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AN ALMOST SURE LIMIT THEOREM FOR THE MAXIMA OF MULTIVARIATE STATIONARY GAUSSIAN SEQUENCES

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Abstract

An almost sure limit theorem for the maxima of multivariate stationary Gaussian sequences is proved under some mild conditions.

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1. Introduction

Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables with $E(X) = 0$, $\mathrm{Var}(X) = 1$ and partial sums $S_n = \sum_{k=1}^{n} X_k$. The simplest version of the almost sure central limit theorem (ASCLT) for partial sums says that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left(\frac{S_k}{\sqrt{k}} \le x\right) = \Phi(x) \quad \text{a.s.} \tag{1.1}$$

for all x, where $I(A)$ is the indicator function of the event A and $\Phi(x)$ stands for the standard Gaussian distribution; see [2, 4, 11, 14]. The functional version of the ASCLT for partial sums is treated in [3]. Fazekas and Chuprunov [9] provide examples where the ASCLT holds but the usual limit theorem does not.

Berkes and Csáki [2] consider the ASCLT for partial maxima of independent and identically distributed random variables; this was also treated previously by Fahrner and Stadtmüller [8] and independently by Cheng *et al.* [6]. The ASCLT for partial

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maxima states that if there exist real sequences $a_n > 0$, $b_n \in \mathbb{R}$ and a nondegenerate distribution $G(x)$ such that $P(M_n < a_n^{-1}x + b_n) \xrightarrow{d} G(x)$, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\big(a_k(M_k - b_k) < x\big) = G(x) \quad \text{a.s.} \tag{1.2}$$

for all x, where $M_n = \max_{1\le i\le n} X_i$. There are a few results on dependent random variables in the literature. For example, in [7] Csáki and Gonchigdanzan considered the ASCLT for the maximum of certain stationary Gaussian sequences under some weak dependence conditions, whereas Chen and Lin [5] extended (1.2) to some nonstationary Gaussian sequences.

In this paper we extend the ASCLT to multivariate stationary Gaussian sequences, under some regularity conditions. To our knowledge, this is the first result to appear on the topic in the multivariate setting.

Throughout the paper, let $\{Z_i = (Z_i(1), Z_i(2), \ldots, Z_i(d)) : i \ge 1\}$ be a standardized stationary Gaussian vector sequence with

$$EZ_n = (EZ_n(p) = 0 : p = 1, \ldots, d), \qquad \mathrm{Var}\, Z_n = (\mathrm{Var}\, Z_n(p) = 1 : p = 1, \ldots, d),$$
$$r_{ij}(p) = \mathrm{Cov}(Z_i(p), Z_j(p)) = r_{|j-i|}(p),$$
$$r_{ij}(p, q) = \mathrm{Cov}(Z_i(p), Z_j(q)) = r_{|j-i|}(p, q).$$

We write $M_n = (M_n(1), \ldots, M_n(d))$ and $M_n(p) = \max_{1\le i\le n} Z_i(p)$, and shall always take $1 \le p \ne q \le d$; $u_n = (u_n(1), \ldots, u_n(d))$ will be a real vector, and $u_n > u_k$ means $u_n(p) > u_k(p)$ for $p = 1, \ldots, d$, while $a \ll b$ stands for $a = O(b)$.

2. The main results

THEOREM 2.1. *Let $Z_1, Z_2, \ldots$ be a standardized stationary Gaussian vector sequence which satisfies:*

(a) $r_n(p) \to 0$ *as* $n \to \infty$ *for* $1 \le p \le d$, *and* $\max_{p\ne q} \big(\sup_{n\ge 0} |r_n(p, q)|\big) < 1$;

(b) *for some* $\gamma > 2$ *and* $\varepsilon > 0$,

$$\frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} (|r_k(p)| \log k) \exp(\gamma |r_k(p)| \log k) \ll (\log\log n)^{-(1+\varepsilon)}, \tag{2.1}$$

$$\frac{1}{n} \sum_{1\le p\ne q\le d} \sum_{k=1}^{n} (|r_k(p, q)| \log k) \exp(\gamma |r_k(p, q)| \log k) \ll (\log\log n)^{-(1+\varepsilon)}. \tag{2.2}$$

