$

This is easy to prove. In the worst position we have $w = -1$ for each face, and the number of faces is even; thus (3.1) is true for the worst position. The change from that position to a position corresponding to instructions $\{j\}$ can be effected step by step, changing one instruction at each step. This change affects only two faces. If it multiplies the weight of one face by k, it multiplies the other by k^{-1}. In fact, any change in instructions leaves Πw invariant Thus, since (3.1) holds for one set of instructions, it holds for all.

THEOREM III. *In the complex plane, $\sum w$ can occupy only lattice points for which the coordinates are even integers.*

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To prove this, we use the notation of (2.5). We have

$$\Pi w = 1^x(-1)^y i^z(-i)^t = i^{2y+z+3t}, \tag{3.2}$$

and so, by (3.1),

$$2y + z + 3t = 4p, \tag{3.3}$$

where p is zero (but only if $\sum w = 2n$) or a positive integer. Solving for (x, y, z, t) the four equations

$$\begin{gathered} x + y + z + t = 2n, \\ x - y = \xi, \qquad z - t = \eta, \\ 2y + z + 3t = 4p, \end{gathered} \tag{3.4}$$

we get

$$\begin{aligned} x &= 2n - 2p - \tfrac{1}{2}\eta, \\ y &= 2n - 2p - \xi - \tfrac{1}{2}\eta, \\ z &= -n + 2p + \tfrac{1}{2}\xi + \eta, \\ t &= -n + 2p + \tfrac{1}{2}\xi. \end{aligned} \tag{3.5}$$

Hence, since (x, y, z, t) are integers, ξ and η are even integers; since

$$\sum w = \xi + i\eta,$$

Theorem III follows.

THEOREM IV. *Assume that for a certain map the four-colour conjecture is false so that a limit line L exists. Let $\sum w = P$ be an attainable point on L. Then the instructions leading to P cannot give to any pair of adjacent faces any one of the following pairs of weights*:

$$(-1, -1), \quad (-1, i), \quad (-1, -i). \tag{3.6}$$

Proof. Let w_1 and w_2 be the two weights in question. There is no essential loss of generality in assuming an incidence index 1 for the common edge and the first face. Let j $(= i, i^2, \text{or } i^3)$ be the instruction on this edge. Let us change this instruction to j', in the same range. This change affects only the two weights in question, changing them into

$$w'_1 = w_1 j'/j, \qquad w'_2 = w_2 j/j'. \tag{3.7}$$

If $j = i$, we may take $j' = i^2$ or i^3, so that $j'/j = i$ or i^2, and so

$$w'_1 = iw_1, \quad w'_2 = -iw_2 \qquad \text{or} \qquad w'_1 = -w_1, \quad w'_2 = -w_2. \tag{3.8}$$

If $j = i^2$, we may take $j' = i$ or i^3, so that $j'/j = -i$ or i, and so

$$w'_1 = -iw_1, \quad w'_2 = iw_2 \qquad \text{or} \qquad w'_1 = iw_1, \quad w'_2 = -iw_2. \tag{3.9}$$

Finally, if $j = i^3$, we may take $j' = i$ or i^2, so that $j'/j = -1$ or $-i$, and so

$$w'_1 = -w_1, \quad w'_2 = -w_2 \qquad \text{or} \qquad w'_1 = -iw_1, \quad w'_2 = iw_2. \tag{3.10}$$

Suppose now that $w_1 = -1$, $w_2 = -1$, as in the first of (3.6), j having any one of the values (i, i^2, i^3). According to the value of j, we use one of the transformations (3.8), (3.9), (3.10). It is clear that we can make Re $(w'_1 + w'_2)$ greater than Re $(w_1 + w_2)$, the increase being either 2 or 4. Thus the pair $(-1, -1)$ cannot occur. The proof for the other pairs in (3.6) is similar.

THEOREM V. *If a map is coloured in four colours, each face (of the dual) has the instruction i^2 on precisely one of its (three) edges, and in all there are precisely n edges carrying the instruction i^2.*

Proof. Allowing for orientations, it is obvious that, to obtain a weight $w = 1$ for a face, one edge must carry the instruction i^2 and the other two must carry instructions taken from the pair (i, i^3). If all three edges have the incidence index 1, the instructions must be (i, i, i^2) or (i^2, i^3, i^3). There are $2n$ faces. Each edge with instruction i^2 is common to two faces, and so the number of such edges is n. The theorem is proved.

When a map can be done in four colours, we may start with the instructions

$$i^2, i^2, \ldots, i^2 \tag{3.11}$$

and change these into

$$j_1, j_2, \ldots, j_{3n}. \tag{3.12}$$

By Theorem V, we know that n of the instructions (3.11) are to remain unchanged. Hence we have

THEOREM VI. *If a map can be done in four colours, the transition from the worst point $\sum w = -2n$ to a solution $\sum w = 2n$ can be effected in $2n$ steps, each step involving a change in one instruction and carrying the representative point 2 units to the right, with possibly simultaneous displacement in the imaginary direction.*

4. The arithmetical isomorph. This isomorph of the four-colour problem may be regarded as logarithmically related to the algebraic isomorph discussed above, but it is clearer to develop it independently with occasional back references.

