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A SCHENSTED ALGORITHM WHICH MODELS TENSOR REPRESENTATIONS OF THE ORTHOGONAL GROUP

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1. Introduction. This paper is concerned with a combinatorial construction which mysteriously "mimics" or "models" the decomposition of certain reducible representations of orthogonal groups. Although no knowledge of representation theory is needed to understand the body of this paper, a little familiarity is necessary to understand the representation theoretic motivation given in the introduction. Details of the proofs will most easily be understood by people who have had some exposure to Schensted's algorithm or jeu de tacquin.

Schensted invented his algorithm in 1961 [**9**] in order to find the longest increasing subsequence of a sequence of k numbers, say from $\{1, 2, \ldots, N\}$. If a such a sequence is input into Schensted's algorithm, the output is a pair of tableaux (P, Q). Here P is a "semistandard Young" tableau with entries from $\{1, 2, \ldots, N\}$, while Q is a "standard Young" tableau with entries from $\{1, 2, \ldots, k\}$, and both P and Q have the same shape, say λ. (Consult the first paragraph of Section 2 for definitions.) In fact there is exactly one input which will produce any such output pair, as long as λ has no more than N rows. Hence one can prove the following polynomial identity with the bijective correspondence given by the algorithm:

$$(x_1 + x_2 + \cdots + x_N)^k = \sum f_\lambda s_\lambda(x_1, \ldots, x_N).$$

Here the sum is over all shapes λ with no more than N rows, f_λ is the number of standard Young tableaux of shape λ, and s_λ is the Schur function of shape λ in N variables (i.e., the multivariate generating function for semistandard Young tableaux of shape λ).

This identity also arises as a character identity when studying representations of GL_N. Let V be the vector space $\mathbf{C}^N$, and consider the action of GL_N on $\otimes^k V$. The axis basis for $\otimes^k V$ is indexed by sequences of length k from the set $\{1, 2, \ldots, N\}$. Decompose this tensor representation into irreducible representations indexed by shapes λ with no more than N rows, and then construct a basis indexed by semistandard Young tableaux for each irreducible component. It is known that here there are f_λ copies of the irreducible GL_N representation whose character is $s_\lambda(x)$. Hence a representation proof of the polynomial identity above can be given. Since Schensted's algorithm produces the same collection of pairs of tableaux, we say that it *models* tensor representations of GL_N. Now let the

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orthogonal group O_N act on $\otimes^k V$. An analogous identity can be written down (Corollary 4) which describes the decomposition of the orthogonal tensor power character into a sum of irreducible characters. The algorithm which we present here models this decomposition.

In [**2**] Berele and Regev announced both a generalization of the above familiar GL_N tensor construction of Schur and an analog of Schensted's algorithm which modelled that generalization. Later, Berele found (following a query from this author inspired by [**2**]) a beautiful analog of Schensted's algorithm which modelled tensor representations of Sp_{2n} [**1**]. His key idea involved the annihilation of inverse pairs of letters and sliding out the resulting empty square via jeu de tacquin. Since then, Stembridge has used [**10**] the same idea to give a nice algorithm which models rational (in addition to polynomial) tensor representations of GL_N. However, until now it has not been possible to extend these methods to tensor representations of both odd and even dimensional orthogonal groups. The recent discovery [**6**] of the "correct" semistandard tableaux for tensor representations of the orthogonal group is the development that has made it possible to give an algorithm which seems to be truly analogous to Berele's, but for the orthogonal cases. These semistandard tableaux are improvements of tableaux found by King [**3**] and by Koike and Terada [**4**]. For the odd orthogonal groups SO_{2n+1} Sundaram has found her own semistandard tableaux and modified Berele's algorithm to obtain a different Schensted-like algorithm [**11**].