Then, as $n(1 - \Phi(u_n(p))) \to \tau_p$ *for* $0 \le \tau_p < \infty$ *and* $p = 1, \ldots, d$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \le u_k) = \prod_{p=1}^{d} e^{-\tau_p} \quad \textit{a.s.}$$

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In particular, for $u_n(p) = a_n^{-1}x_p + b_n$ *where* $x_p \in \mathbb{R}$ *and* $p = 1, 2, \ldots, d$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \le u_k) = \prod_{p=1}^{d} e^{-e^{-x_p}} \quad a.s.,$$

with $a_n = (2\log n)^{1/2}$ *and* $b_n = a_n - (1/2)a_n^{-1}\log(4\pi \log n)$.

THEOREM 2.2. *Let* $Z_1, Z_2, \ldots$ *be a standardized stationary Gaussian vector sequence with*

$$r_n(p) \log n(\log\log n)^{1+\varepsilon} = O(1), \tag{2.3}$$

$$\max_{p\neq q}\Big(\sup_{n\ge 0} |r_n(p,q)|\Big) < 1, \quad r_n(p,q) \log n(\log\log n)^{1+\varepsilon} = O(1), \tag{2.4}$$

for some $\varepsilon > 0$ *when* $1 \le p \ne q \le d$. *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \le u_k) = \prod_{p=1}^{d} e^{-\tau_p} \quad a.s. \tag{2.5}$$

as $n(1 - \Phi(u_n(p))) \to \tau_p$ *for* $0 \le \tau_p < \infty$ *and* $p = 1, \ldots, d$. *In particular, for* $u_n(p) = a_n^{-1}x_p + b_n$ *where* $x_p \in \mathbb{R}$ *and* $p = 1, 2, \ldots d$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \le u_k) = \prod_{p=1}^{d} \exp(-e^{-x_p}) \quad a.s. \tag{2.6}$$

REMARK. Theorems 2.1 and 2.2 are multivariate versions of [7, Theorem 1.1], and Theorem 2.2 is an obvious consequence of Theorem 2.1.

3. Proofs

For the proof of the main results, we need the following lemmas.

LEMMA 3.1. *Let* $\{\xi_n\}_{n=1}^{\infty}$ *and* $\{\eta_n\}_{n=1}^{\infty}$ *be* d*-dimensional standardized stationary Gaussian sequences with*

$$r_{ij}^0(p) = \mathrm{Cov}(\xi_i(p), \xi_j(p)), \quad r_{ij}^0(p,q) = \mathrm{Cov}(\xi_i(p), \xi_j(q))$$

and

$$r'_{ij}(p) = \mathrm{Cov}(\eta_i(p), \eta_j(p)), \quad r'_{ij}(p,q) = \mathrm{Cov}(\eta_i(p), \eta_j(q)).$$

Write

$$\rho_{ij}(p) = \max(|r_{ij}^0(p)|, |r'_{ij}(p)|), \quad \rho_{ij}(p,q) = \max(|r_{ij}^0(p,q)|, |r'_{ij}(p,q)|),$$

and let $u_i = (u_i(1), u_i(2), \ldots, u_i(d))$, $i \ge 1$, *be real vectors. If*

$$\max_{\substack{1\le i<j\le n\\ 1\le p\le d}} (\rho_{ij}(p)) < 1 \quad \text{and} \quad \max_{\substack{1\le i\le j\le n\\ 1\le p\ne q\le d}} (\rho_{ij}(p,q)) < 1,$$

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then

$$
\begin{aligned}
&|P(\xi_j \le u_j \ \forall j = 1, \ldots, n) - P(\eta_j \le u_j \ \forall j = 1, \ldots, n)| \\
&\quad \le K_1 \sum_{p=1}^{d} \sum_{1 \le i < j \le n} |r_{ij}^0(p) - r'_{ij}(p)| \exp\left(-\frac{u_i^2(p) + u_j^2(p)}{2(1 + \rho_{ij}(p))}\right) \\
&\qquad + K_2 \sum_{1 \le p \ne q \le d} \sum_{1 \le i \le j \le n} |r_{ij}^0(p, q) - r'_{ij}(p, q)| \exp\left(-\frac{u_i^2(p) + u_j^2(q)}{2(1 + \rho_{ij}(p, q))}\right) \quad (3.1)
\end{aligned}
$$

where K_1, K_2 are absolute constants.

