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CHARACTERIZING WARPED-PRODUCT LAGRANGIAN IMMERSIONS IN COMPLEX PROJECTIVE SPACE

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Abstract Starting from two Lagrangian immersions and a horizontal curve in $S^3(1)$, it is possible to construct a new Lagrangian immersion, which we call a warped-product Lagrangian immersion. In this paper, we find two characterizations of warped-product Lagrangian immersions. We also investigate Lagrangian submanifolds which attain at every point equality in the improved version of Chen's inequality for Lagrangian submanifolds of $\mathbb{C}P^n(4)$ as discovered by Oprea. We show that, for $n \geqslant 4$, an n-dimensional Lagrangian submanifold in $\mathbb{C}P^n(4)$ for which equality is attained at all points is necessarily minimal.

Keywords: Lagrangian submanifold; complex projective space; Chen's δ invariant

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1. Introduction

In the early 1990s Chen [**6**] introduced a new invariant, δ_M, for a Riemannian manifold M. Specifically, $\delta_M : M \to \mathbb{R}$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

where

$$(\inf K)(p) = \inf\{K(\pi) \mid \pi \text{ is a two-dimensional subspace of } T_pM\},$$

with $K(\pi)$ being the sectional curvature of π, and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature defined in terms of an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM of M at p. In the same paper, he discovered, for submanifolds of real space forms, an inequality relating this invariant with the length of the mean curvature vector H. A similar inequality was proved in [**7**, **8**] for n-dimensional Lagrangian submanifolds

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of a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Indeed, it was shown that

$$\delta_M \leqslant \frac{(n-2)(n+1)}{2}\frac{c}{4} + \frac{n^2}{2}\frac{n-2}{n-1}\|H\|^2. \tag{1.1}$$

Note that, for $n = 2$, both sides of the above inequality are zero.

Let $\mathbb{C}P^n(4)$ denote complex projective n-space of constant holomorphic sectional curvature 4. For $n \geqslant 3$, Lagrangian submanifolds of $\mathbb{C}P^n(4)$ attaining equality in (1.1) at every point were studied in, amongst others, [**2**, **3**, **7**, **8**]. In particular, in [**7**, **8**], it was shown that such submanifolds are minimal, and a complete classification was obtained of three-dimensional Lagrangian submanifolds of $\mathbb{C}P^3(4)$ attaining at each point equality in (1.1). Such submanifolds are obtained by starting from minimal surfaces with ellipse of curvature a circle in the unit 5-sphere.

However, Oprea [**9**] has recently shown that the inequality (1.1) is not optimal and, for $n \geqslant 3$, can be improved to

$$\delta_M \leqslant \frac{(n-2)(n+1)}{2}\frac{c}{4} + \frac{n^2}{2}\frac{2n-3}{2n+3}\|H\|^2. \tag{1.2}$$

A careful analysis of [**9**] shows that equality is attained if the second fundamental form satisfies certain properties (described in detail in § 3).

It turns out that many of these properties are also satisfied by *warped-product* Lagrangian immersions into $\mathbb{C}P^n(4)$. We construct such immersions from two Lagrangian immersions $\phi_i : N_i \to \mathbb{C}P^{n_i}(4)$, where, for future convenience, we take $i = 2, 3$, and a horizontal curve $\tilde{\alpha}(t) = (\tilde{\alpha}_2(t), \tilde{\alpha}_3(t)) : I \to S^3(1) \subset \mathbb{R}^4 = \mathbb{C}^2$ by taking $\phi = [(\tilde{\alpha}_2\phi_2, \tilde{\alpha}_3\phi_3)] : I \times N_2 \times N_3 \to \mathbb{C}P^{1+n_2+n_3}(4)$, where, for $i = 2, 3$, $\tilde{\phi}_i$ is a horizontal lift [**10**] to $S^{2n_i+1}(1) \subset \mathbb{R}^{2n_i+2} = \mathbb{C}^{n_i+1}$ of ϕ_i. This construction slightly generalizes the construction of [**4**], which itself is analogous to the well-known Calabi product in affine differential geometry.

The purpose of this paper is twofold. Firstly, we will give two characterizations of a generic warped-product immersion in terms of the second fundamental form. Secondly, we will show that, for $n \geqslant 4$, an n-dimensional, Lagrangian submanifold in $\mathbb{C}P^n(4)$ attaining equality in (1.2) at every point is necessarily minimal (in which case (1.2) reduces to the original inequality (1.1) obtained in [**7**, **8**]). Note that this is not the case in dimension 3; a complete classification of the non-minimal three-dimensional case was given in [**1**], while the minimal case was treated in [**2**, **3**]. In particular, the non-minimal examples may be regarded as warped-product immersions for which the second factor degenerates to a point.

The paper is organized as follows. In the next section we use the Codazzi equations to study immersions whose second fundamental form has properties that are common to both problems. In the following sections we then classify submanifolds attaining at every point equality in (1.2), non-minimal warped-product immersions and minimal warped-product immersions, respectively.

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2. A particular second fundamental form

Throughout this section we shall assume that M is a Lagrangian submanifold of the complex projective space $\mathbb{C}P^n(4)$. We shall moreover assume that M admits three mutually orthogonal differentiable distributions, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ of dimension 1, n_2 and n_3, respectively, with $1+n_2+n_3=n$ and

$$\left.\begin{aligned} h(T,T) &= \lambda_1 JT,\\ h(T,V) &= \lambda_2 JV,\\ h(T,W) &= \lambda_3 JW,\\ h(V,W) &= 0 \end{aligned}\right\} \tag{2.1}$$

for all $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$, with T being a unit vector spanning $\mathcal{D}_1$. Here, λ_1, λ_2 and λ_3 are smooth functions on M, and we assume that λ_1, $2\lambda_2$ and $2\lambda_3$ are mutually distinct real numbers at each point of M. It is clear that $\lambda_2 \neq \lambda_3$ is necessary in order to distinguish the distributions $\mathcal{D}_2$ and $\mathcal{D}_3$, but the other conditions are more technical and it would be worthwhile investigating what happens when they are not satisfied.