We will present our results at three levels. The simple *coarse* level gives an orthogonal analog to the "integer" specialization of the above identity, viz. the coarser identity obtained by setting all $x_i = 1$. The more complicated *fine* level gives an orthogonal analog to the original identity above. It is possible to mix these two levels; this will be referred to as the *general* case. In Section 2, which can be regarded as a more detailed introduction, we give both Berele's symplectic and our coarse orthogonal analogs to the above identity. In Section 3 we present our coarse orthogonal Schensted algorithm and describe the bijection it produces. Section 4 describes the fine and general orthogonal analogs of the above identity. Section 5 treats the general mixed situation. The main result of this paper, Theorem 5, states that the general algorithm gives a bijection involving orthogonal tableaux which is analogous to the bijection described by the Schensted algorithm. The proofs of the coarse and general bijections are presented in Sections 6 and 7. Section 8 provides an interpretation of the combinatorial results from the vantage point of representation theory. Readers who have some familiarity with representation theory are encouraged to refer to this section while reading Sections 2 and 4 so that the presentation will be more motivated. Section 9 contains some concluding remarks and open problems.

2. The symplectic and coarse orthogonal identities. A *shape* λ is a diagram which has λ_i (left-justified) squares in its *ith* row, where $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_r > 0$. An *N-semistandard (Young) tableau of shape* λ is a filling of the squares in the shape λ with letters from the alphabet $\{1, 2, \ldots, N\}$ such that the letters strictly increase down each column and weakly increase across each row. The *weight monomial* of an N-semistandard tableau is

$$x_1^{\#1\text{'s}} x_2^{\#2\text{'s}} \cdots x_N^{\#N\text{'s}},$$

where $\#q$'s is the number of times the letter q appears as an entry of the tableau. The *Schur function* $s_\lambda(x_1, \ldots, x_N)$ is defined to be the sum of the weight monomials of all N-semistandard tableaux of shape λ. Suppose that a shape λ has k squares. A *standard (Young) tableau* of shape λ is a k-semistandard tableau of shape λ which uses each of the numbers from $\{1, 2, \ldots, k\}$ exactly once each. Such a standard tableau can also be viewed as a sequence of shapes $\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)} = \lambda$, wherein each shape is obtained from the preceding one by adjoining one square.

A *$2n$-symplectic tableau of shape* λ is a $2n$-semistandard tableau such that the letters $2i - 1$ and $2i$ do not occur below the *ith* row. (Hence $2n$-symplectic tableaux cannot have more than n rows.) The *weight monomial* of an $2n$-symplectic tableau is

$$x_1^{\#1\text{'s}-\#2\text{'s}} x_2^{\#3\text{'s}-\#4\text{'s}} \cdots x_n^{\#(2n-1)\text{'s}-2\#n\text{'s}}.$$

The *symplectic Schur function* $Sp_{2n}(\lambda; x_1, \ldots, x_n)$ is defined to be the sum of the weight monomials of all $2n$-symplectic tableaux of shape λ. An *n-oscillating tableau of final shape λ and length k* is a sequence of $k + 1$ shapes $\emptyset$, $\lambda^{(1)}$, $\lambda^{(2)}, \ldots, \lambda^{(k)} = \lambda$, each of which has no more than n rows, such that each shape is obtained from the preceding one by adjoining or removing one square. Let $g_\lambda(n, k)$ be the number of such tableaux. Berele obtained the following (Laurent) polynomial identity as a consequence of his algorithmic bijection:

$$(x_1 + x_1^{-1} + x_2 + x_2^{-1} + \cdots + x_n + x_n^{-1})^k = \sum g_\lambda(k, n) Sp_{2n}(\lambda; x_1, x_2, \ldots, x_n).$$

Here the sum is over all shapes λ with $k - 2c$ squares, $c \geqq 0$, and no more than n rows.