PROOF. This is a special case of the normal comparison lemma [12, see Theorem 4.2.1]. □

LEMMA 3.2. *Let $Z_1, Z_2, \ldots$ be a standardized stationary Gaussian vector sequence such that conditions (a) and (b) of Theorem 2.1 hold, and further suppose that $n(1 - \Phi(u_n(p)))$ is bounded for $p = 1, \ldots, d$. Then, for some $\varepsilon > 0$,*

$$
\sup_{1 \le k \le n} k \sum_{p=1}^{d} \sum_{j=1}^{n} |r_j(p)| \exp\left(-\frac{u_k^2(p) + u_n^2(p)}{2(1 + |r_j(p)|)}\right) \ll (\log \log n)^{-(1+\varepsilon)}, \quad (3.2)
$$

$$
\sup_{1 \le k \le n} k \sum_{1 \le p \ne q \le d} \sum_{j=1}^{n} |r_j(p, q)| \exp\left(-\frac{u_k^2(p) + u_n^2(q)}{2(1 + |r_j(p, q)|)}\right) \ll (\log \log n)^{-(1+\varepsilon)}. \quad (3.3)
$$

PROOF. This is the multivariate version of [7, Lemma 2.1]. Here we may assume, for each $p = 1, 2, \ldots, d$, that $n(1 - \Phi(u_n(p))) = C_p$ for some constant C_p [12, Lemma 4.3.2]. Also,

$$
\begin{aligned}
&k \sum_{1 \le p \ne q \le d} \sum_{j=1}^{n} |r_j(p, q)| \exp\left(-\frac{u_k^2(p) + u_n^2(q)}{2(1 + |r_j(p, q)|)}\right) \\
&\quad = k \sum_{1 \le p \ne q \le d} \sum_{j=1}^{[n^\beta]} |r_j(p, q)| \exp\left(-\frac{u_k^2(p) + u_n^2(q)}{2(1 + |r_j(p, q)|)}\right) \\
&\qquad + k \sum_{1 \le p \ne q \le d} \sum_{j=[n^\beta]+1}^{n} |r_j(p, q)| \exp\left(-\frac{u_k^2(p) + u_n^2(q)}{2(1 + |r_j(p, q)|)}\right) \\
&\quad =: A_1 + B_1,
\end{aligned}
$$

where $0 < \beta < 2/\gamma$. Similarly to [7, proof of Lemma 2.1], we find that

$$
A_1 \ll (\log \log n)^{-(1+\varepsilon)}
$$

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and

$$\begin{aligned} B_1 &\leq \frac{1}{n\beta} \sum_{j=1}^{n} |r_j(p,q)|(\log j) \exp(\gamma |r_j(p,q)| \log j) \\ &\ll (\log\log n)^{-(1+\varepsilon)}. \end{aligned}$$

Thus, (3.3) holds. The proof of (3.2) follows similar arguments and is therefore omitted. □

LEMMA 3.3. *Let $Z_1, Z_2, \ldots$ be a d-dimensional standardized stationary Gaussian sequence with $r_n(p) \to 0$ as $n \to \infty$ for $1 \leq p \leq d$, and $\max_{p\neq q}(\sup_{n\geq 0} |r_n(p,q)|) < 1$. Assume that, for some constant $\gamma > 2$,*

$$\frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} |r_k(p)| \log k \exp(\gamma |r_k(p)| \log k) \to 0, \tag{3.4}$$

$$\frac{1}{n} \sum_{1\leq p\neq q\leq d} \sum_{k=1}^{n} |r_k(p,q)| \log k \exp(\gamma |r_k(p,q)| \log k) \to 0. \tag{3.5}$$