In the following, V, $\tilde{V}$ and V^* will be vector fields belonging to $\mathcal{D}_2$, and $\{V_1,\dots,V_{n_2}\}$ will be a local smooth moving orthonormal framing of $\mathcal{D}_2$. Similarly, W, $\tilde{W}$, W^* will be vector fields belonging to $\mathcal{D}_3$, and $\{W_1,\dots,W_{n_3}\}$ will be a local smooth moving orthonormal framing of $\mathcal{D}_3$. We recall that, for a Lagrangian immersion, the cubic form

$$C(X,Y,Z) = \langle h(X,Y), JZ\rangle \tag{2.2}$$

is symmetric in X, Y and Z (see [**5**]), while the Codazzi equation states that the cubic form

$$(\nabla h)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \tag{2.3}$$

is also symmetric in X, Y and Z. We now investigate the consequences of the above for the functions λ_1, λ_2, λ_3 and for the connection ∇ of M.

Lemma 2.1. *For any vector $V \in \mathcal{D}_2$, we have*

$$\operatorname{grad}\lambda_1 - (\lambda_1 - 2\lambda_2)\nabla_T T \equiv 0 \bmod \mathcal{D}_2^{\perp}, \tag{2.4}$$

$$(\lambda_1 - 2\lambda_2)\nabla_V T - T(\lambda_2)V - Jh(V,\nabla_T T) \equiv 0 \bmod \mathcal{D}_2^{\perp}, \tag{2.5}$$

$$(\lambda_1 - 2\lambda_3)\nabla_V T - (\lambda_2 - \lambda_3)\nabla_T V \equiv 0 \bmod \mathcal{D}_3^{\perp}. \tag{2.6}$$

Proof. We use the symmetry of (2.2) and (2.3), together with our assumptions (2.1) on the second fundamental form of M. Firstly, using (2.1) and (2.3), we have

$$\begin{aligned}(\nabla h)(V,T,T) &= \nabla^{\perp}_V h(T,T) - 2h(\nabla_V T,T)\\ &= V(\lambda_1)JT + \lambda_1 J\nabla_V T - 2h(\nabla_V T,T).\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}(\nabla h)(T,V,T) &= \nabla^{\perp}_T h(V,T) - h(\nabla_T V,T) - h(V,\nabla_T T)\\ &= T(\lambda_2)JV + \lambda_2 J\nabla_T V - h(\nabla_T V,T) - h(V,\nabla_T T).\end{aligned} \tag{2.8}$$

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Taking inner products of the right-hand sides of (2.7) and (2.8) with JT and comparing the results, we obtain, using the symmetry of (2.2) and (2.3),

$$V(\lambda_1) = (\lambda_1 - 2\lambda_2)\langle \nabla_T T, V\rangle,$$

which implies (2.4). Similarly, comparing the inner products of the right-hand sides of (2.7) and (2.8) with $J\tilde{V}$ for any $\tilde{V} \in \mathcal{D}_2$ shows that the left-hand side of (2.5) is orthogonal to $\mathcal{D}_2$. Finally, comparing the inner products of the right-hand sides of (2.7) and (2.8) with $J\tilde{W}$ for any $\tilde{W} \in \mathcal{D}_3$ shows that the left-hand side of (2.6) is orthogonal to $\mathcal{D}_3$. □

Interchanging the roles of the distributions $\mathcal{D}_2$ and $\mathcal{D}_3$, we obtain the following result.

Lemma 2.2. *For any vector $W \in \mathcal{D}_3$, we have*

$$\operatorname{grad}\lambda_1 - (\lambda_1 - 2\lambda_3)\nabla_T T \equiv 0 \bmod \mathcal{D}_3^{\perp}, \tag{2.9}$$

$$(\lambda_1 - 2\lambda_3)\nabla_W T - T(\lambda_3)W - Jh(W, \nabla_T T) \equiv 0 \bmod \mathcal{D}_3^{\perp}, \tag{2.10}$$

$$(\lambda_1 - 2\lambda_2)\nabla_W T + (\lambda_2 - \lambda_3)\nabla_T W \equiv 0 \bmod \mathcal{D}_2^{\perp}. \tag{2.11}$$

Using similar arguments, from the symmetry of $(\nabla h)(T, V, W)$ we have the following.

Lemma 2.3. *For any vectors $V \in \mathcal{D}_2$, and $W \in \mathcal{D}_3$ we have*

$$\begin{aligned}
(\lambda_2 - \lambda_3)\nabla_V W &\equiv -Jh(V, \nabla_T W) \bmod \mathcal{D}_2^{\perp} && (2.12)\\
&\equiv -W(\lambda_2)V - Jh(V, \nabla_W T) \bmod \mathcal{D}_2^{\perp}, && (2.13)\\
(\lambda_3 - \lambda_2)\nabla_W V &\equiv -Jh(W, \nabla_T V) \bmod \mathcal{D}_3^{\perp} && (2.14)\\
&\equiv -V(\lambda_3)W - Jh(W, \nabla_V T) \bmod \mathcal{D}_3^{\perp}. && (2.15)
\end{aligned}$$

Finally, considering the following consequences of the Codazzi equations,

$$\begin{gathered}
\langle(\nabla h)(T, V, \tilde{V}), JV^*\rangle = \langle(\nabla h)(V, T, \tilde{V}), JV^*\rangle,\\
\langle(\nabla h)(T, W, \tilde{W}), JW^*\rangle = \langle(\nabla h)(W, T, \tilde{W}), JW^*\rangle,\\
\langle(\nabla h)(W, V, \tilde{V}), J\tilde{W}\rangle = \langle(\nabla h)(V, W, \tilde{V}), J\tilde{W}\rangle,
\end{gathered}$$

we may prove the following lemma.

Lemma 2.4. *For any vectors $V, \tilde{V}, V^* \in \mathcal{D}_2$ and $W, \tilde{W}, W^* \in \mathcal{D}_3$ we have*

$$\begin{aligned}
&T(\langle h(V, \tilde{V}), JV^*\rangle) - V(\lambda_2)\langle \tilde{V}, V^*\rangle\\
&\qquad = \sigma(\langle h(V, \tilde{V}), J\nabla_T V^*\rangle) - \langle h(\tilde{V}, V^*), J\nabla_V T\rangle, && (2.16)\\
&T(\langle h(W, \tilde{W}), JW^*\rangle) - W(\lambda_3)\langle \tilde{W}, W^*\rangle\\
&\qquad = \sigma(\langle h(W, \tilde{W}), J\nabla_T W^*\rangle) - \langle h(\tilde{W}, W^*), J\nabla_W T\rangle, && (2.17)\\
&\langle h(V, \tilde{V}), J\nabla_W \tilde{W}\rangle = \langle h(W, \tilde{W}), J\nabla_V \tilde{V}\rangle, && (2.18)
\end{aligned}$$

where σ denotes cyclic summation over V, $\tilde{V}$, V^* and W, $\tilde{W}$, W^*, respectively.