A shape λ is said to be *N-orthogonal* if the sum of its first two column lengths does not exceed N. An N-semistandard tableau which has no more than a total of q entries in the first two columns which are $\leqq q$ is said to satisfy the *qth orthogonal condition*, or the *value q case of Condition A*. The 5-semistandard tableau shown here satisfies the value 1, 3, and 5 cases of this condition, but not the value 2 and 4 cases:

1	2	2	4	4
2	4	5		
4				

An *N-orthogonal tableau* is a N-semistandard tableau which satisfies the *qth* orthogonal condition for all $1 \leqq q \leqq N$. Let $O_N(\lambda)$ be the number of N-orthogonal tableaux of shape λ. An *oscillating N-orthogonal tableau of final* *shape* λ *and length* k is an N-oscillating tableau of final shape λ and length k such that each shape in the sequence is N-orthogonal. Let $h_\lambda(N,k)$ be the number of such tableaux. The following result is an immediate consequence of either Theorem 3 or Corollary 4:

COROLLARY 2.

$$N^k = \sum h_\lambda(k,n) O_N(\lambda).$$

Here the sum is over all N*-orthogonal shapes* λ *with* $k-2c$ *squares,* $c \geqq 0$.

3. The coarse orthogonal bijection and algorithm.

THEOREM 3. *The coarse orthogonal Schensted algorithm described below gives a bijection between the sets:*

$$\{1,2,\ldots,N\}^k \leftrightarrow \cup \mathcal{P}(\lambda) \times \mathcal{Q}(\lambda,k),$$

where $\mathcal{P}(\lambda)$ *is the set of all* N*-orthogonal tableaux of shape* λ*,* $\mathcal{Q}(\lambda,k)$ *is the set of all oscillating* N*-orthogonal tableaux of final shape* λ *and length* k*, and the union is over all* N*-orthogonal shapes* λ *with* $k-2c$ *squares,* $c \geqq 0$.

Before describing our algorithm, we first review the original Schensted algorithm [**9**]. Given a word of length k from $\{1,2,\ldots,N\}$, the algorithm creates a pair of tableaux in k steps, each of which is called an *insertion*. One begins with the following pair: the left member $P^{(0)}$ is the semistandard tableau on the empty shape and the right member $Q^{(0)}$ is the history sequence (or tableau) consisting of the empty shape. Suppose that after $h-1$ "insertions" of letters from the input word one has a pair consisting of a semistandard tableau $T = P^{(h-1)}$ and a history sequence of shapes $Q^{(h-1)}$. Let b be the next (the *hth*) letter in the input word. The insertion process consists of a series of "bumps". First replace the leftmost copy of the smallest letter c in the first row of T which is $> b$ with b to produce a new semistandard tableau U. We say that b *bumps* c out of the first row. Repeat this procedure with the bumped letter c being inserted into the second row of U to produce a tableau V and a *loose* letter d. We say that c *lands in* U *converting it to* V; or briefly, c *lands in* U/V. The generic symbols we will use are:

$$U \xrightarrow{c} V \xrightarrow{d} W$$

If at some stage an element g is $\geqq$ all of the elements in its target row in Y, then the new left output tableau $P^{(h)} = Z$ is obtained by placing g at the end of that row. After the bumping process has finished in the left tableau, the right tableau (or history sequence) is updated by appending the shape of Z, thereby creating $Q^{(h)}$.

The overall structure of the symplectic and coarse orthogonal algorithms are the same as for Schensted's algorithm. Letters from the input word are successively *inserted* into the left tableau. The right member of the output pair is the history sequence of shapes. Hence we need to specify only the method of inserting one input letter into a current left tableau T. As before [**1**] [**10**], the philosophy within one insertion is to do ordinary Schensted bumping until a bad left tableau is produced. Then the pair of trouble-making letters are *annihilated*. This is accomplished by backing up one bump step and discarding the bumping letter just before it is to be placed in its offending position, and by erasing one other letter in the left tableau. The empty square thereby created in the left tableau is denoted Δ. This empty square Δ is *slid out* to the southeastern boundary with a series of *jdt steps*: Repeatedly interchange Δ with the smaller of the two entries which are below or to the right of Δ. (Pick the entry below if the two choices are equal.) Forget about Δ once it reaches the southeastern boundary.