We code the four colours by means of the integers 1, 2, 3, 4 (mod 4). The colour instructions on the oriented edges are the integers 1, 2, 3. These instructions are used as follows.

Let v, v' be two vertices. Let the edge joining them carry the instruction e. Let the colour c be assigned to v. Then we are to give to v' the colour $c' = c + e$ or $c' = c - e$ according as the sense $v \rightarrow v'$ agrees or disagrees with the positive sense of the edge.

As for the algebraic isomorph, random instructions will, in general, be inconsistent. To find the consistency conditions, we use the incidence matrix familiar in electrical circuit theory.

It may be remarked here that, in the interests of simplicity, in §1 we treated orientation rather summarily. It is possible to assign orientations arbitrarily to all three elements—faces, edges, and vertices. Then we have three incidence matrices, conveniently denoted as follows:

$$[FE] = \text{face-edge incidence matrix},$$

$$[EV] = \text{edge-vertex incidence matrix},$$

$$[VF] = \text{vertex-face matrix}.$$

The elements of these matrices are 0, ± 1, the sign depending on relative orientations.

Here we use only the matrix $[FE]$. It has $2n$ rows (one for each face) and $3n$ columns (one for each edge). The elements are the incidence indices (0, ± 1) of faces and edges, as discussed earlier. Since each edge belongs to two faces, and each face has three edges, the matrix $[FE]$ has precisely two non-zero entries in each column and precisely three non-zero entries in each row. Adopting the previous plan for orientation of the faces by the counterclockwise rule, we see that of the two column entries, one is $+1$ and the other -1.

There is another property arising from the fact that two faces have at most one edge in common: when $[FE]$ is written out in the usual matrix form, no rectangle is formed by four non-zero elements.

In the algebraic isomorph we defined the weight of a face as the product of the instructions on its three edges, with due allowance for orientations by use of incidence indices. We proceed similarly now, replacing multiplication by addition. However, we can save a lot of verbiage by making use of the formula

$$F = [FE]E, \tag{4.1}$$

properly interpreted.

To interpret (4.1), we recall that in (1.1) F and E stood for the numbers of faces and edges. But we do not really need that symbolism since the numbers in question are expressible in terms of the order n. Let us henceforth take E to represent a column matrix with $3n$ elements, these elements being the instructions (1, 2, or 3) assigned to the edges, which we suppose numbered off $1, 2, \ldots, 3n$. Now (4.1) instructs us to multiply this column matrix E by the matrix $[FE]$. The result is a shorter column matrix F with $2n$ elements. Thus, if the faces are numbered off $1, 2, \ldots, 2n$, F gives us a number for each face. These numbers generated by E, we call the *weights* of the faces in the arithmetical isomorph.

But when we follow colouring instructions round the three edges which bound a face, and wonder whether we return to the same colour as that we started with, we find the answer directly in F. We have in fact the consistency condition

$$F = (0, \bmod 4), \tag{4.2}$$

this notation meaning that each element is zero (mod 4). If (4.2) is satisfied, the original colour is restored on passing round any loop, and the condition is necessary for this restoration. Hence we have

THEOREM VII. *The four-colour problem is isomorphic to the problem of finding a column matrix E with $3n$ elements in the range* (1, 2, 3) *such that the elements of the column matrix $F = [FE]\,E$ are all zero* (mod 4).

In the algebraic isomorph, we considered the worst position ($\sum w = -2n$) in the complex plane, and the problem of proceeding from that point to the desired point ($\sum w = 2n$). The analogue in the arithmetical isomorph is the worst matrix

$$E' = (2), \tag{4.3}$$

that is, a column with each element 2. From the stated properties of the matrix $[FE]$ it follows that

$$F' = [FE]E' = (2, \bmod 4), \tag{4.4}$$

which is about as far as possible from the desired result (4.2).

Let us now set $E = E' + E''$ with E' as above and E'' arbitrary, and apply (4.1) to obtain the corresponding weights. We get

$$F = F' + F'', \tag{4.5}$$

where F' is as in (4.4) and

$$F'' = [FE]E''. \tag{4.6}$$

In view of (4.4), the consistency condition (4.2) demands

$$F'' = (2, \bmod 4). \tag{4.7}$$

We had the elements of E' equal to 2, and we require the elements of E to be in the range (1, 2, 3). Hence the elements of E'' are to be in the range $(0, \pm 1)$, and we have

THEOREM VIII. *The four-colour problem is isomorphic to the problem of finding a column matrix E'' with $3n$ elements in the range* $(0, \pm 1)$ *such that when we calculate*

$$F'' = [FE]E'', \tag{4.8}$$

each element of F'' is 2 (mod 4).

5. Conclusion. It is a pleasure to dedicate this note to Professor H. S. M. Coxeter, F.R.S., in recognition of his achievements in geometry and as a tribute to a long friendship in which he has shown sympathy and consideration to a rather old-fashioned geometer.

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