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We now prove a lemma that will be useful later on. In this lemma, we let $\bar{V}$ denote the $\mathcal{D}_2$ component of $\operatorname{grad}\lambda_1$, and $\bar{W}$ the $\mathcal{D}_3$ component of $\operatorname{grad}\lambda_1$.

Lemma 2.5. *For any vectors $V, \tilde{V} \in \mathcal{D}_2$ and $W, \tilde{W} \in \mathcal{D}_3$, we have*

$$W(\lambda_2)V - \frac{\lambda_2}{\lambda_1 - 2\lambda_3}W(\lambda_1)V + \frac{\lambda_1 - \lambda_2 - \lambda_3}{\lambda_1 - 2\lambda_2}\sum_{k=1}^{n_2}\langle \nabla_T W, V_k\rangle Jh(V, V_k) \equiv 0 \bmod \mathcal{D}_2^{\perp},$$

$$V(\lambda_3)W - \frac{\lambda_3}{\lambda_1 - 2\lambda_2}V(\lambda_1)W + \frac{\lambda_1 - \lambda_2 - \lambda_3}{\lambda_1 - 2\lambda_3}\sum_{p=1}^{n_3}\langle \nabla_T V, W_p\rangle Jh(W, W_p) \equiv 0 \bmod \mathcal{D}_3^{\perp},$$

and

$$\begin{aligned}\left(\frac{\lambda_3 T(\lambda_2)}{(\lambda_1 - 2\lambda_2)} - \frac{\lambda_2 T(\lambda_3)}{(\lambda_1 - 2\lambda_3)}\right)&\langle V, \tilde{V}\rangle\langle W, \tilde{W}\rangle \\ &= \frac{\lambda_2(\lambda_1 - \lambda_2 - \lambda_3)\langle h(W, \tilde{W}), J\bar{W}\rangle\langle V, \tilde{V}\rangle}{(\lambda_1 - 2\lambda_3)^2(\lambda_2 - \lambda_3)} + \frac{\lambda_3(\lambda_1 - \lambda_2 - \lambda_3)\langle h(V, \tilde{V}), J\bar{V}\rangle\langle W, \tilde{W}\rangle}{(\lambda_1 - 2\lambda_2)^2(\lambda_2 - \lambda_3)}.\end{aligned}$$

Proof. The first equation is obtained as follows. From (2.12) and (2.13) we have that, for all $\tilde{V} \in \mathcal{D}_2$,

$$W(\lambda_2)\langle V, \tilde{V}\rangle = \langle h(V, \nabla_W T), J\tilde{V}\rangle - \langle h(V, \nabla_T W), J\tilde{V}\rangle. \tag{2.19}$$

However, using (2.1) and the symmetry of (2.2),

$$\begin{aligned}\langle h(V, \nabla_T W), J\tilde{V}\rangle &= \langle h(V, \tilde{V}), J\nabla_T W\rangle \\ &= \sum_{k=1}^{n_2}\langle h(V, \tilde{V}), JV_k\rangle\langle V_k, \nabla_T W\rangle + \langle h(V, \tilde{V}), JT\rangle\langle T, \nabla_T W\rangle \\ &= \sum_{k=1}^{n_2}\langle h(V, \tilde{V}), JV_k\rangle\langle V_k, \nabla_T W\rangle - \lambda_2\langle V, \tilde{V}\rangle\langle W, \nabla_T T\rangle.\end{aligned}$$

Thus, using (2.9), we find that

$$\langle h(V, \nabla_T W), J\tilde{V}\rangle = \sum_{k=1}^{n_2}\langle h(V, \tilde{V}), JV_k\rangle\langle V_k, \nabla_T W\rangle - \frac{\lambda_2}{\lambda_1 - 2\lambda_3}W(\lambda_1)\langle V, \tilde{V}\rangle. \tag{2.20}$$

Similarly, using (2.1),

$$\begin{aligned}\langle h(V, \nabla_W T), J\tilde{V}\rangle &= \langle h(V, \tilde{V}), J\nabla_W T\rangle \\ &= \sum_{k=1}^{n_2}\langle h(V, \tilde{V}), JV_k\rangle\langle V_k, \nabla_W T\rangle.\end{aligned} \tag{2.21}$$

Thus, from (2.19), (2.20) and (2.21), we obtain

$$W(\lambda_2)\langle V, \tilde{V}\rangle = \frac{\lambda_2}{\lambda_1 - 2\lambda_3}W(\lambda_1)\langle V, \tilde{V}\rangle + \sum_{k=1}^{n_2}\langle h(V, \tilde{V}), JV_k\rangle\langle V_k, \nabla_W T - \nabla_T W\rangle. \tag{2.22}$$

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However, using (2.11), we obtain

$$\langle V_k, \nabla_W T\rangle = \frac{\lambda_3-\lambda_2}{\lambda_1-2\lambda_2}\langle V_k, \nabla_T W\rangle,$$

so that

$$\langle V_k, \nabla_W T - \nabla_T W\rangle = \frac{\lambda_1-\lambda_2-\lambda_3}{\lambda_1-2\lambda_2}\langle \nabla_T V_k, W\rangle.$$

The first equation of Lemma 2.5 now follows from this and (2.22).