Then, as $n(1-\Phi(u_n(p))) \to \tau_p$ for $0 \leq \tau_p < \infty$ and $p = 1, \ldots, d$,

$$\lim_{n\to\infty} P(M_n \leq u_n) = \prod_{p=1}^{d} e^{-\tau_p}. \tag{3.6}$$

Also, for $u_n(p) = a_n^{-1} x_p + b_n$ where $x_p \in \mathbb{R}$ and $p = 1, \ldots, d$,

$$\lim_{n\to\infty} P(M_n \leq u_n) \to \prod_{p=1}^{d} \exp(-e^{-x_p}). \tag{3.7}$$

PROOF. From Lemma 3.1 and arguments similar to those used in [12, proof of Lemma 4.5.1], we obtain

$$\lim_{n\to\infty} \left| P(M_n \leq u_n) - \prod_{p=1}^{d} \Phi^n(u_n(p)) \right| = 0.$$

Note that

$$\lim_{n\to\infty} \prod_{p=1}^{d} \Phi^n(u_n(p)) = \prod_{p=1}^{d} \exp(-e^{-x_p}),$$

and the result follows.

From Lemma 3.2 and arguments similar to those used in [7, proofs of Lemmas 2.3 and 2.4], we get the following two lemmas. □

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LEMMA 3.4. *Under the conditions of Theorem 2.1,*

$$\Big|\mathrm{Cov}\big(I(M_k \le u_k),\, I(M_{k,n} \le u_n)\big)\Big| \ll (\log\log n)^{-(1+\varepsilon)},$$

where $M_{k,n} = (M_{k,n}(p) = \max_{k+1\le i\le n}(Z_i(p)) : p = 1, 2, \ldots, d)$.

LEMMA 3.5. *Under the conditions of Theorem 2.1,*

$$E\Big|I(M_n \le u_n) - I(M_{k,n} \le u_n)\Big| \ll \frac{k}{n} + (\log\log n)^{-(1+\varepsilon)}.$$

PROOF OF THEOREM 2.1. If

$$\mathrm{Var}\bigg(\sum_{k=1}^{n} \frac{1}{k}\alpha_k\bigg) \ll (\log n)^2(\log\log n)^{-(1+\varepsilon)}, \tag{3.8}$$

where $\alpha_k = I(M_k \le u_k)$, then the result follows from [7, Lemma 3.1] and Lemma 3.3 above.

From Lemmas 3.4 and 3.5 above, we have, for $1 \le k < l \le n$,

$$\begin{aligned}
|\mathrm{Cov}(\alpha_k, \alpha_l)| &= |\mathrm{Cov}(I(M_k \le u_k), I(M_l \le u_l))| \\
&\le |\mathrm{Cov}(I(M_k \le u_k), I(M_l \le u_l) - I(M_{k,l} \le u_l))| \\
&\quad + |\mathrm{Cov}(I(M_k \le u_k), I(M_{k,l} \le u_l))| \\
&\le 2E|I(M_l \le u_l) - I(M_{k,l} \le u_l)| + |\mathrm{Cov}(I(M_k \le u_k), I(M_{k,l} \le u_l))| \\
&\ll \frac{k}{l} + (\log\log n)^{-(1+\varepsilon)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{Var}\bigg(\sum_{k=1}^{n} \frac{1}{k}\alpha_k\bigg) &= \sum_{k=1}^{n} \frac{1}{k^2}\mathrm{Var}(\alpha_k) + 2\sum_{1\le k<l\le n} \frac{\mathrm{Cov}(\alpha_k, \alpha_l)}{kl} \\
&\le \sum_{k=1}^{n} \frac{1}{k^2} + 2\sum_{1\le k<l\le n} \frac{\mathrm{Cov}(\alpha_k, \alpha_l)}{kl} \\
&\le \sum_{k=1}^{n} \frac{1}{k^2} + 2\sum_{1\le k<l\le n} \frac{1}{kl}\bigg(\frac{k}{l} + (\log\log l)^{-(1+\varepsilon)}\bigg) \\
&\ll \log n + \sum_{l=3}^{n} \frac{1}{l(\log\log l)^{(1+\epsilon)}} \sum_{k=1}^{l-1} \frac{1}{k} \\
&\ll (\log n)^2(\log\log n)^{-(1+\varepsilon)}.
\end{aligned}$$

The proof of Theorem 2.1 is therefore complete.

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