The second equation is obtained by interchanging the roles of the distributions $\mathcal{D}_2$ and $\mathcal{D}_3$. In order to obtain the third equation, we use (2.18). On the one hand we have

$$\langle h(V,\tilde{V}), J\nabla_W \tilde{W}\rangle = \langle h(V,\tilde{V}), JT\rangle\langle T, \nabla_W\tilde{W}\rangle + \sum_{k=1}^{n_2}\langle \nabla_W\tilde{W}, V_k\rangle\langle h(V,\tilde{V}), JV_k\rangle.$$

However, $\langle h(V,\tilde{V}), JT\rangle = \lambda_2\langle V,\tilde{V}\rangle$, while, using (2.9) and (2.10), we obtain

$$\begin{aligned}\langle T, \nabla_W\tilde{W}\rangle &= -\langle \nabla_W T, \tilde{W}\rangle\\ &= \frac{1}{\lambda_1-2\lambda_3}(-T(\lambda_3)\langle W,\tilde{W}\rangle - \langle Jh(W,\nabla_T T),\tilde{W}\rangle)\\ &= -\frac{1}{\lambda_1-2\lambda_3}T(\lambda_3)\langle W,\tilde{W}\rangle - \frac{1}{(\lambda_1-2\lambda_3)^2}\langle Jh(W,\tilde{W}),\bar{W}\rangle.\end{aligned}$$

From (2.14) and (2.4), we also have

$$\begin{aligned}\langle \nabla_W\tilde{W}, V_k\rangle &= -\langle \nabla_W V_k, \tilde{W}\rangle\\ &= \frac{1}{\lambda_2-\lambda_3}\langle h(W,\nabla_T V_k), J\tilde{W}\rangle\\ &= \frac{1}{\lambda_2-\lambda_3}\biggl\{\langle h(W,\tilde{W}), JT\rangle\langle \nabla_T V_k, T\rangle + \sum_{r=1}^{n_3}\langle \nabla_T V_k, W_r\rangle\langle h(W,\tilde{W}), JW_r\rangle\biggr\}\\ &= \frac{-1}{(\lambda_2-\lambda_3)(\lambda_1-2\lambda_2)}\langle h(W,\tilde{W}), JT\rangle\langle \bar{V}, V_k\rangle\\ &\quad + \frac{1}{\lambda_2-\lambda_3}\sum_{r=1}^{n_3}\langle \nabla_T V_k, W_r\rangle\langle h(W,\tilde{W}), JW_r\rangle.\end{aligned}$$

Combining the above equations, we see that

$$\begin{aligned}\langle h(V,\tilde{V}), J\nabla_W\tilde{W}\rangle &= -\frac{\lambda_2 T(\lambda_3)}{\lambda_1-2\lambda_3}\langle V,\tilde{V}\rangle\langle W,\tilde{W}\rangle + \frac{\lambda_2}{(\lambda_1-2\lambda_3)^2}\langle h(W,\tilde{W}), J\bar{W}\rangle\langle V,\tilde{V}\rangle\\ &\quad + \frac{1}{\lambda_2-\lambda_3}\sum_{k=1}^{n_2}\sum_{r=1}^{n_3}\langle \nabla_T V_k, W_r\rangle\langle h(W,\tilde{W}), JW_r\rangle\langle h(V,\tilde{V}), JV_k\rangle\\ &\quad - \frac{\lambda_3}{(\lambda_2-\lambda_3)(\lambda_1-2\lambda_2)}\langle W,\tilde{W}\rangle\langle h(V,\tilde{V}), J\bar{V}\rangle.\end{aligned}$$

On the other hand, if we interchange the role of the two distributions, and use (2.18) to equate the two expressions, the double summations cancel from the resulting equation, which then simplifies to give the third equation of Lemma 2.5. $\square$

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3. Submanifolds attaining equality

Throughout this section we assume that M is an n-dimensional Lagrangian submanifold of the complex projective space $\mathbb{C}P^n(4)$ attaining equality in (1.2) at every point. A careful analysis of the arguments of Oprea [**9**] shows that equality at a certain point is attained in (1.2) if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space such that the second fundamental form satisfies

$$\begin{aligned} &h(e_1, e_1) = 12\lambda Je_1, && h(e_1, e_p) = 4\lambda Je_p, && h(e_p, e_q) = \delta_{pq} 4\lambda Je_1, \\ &h(e_2, e_2) = 3\lambda Je_1 + aJe_2, && h(e_2, e_3) = -aJe_3, && h(e_j, e_p) = 0, \\ &h(e_3, e_3) = 3\lambda Je_1 - aJe_2, && h(e_j, e_1) = 3\lambda Je_j \end{aligned}$$

for some real number λ, where $j \in \{2, 3\}$ and $p, q \in \{4, \dots, n\}$. Note that, with respect to the above basis, the plane with the smallest sectional curvature is that determined by e_2 and e_3.

We shall assume that $n \geqslant 4$ and that M has no minimal points, that is to say λ is nowhere zero. In this case, Je_1 is a multiple of the mean curvature vector, so that λ is a globally defined differentiable function and the vector field corresponding to e_1, which, in accordance with § 2, we denote by T, is a globally defined differentiable vector field. We let $\mathcal{D}_1$ denote the distribution spanned by T.

At every point, the linear symmetric operator A_{JT} has three distinct eigenvalues 12λ, 3λ, 4λ of respective multiplicities 1, 2 and $n-3$ (where the first eigenspace is spanned by T). Again in accordance with § 2, we let $\mathcal{D}_2$ be the two-dimensional distribution and let $\mathcal{D}_3$ be the orthogonal $(n-3)$-dimensional distribution corresponding to the two other eigenspaces. Let $\{V_1, V_2\}$ be a local smooth moving orthonormal framing of $\mathcal{D}_2$ and let $\{W_1, W_2, \dots, W_{n_3}\}$, where $n_3 = n - 3$, be a local smooth moving orthonormal framing of $\mathcal{D}_3$.

As M attains equality in (1.2) at every point, it follows that there exist smooth local functions b and c such that the second fundamental form has the following form:

$$\begin{aligned} &h(T, T) = 12\lambda JT, && h(T, W_p) = 4\lambda JW_p, && h(W_p, W_q) = \delta_{pq} 4\lambda JT, \\ &h(V_1, V_1) = 3\lambda JT + bJV_1 + cJV_2, && h(V_1, V_2) = cJV_1 - bJV_2, && h(V_j, W_p) = 0, \\ &h(V_2, V_2) = 3\lambda JT - bJV_1 - cJV_2, && h(V_j, T) = 3\lambda JV_j. && \end{aligned}$$

Hence, we may apply all the formulae of § 2 with $\lambda_1 = 12\lambda$, $\lambda_2 = 3\lambda$ and $\lambda_3 = 4\lambda$.

Theorem 3.1. *Let M be a Lagrangian submanifold of $\mathbb{C}P^n(4)$ attaining equality in (1.2) at every point. If $n \geqslant 4$, then M is minimal. Hence, M is one of the submanifolds of $\mathbb{C}P^n(4)$ discussed in [**7, 8**].*

Proof. Assume that there exists a point x and hence a neighbourhood of x on which M is not minimal. We construct a local smooth moving orthonormal framing $\{T, V_1, V_2, W_1, \dots W_{n-3}\}$, on a (possibly smaller) neighbourhood of x as described above.

We first show that λ is constant. To do this, we take $V^* = \tilde{V} = V_1$ in (2.16) and add this to (2.16) with $V^* = \tilde{V} = V_2$. Using the special form of the second fundamental form,

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this gives

$$6V(\lambda) = -6\lambda\langle T, \nabla_T V\rangle - 2\langle h(V, V_1), J\nabla_T V_1\rangle - 2\langle h(V, V_2), J\nabla_T V_2\rangle.$$

Thus, using (2.4), we find that

$$\begin{aligned}6V(\lambda) &= 12V(\lambda) - 2\langle h(V, V_1), JT\rangle\langle T, \nabla_T V_1\rangle - 2\langle h(V, V_2), JT\rangle\langle T, \nabla_T V_2\rangle \\ &\quad - 2\langle h(V, V_1), JV_2\rangle(\langle V_2, \nabla_T V_1\rangle + \langle \nabla_T V_2, V_1\rangle) \\ &= 12V(\lambda) + 6\lambda\langle \nabla_T T, V\rangle = 24V(\lambda).\end{aligned}$$

Hence, $V(\lambda) = 0$, that is to say, $\bar{V} = 0$. Similarly, it follows from (2.17) that $\bar{W} = 0$. Equations (2.4) and (2.9) now show that $\nabla_T T = 0$, while the third equation of Lemma 2.5 shows that $T(\lambda) = 0$, so that λ is indeed constant.

Since $\nabla_T T = 0$, the first equation of Lemma 2.5 now implies that, for all $V, \tilde{V} \in \mathcal{D}_2$, $W, \tilde{W} \in \mathcal{D}_3$, we have

$$\langle h(V, \tilde{V}), J\nabla_T W\rangle = 0. \tag{3.1}$$

The symmetry of (2.2), together with (2.12), now shows that $\nabla_V W$ is orthogonal to $\mathcal{D}_2$. Using (2.5), we also have that $\langle \nabla_V \tilde{V}, T\rangle = 0$, so that

$$\nabla_{\mathcal{D}_2}\mathcal{D}_2 \subset \mathcal{D}_2, \qquad \nabla_{\mathcal{D}_3}\mathcal{D}_3 \subset \mathcal{D}_3. \tag{3.2}$$

Since n is assumed to be greater than 4, we may choose non-zero vector fields $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$, such that $\langle \nabla_T V, W\rangle = 0$. From (2.6) and (2.11) we then get that

$$\langle \nabla_V W, T\rangle = \langle \nabla_W V, T\rangle = 0.$$

From the Gauss equation we have that

$$\langle R(V, W)W, V\rangle = (1 + 12\lambda^2)\langle V, V\rangle\langle W, W\rangle,$$

while computing directly using (3.2), we find that

$$\langle R(V, W)W, V\rangle = \langle \nabla_V \nabla_W W, V\rangle - \langle \nabla_W \nabla_V W, V\rangle - \langle \nabla_{\nabla_V W - \nabla_W V} W, V\rangle = 0.$$

This contradiction completes the proof. □

4. Warped-product immersions

In this section we characterize warped-product Lagrangian immersions into $\mathbb{C}P^n(4)$. We begin by showing that the second fundamental form of such an immersion has the form given in (2.1), with some extra conditions.

Taking our notation from § 1, we let ϕ be a warped-product Lagrangian immersion into $\mathbb{C}P^n(4)$. For tangent vectors v to N_1 and w to N_2 we introduce local vector fields E_t, E_v and E_w by

$$E_t = \frac{\phi_*(\partial/\partial t)}{|\phi_*(\partial/\partial t)|}, \qquad E_v = \frac{\phi_*(0, v, 0)}{|\phi_*(0, v, 0)|}, \qquad E_w = \frac{\phi_*(0, 0, w)}{|\phi_*(0, 0, w)|}.$$

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It is easily verified that the second fundamental form satisfies

$$h(E_t, E_t) = \lambda_1 J E_t, \qquad h(E_t, E_v) = \lambda_2 J E_v,$$
$$h(E_t, E_w) = \lambda_3 J E_w, \qquad h(E_v, E_w) = 0,$$

where

$$\lambda_1 = \frac{\langle \alpha'', J\alpha' \rangle}{|\alpha'|^3}, \qquad \lambda_2 = \frac{\langle \alpha_1', J\alpha_1 \rangle}{|\alpha'|\,|\alpha_1|^2}, \qquad \lambda_3 = \frac{\langle \alpha_2', J\alpha_2 \rangle}{|\alpha'|\,|\alpha_2|^2}.$$

In particular, we see that the second fundamental form has the properties given in (2.1). We note, however, that in this case the functions λ_1, λ_2 and λ_3 are functions of t only, and we also note that

$$\langle [E_v, E_w], E_t \rangle = 0.$$

We will now characterize warped-product immersions. We use the notation developed in §§ 1 and 2, and assume in this section that M is a Lagrangian submanifold of $\mathbb{C}P^n(4)$ whose second fundamental form satisfies (2.1) with, as usual, λ_1, $2\lambda_2$ and $2\lambda_3$ being mutually distinct real numbers at each point of M. We further assume, as indicated by the above, that $\operatorname{grad}\lambda_1$ lies in $\mathcal{D}_1$ and that $\langle [V, W], T \rangle = 0$ for all $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$. Note that the former condition implies the latter on the open subset of M on which $\operatorname{grad}\lambda_1 \neq 0$ and, as we shall see, these conditions are sufficient to prove that the distributions $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_2 \oplus \mathcal{D}_3$ are all integrable.

Lemma 4.1. *The above assumptions imply that $\nabla_T T = 0$ and, for each $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$, we have that $\nabla_T V \perp \mathcal{D}_3$, $\nabla_T W \perp \mathcal{D}_2$ $\nabla_V W \in \mathcal{D}_3$, $\nabla_W V \in \mathcal{D}_2$.*

Proof. It follows from (2.4) and (2.9) that $\nabla_T T = 0$ if and only if $\operatorname{grad}\lambda_1$ is parallel to T. Furthermore, using (2.6) and (2.11), we have

$$0 = \langle [V, W], T \rangle = \langle \nabla_V W - \nabla_W V, T \rangle = -\frac{\lambda_2 - \lambda_3}{\lambda_1 - 2\lambda_3} \langle \nabla_T V, W \rangle + \frac{\lambda_2 - \lambda_3}{\lambda_1 - 2\lambda_2} \langle \nabla_T V, W \rangle,$$

which implies that $\langle \nabla_T V, W \rangle = 0$. That $\langle \nabla_W V, T \rangle = 0$ and $\langle \nabla_V W, T \rangle = 0$ follow immediately from (2.11) and (2.6). Finally, the fact that $\langle \nabla_V W, \tilde{V} \rangle = 0$ and $\langle \nabla_W V, \tilde{W} \rangle = 0$ for all $\tilde{V} \in \mathcal{D}_2$ and all $\tilde{W} \in \mathcal{D}_3$ follows from (2.13) and (2.15), using our assumptions (2.1) on the second fundamental form of M. □

Corollary 4.2. *The distributions $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_2 \oplus \mathcal{D}_3$ are all integrable.*

Proof. For each V, $\tilde{V}$ in $\mathcal{D}_2$, using (2.5), we have

$$\langle [V, \tilde{V}], T \rangle = \langle \nabla_{\tilde{V}} T, V \rangle - \langle \nabla_V T, \tilde{V} \rangle = \frac{T(\lambda_2)}{\lambda_1 - 2\lambda_2} \{ \langle \tilde{V}, V \rangle - \langle V, \tilde{V} \rangle \} = 0.$$

That $\langle [V, \tilde{V}], W \rangle = 0$ for all $W \in \mathcal{D}_3$ follows similarly from Lemma 4.1. Thus, $\mathcal{D}_2$ is integrable, and the integrability of $\mathcal{D}_3$ follows in a similar way. It is now clear that $\mathcal{D}_2 \oplus \mathcal{D}_3$ is also integrable. □

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Lemma 4.3. *In addition to the standard assumptions of this section, assume that both of the following conditions are satisfied:*

(i) $\dim \mathcal{D}_2 > 1$ *or* $\langle h(\mathcal{D}_2, \mathcal{D}_2), J\mathcal{D}_2\rangle = 0$;

(ii) $\dim \mathcal{D}_3 > 1$ *or* $\langle h(\mathcal{D}_3, \mathcal{D}_3), J\mathcal{D}_3\rangle = 0$.

Then $\operatorname{grad}\lambda_2$ *and* $\operatorname{grad}\lambda_3$ *are parallel to* T.

Proof. Using the symmetry of the cubic form C in (2.2), we may deduce immediately from (2.16) that

$$V(\lambda_2)\langle \tilde{V}, V^*\rangle - \langle h(\tilde{V}, V^*), J\nabla_V T\rangle = \tilde{V}(\lambda_2)\langle V, V^*\rangle - \langle h(V, V^*), J\nabla_{\tilde{V}} T\rangle.$$

Since, from Lemma 4.1, $\nabla_T T = 0$, we may use (2.5) to rewrite the above equation as

$$V(\lambda_2)\langle \tilde{V}, V^*\rangle = \tilde{V}(\lambda_2)\langle V, V^*\rangle.$$

If $\dim \mathcal{D}_2 > 1$, it follows immediately that $\operatorname{grad}\lambda_2$ is orthogonal to $\mathcal{D}_2$. The same conclusion follows directly from (2.16), if $\langle h(\mathcal{D}_2, \mathcal{D}_2), J\mathcal{D}_2\rangle = 0$. Similar arguments show that $\operatorname{grad}\lambda_3$ is orthogonal to $\mathcal{D}_3$.

The fact that $\operatorname{grad}\lambda_2$ is perpendicular to $\mathcal{D}_3$ and $\operatorname{grad}\lambda_3$ is perpendicular to $\mathcal{D}_2$ now follows immediately from the first two equations of Lemma 2.5. □

Theorem 4.4. *Let* $\phi : M^n \to \mathbb{C}P^n(4)$ *be a Lagrangian immersion of an* n*-dimensional Riemannian manifold* M*. Assume that* M *admits three mutually orthogonal distributions* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *and* $\mathcal{D}_3$ *of dimension* 1*,* n_2 *and* n_3*, respectively, with* $1 + n_2 + n_3 = n$ *and, for all vectors* $V \in \mathcal{D}_2$ *and* $W \in \mathcal{D}_3$*,*

$$\begin{aligned} h(T,T) &= \lambda_1 JT, & h(T,V) &= \lambda_2 JV, \\ h(T,W) &= \lambda_3 JW, & h(V,W) &= 0, \end{aligned}$$

where T *is a unit vector spanning* $\mathcal{D}_1$*, with* λ_i*,* $i = 1, 2, 3$*, functions on* M *such that* λ_1*,* $2\lambda_2$ *and* $2\lambda_3$ *are mutually distinct real numbers at each point of* M*. Suppose moreover that* $\operatorname{grad}\lambda_1$ *lies in* $\mathcal{D}_1$ *and that* $\langle [V, W], T\rangle = 0$ *for all* $V \in \mathcal{D}_2$*,* $W \in \mathcal{D}_3$ *on any open set on which* $\operatorname{grad}\lambda_1 = 0$*. If the hypotheses of Lemma 4.3 are satisfied, then* ϕ *is locally congruent to a warped-product Lagrangian immersion.*

Proof. The integrability of the distributions $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_2 \oplus \mathcal{D}_3$ implies the existence of local coordinates (t, p, q) for M based on an open subset containing the origin of $\mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, such that $\mathcal{D}_1$ is given by $\mathrm{d}p = \mathrm{d}q = 0$, $\mathcal{D}_2$ is given by $\mathrm{d}t = \mathrm{d}q = 0$, and $\mathcal{D}_3$ is given by $\mathrm{d}t = \mathrm{d}p = 0$. In this case, it follows from Lemma 4.3 that λ_1, λ_2, λ_3 are functions of t only. We let λ_i' denote the derivative of λ_i with respect to t.

Since $\bar{V} = \bar{W} = 0$, it is immediate from Lemma 2.5 that

$$\frac{\lambda_2\lambda_3'}{\lambda_1 - 2\lambda_3} = \frac{\lambda_3\lambda_2'}{\lambda_1 - 2\lambda_2}, \tag{4.1}$$

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and we now derive a further differential equation satisfied by λ_i, $i = 1, 2, 3$. Using the definition of the curvature tensor, together with (2.5) and (2.10), we find that for $V \in \mathcal{D}_2$ and $W \in \mathcal{D}_3$ we have

$$R(V, W, W, V) = -\frac{\lambda_2' \lambda_3' \langle V, V\rangle\langle W, W\rangle}{(\lambda_1 - 2\lambda_2)(\lambda_1 - 2\lambda_3)}.$$

On the other hand, it follows from the Gauss equation that

$$R(V, W, W, V) = (1 + \lambda_2\lambda_3)\langle V, V\rangle\langle W, W\rangle.$$

By comparing both expressions, we deduce that

$$-\frac{\lambda_2' \lambda_3'}{(\lambda_1 - 2\lambda_2)(\lambda_1 - 2\lambda_3)} = 1 + \lambda_2\lambda_3. \tag{4.2}$$

We consider a horizontal lift $\tilde{\phi}$ of ϕ, and identify T and $\tilde{\phi}_* T$. We define $\tilde{\phi}_i : M \to \mathbb{C}^{n+1}$, $i = 2, 3$, by

$$\tilde{\phi}_2 = f\tilde{\phi} + gT, \qquad \tilde{\phi}_3 = h\tilde{\phi} + jT,$$

where the functions f, g, h and j, which depend only on t, are determined by

$$g' = -f - g\lambda_1 \mathrm{i},$$
$$j' = -h - j\lambda_1 \mathrm{i},$$

and

$$f = -g\biggl(\mathrm{i}\lambda_3 + \frac{\lambda_3'}{\lambda_1 - 2\lambda_3}\biggr), \qquad h = -j\biggl(\mathrm{i}\lambda_2 + \frac{\lambda_2'}{\lambda_1 - 2\lambda_2}\biggr).$$

Here, as before, the prime denotes differentiation with respect to t.

We also assume the following initial conditions:

$$|g|^2\biggl(\lambda_3^2 + 1 + \frac{\lambda_3'^2}{(\lambda_1 - 2\lambda_3)^2}\biggr) = |j|^2\biggl(\lambda_2^2 + 1 + \frac{\lambda_2'^2}{(\lambda_1 - 2\lambda_2)^2}\biggr) = 1, \qquad f\bar{h} + g\bar{j} = 0.$$

Using (4.1) and (4.2), it is straightforward to verify that if f, g, h and j satisfy the initial conditions at one point, they satisfy them everywhere. We also find that

$$f' = g, \qquad h' = j.$$

Using this, it is not difficult to see that

$$T(\tilde{\phi}_2) = 0, \qquad W(\tilde{\phi}_2) = 0, \qquad T(\tilde{\phi}_3) = 0, \qquad V(\tilde{\phi}_3) = 0,$$

and

$$\langle\tilde{\phi}_2, \tilde{\phi}_3\rangle = \langle\tilde{\phi}_2, J\tilde{\phi}_3\rangle = 0.$$

So in particular, we recover $\tilde{\phi}$ in terms of $\tilde{\phi}_i$ by

$$\tilde{\phi}(t, p, q) = (\tilde{\alpha}_2(t)\tilde{\phi}_2(0, p, 0), \tilde{\alpha}_3(t)\tilde{\phi}_3(0, 0, q)),$$

for any $(t, p, q) \in I \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, where $\tilde{\alpha} = (\tilde{\alpha}_2, \tilde{\alpha}_3) : I \to \mathbb{S}^3(1) \subset \mathbb{C}^2$ is a regular curve.

The properties of the lift $\tilde{\phi}$ imply that $\tilde{\phi}_i$ are horizontal immersions in the corresponding $\mathbb{S}^{2n_i+1}(1) \subset \mathbb{C}^{n_i+1}$, which means that $\tilde{\phi}_i$ are horizontal lifts of Lagrangian immersions $\phi_i = \Pi(\tilde{\phi}_i)$, $i = 2, 3$, in $\mathbb{CP}^{n_i}$ and also $\tilde{\alpha}$ is a horizontal lift of a regular curve $\alpha = \Pi(\tilde{\alpha})$ in $\mathbb{C}P^1(4)$. Hence, $\phi = [\tilde{\phi}]$ is indeed a warped-product Lagrangian immersion. □

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5. Warped-product minimal immersions

In this section we consider warped-product immersions which are also minimal. In this case, we show that the assumptions of Lemma 4.1 are automatically satisfied. Hence, we prove the following.

Theorem 5.1. *Let $\phi : M^n \to \mathbb{C}P^n(4)$ be a minimal Lagrangian immersion of an n-dimensional Riemannian manifold M. Assume that M admits three mutually orthogonal distributions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ of dimension 1, $n_2 > 1$ and $n_3 > 1$, respectively, with $1 + n_2 + n_3 = n$ and, for all vectors $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$,*

$$h(T,T) = \lambda_1 JT, \qquad h(T,V) = \lambda_2 JV,$$
$$h(T,W) = \lambda_3 JW, \qquad h(V,W) = 0,$$

where T is a unit vector spanning $\mathcal{D}_1$, with λ_i, $i = 1,2,3$, functions on M such that λ_1, $2\lambda_2$ and $2\lambda_3$ are mutually distinct real numbers at each point of M with $\lambda_2\lambda_3$ and $\lambda_1 - \lambda_2 - \lambda_3$ never zero. Then ϕ is locally congruent to a warped-product Lagrangian immersion.

Proof. We first show that, in the minimal case, $\bar{V} = \bar{W} = 0$.

As before, we let $\{V_1, \dots, V_{n_2}\}$, $\{W_1, \dots, W_{n_3}\}$ be local smooth moving orthonormal framings of $\mathcal{D}_2$,$\mathcal{D}_3$, respectively. From the third equation of Lemma 2.5, it follows for $i \neq j$ that

$$\langle h(V_i, V_j), J\bar{V}\rangle = 0,$$
$$\langle h(V_i, V_i) - h(V_j, V_j), J\bar{V}\rangle = 0.$$

From the symmetry of (2.2) and the assumption of minimality, we obtain

$$\sum_{k=1}^{n_2} \langle h(V_k, V_k), J\bar{V}\rangle = 0.$$

This implies that for vector fields $V, V^* \in \mathcal{D}_2$ we have that

$$\langle h(V, V^*), J\bar{V}\rangle = 0, \tag{5.1}$$

and similarly, for vector fields $W, W^* \in \mathcal{D}_3$,

$$\langle h(W, W^*), J\bar{W}\rangle = 0. \tag{5.2}$$

It follows from (2.4) that

$$\bar{V}(\lambda_1) = (\lambda_1 - 2\lambda_2)\langle \nabla_T T, \bar{V}\rangle, \tag{5.3}$$

whereas from the second equation of Lemma 2.5, (2.4) and (5.2), it follows that

$$\bar{V}(\lambda_3) = \lambda_3 \langle \nabla_T T, \bar{V}\rangle. \tag{5.4}$$

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Finally, from (2.16) and (5.1) it follows that

$$\bar{V}(\lambda_2)\langle \bar{V}, \bar{V}\rangle = -3\lambda_2\langle \nabla_T \bar{V}, T\rangle\langle \bar{V}, \bar{V}\rangle.$$

Hence,

$$\bar{V}(\lambda_2) = 3\lambda_2\langle \nabla_T T, \bar{V}\rangle. \tag{5.5}$$

On the other hand, it follows from minimality that $\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = 0$ so, applying $\bar{V}$ to this and using (5.3)–(5.5), it follows that

$$3n_2\lambda_2\langle \nabla_T T, \bar{V}\rangle = (n_2 + 2)\lambda_2\langle \nabla_T T, \bar{V}\rangle.$$

As $n_2 \neq 1$ and $\lambda_2 \neq 0$, it follows that $\langle \nabla_T T, \bar{V}\rangle = 0$ and hence $\bar{V} = 0$. Interchanging the roles played by the two distributions, we find that $\bar{W} = 0$ also.

To complete the proof, we now assume that λ_1 is constant on an open set, and show that $\langle [V, W], T\rangle = 0$ for all $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$. Our assumption, together with (2.4) and (2.9), implies that $\nabla_T T = 0$, so, by (2.5), $\langle h(\tilde{V}, V^*), J\nabla_V T\rangle$ is symmetric in V, $\tilde{V}$, V^*. Equation (2.16) now shows that $V(\lambda_2)\langle \tilde{V}, V^*\rangle$ is also symmetric in $V, \tilde{V}, V^*$, so, since $n_2 > 1$, it now follows that $V(\lambda_2) = 0$ for all $V \in \mathcal{D}_2$. Similarly, using (2.17), $W(\lambda_3) = 0$ for all $W \in \mathcal{D}_3$. Minimality of the immersion now implies that $W(\lambda_2) = V(\lambda_3) = 0$. If either $T(\lambda_2) \neq 0$ or $T(\lambda_3) \neq 0$, it follows straight away that $\langle [V, W], T\rangle = 0$. Therefore, we may assume that λ_2 and λ_3 are also constant. However, this is sufficient to carry out the proof of (3.2), which thus holds in this situation.

For $W \in \mathcal{D}_3$ and $V \in \mathcal{D}_2$ we now find, using (2.6), (2.11) and (3.2) that

$$\begin{aligned}
\langle R(V, W)W, V\rangle &= \langle \nabla_V \nabla_W W, V\rangle - \langle \nabla_W \nabla_V W, V\rangle - \nabla_{\nabla_V W - \nabla_W V} W, V\rangle \\
&= -\langle \nabla_V W, T\rangle\langle \nabla_W T, V\rangle - \langle \nabla_V W, T\rangle\langle \nabla_T W, V\rangle + \langle \nabla_W V, T\rangle\langle \nabla_T W, V\rangle \\
&= \left(-\frac{(\lambda_2 - \lambda_3)^2}{(\lambda_1 - 2\lambda_2)(\lambda_1 - 2\lambda_3)} - \frac{(\lambda_2 - \lambda_3)}{(\lambda_1 - 2\lambda_3)} + \frac{(\lambda_2 - \lambda_3)}{(\lambda_1 - 2\lambda_2)}\right)\langle \nabla_T V, W\rangle^2 \\
&= \frac{(\lambda_2 - \lambda_3)^2}{(\lambda_1 - 2\lambda_2)(\lambda_1 - 2\lambda_3)}\langle \nabla_T V, W\rangle^2.
\end{aligned}$$

On the other hand, by the Gauss equation we have

$$\langle R(V, W)W, V\rangle = 1 + \lambda_2\lambda_3.$$

Hence,

$$\frac{(\lambda_2 - \lambda_3)^2}{(\lambda_1 - 2\lambda_2)(\lambda_1 - 2\lambda_3)}\langle \nabla_T V, W\rangle^2 = 1 + \lambda_2\lambda_3.$$

Since $n_3 > 1$, we can find non-zero vectors $V \in \mathcal{D}_2$, $W \in \mathcal{D}_3$ such that $\langle \nabla_T V, W\rangle = 0$. Hence, we must have $\lambda_2\lambda_3 = -1$ and

$$\langle \nabla_T V, W\rangle = 0$$

for arbitrary V and W. From (2.6) and (2.11) we now get that

$$\langle \nabla_V W, T\rangle = \langle \nabla_W V, T\rangle = 0,$$

and hence also in this case the assumptions of Lemma 4.1 are satisfied. □

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Remark 5.2. It is easy to verify that a warped-product immersion is minimal if and only if both original immersions are minimal and the horizontal curve α satisfies

$$\frac{\langle \alpha'', J\alpha' \rangle}{|\alpha'|^3} + n_2 \frac{\langle \alpha_2', J\alpha_2 \rangle}{|\alpha'|\,|\alpha_2|^2} + n_3 \frac{\langle \alpha_3', J\alpha_3 \rangle}{|\alpha'|\,|\alpha_3|^2} = 0.$$

We note that an example of such a horizontal curve α is given by

$$\alpha(t) = \left(\sqrt{\frac{1+n_2}{2+n_2+n_3}} \exp\left(\mathrm{i}t \sqrt{\frac{1+n_3}{1+n_2}} \right), \sqrt{\frac{1+n_3}{2+n_2+n_3}} \exp\left(-\mathrm{i}t \sqrt{\frac{1+n_2}{1+n_3}} \right) \right).$$

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