COARSE ORTHOGONAL INSERTION PROCEDURE. *Do ordinary Schensted bumping until a letter $d = r$ lands in the first (second) column of V/W and participates in a violation of the rth orthogonal condition. (If this never happens then the usual output tableau Z becomes the new left tableau.) Replace the other occurrence of r in the second (respectively first) column of V with Δ and slide it out to produce the new left tableau Z.*

Suppose that the first 11 letters (reading from left to right) of the input word 5 3 1 6 4 3 6 5 4 7 6, 2, 7 7 have been inserted so far: Condition A is never violated, and therefore by regular Schensted insertion the left tableaux at this point $P^{(11)} = T = U$ is:

1	3	4	6
3	4	5	7
5	6	6	

Now insert the 2. It bumps out the 3 from the first row, thereby creating V. The 3 then lands in the second row, thereby creating W. But now W has 4 entries which are each $\leqq 3$ in its first two columns, which violates the $3rd$ orthogonal condition. As implicitly claimed by the procedure, there is another occurrence of a 3 at (2, 1) in V. Replacing this 3 by Δ in V and discarding the loose bumping 3 we have:

1	2	4	6
Δ	4	5	7
5	6	6	

Now Δ successively trades places with the 4, the 5, and then the 6. The $12th$ insertion step is complete, and for $P^{(12)}$ we have:

1	2	4	6
4	5	6	7
5	6		

Lemma 6.1 confirms that the procedure makes sense, i.e., that there always necessarily exists another occurrence of r in the "complementary" column. This lemma will also confirm that when a letter is inserted into an N-orthogonal tableau, another N-orthogonal tableau is produced.

In the original notation of [**1**], the input sequences (or *words*) for Berele's algorithm were formed of letters from the alphabet $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$. When inserting into the left (symplectic) tableau, annihilations always occurred between a letter i and its opposite $\bar{\imath}$. This was reflected in the generating function context (after weights x_i and x_i^{-1} were assigned to the letters i and $\bar{\imath}$) by $x_n x_n^{-1} = 1$. This will also be the case in the fine orthogonal algorithm. However, in the coarse algorithm, one has each letter cancelling with itself in "bad" situations; hence there is no polynomial identity implied by the coarse correspondence of Theorem 3.

4. The fine and general orthogonal identities. When the number of letters in the alphabet $\{1, 2, \ldots, N\}$ is even, $N = 2n$, the terminology "*fine situation*" will refer to the following choices. The letters are *paired* as follows: $1 \longleftarrow 2$, $3 \longleftarrow 4$, ..., $2n - 1 \longleftarrow 2n$. When $N = 2n + 1$ it is impossible for every letter to be a member of a pair. One might specify that the letter $2n + 1$ (or 1) behave as in the coarse algorithm and then pair off the other $2n$ letters. More generally, one could choose any letter $2i + 1$ to be unpaired and then pair off the other $2n$ letters in the obvious fashion. Even more generally, the terminology "*general situation*" refers to any fixed set of choices of the following form: There are a total of $N = 2m + g$ letters in the alphabet, which are still denoted $\{1, 2, \ldots, N\}$. There are m pairs of consecutive letters $q - 1 \longleftarrow q$ specified; the remaining g letters are said to be *unpaired*. When $N = 2n + 1$, the *fine situation* refers to taking the letter $2n + 1$ to be unpaired and the other letters to be paired.

At this point it is natural to hope that we will simply define an orthogonal Schur function of shape λ to be the sum of the same weight monomials as were used in Section 2 for symplectic Schur functions, where the sum is taken over all N-orthogonal tableaux of shape λ. Unfortunately small examples indicate that the resulting Laurent polynomials are not invariant under the interchanges $x_i \leftrightarrow x_i^{-1}$ and $x_i \leftrightarrow x_j$, which is a necessary property for characters of the orthogonal group. However, the number of N-orthogonal tableaux of shape λ is equal to the dimension of the corresponding irreducible representation of the N*th* orthogonal group. The problem described above could therefore be remedied by modifying the definition of weight monomial. Alternatively, with the same amount of effort, we will retain the definition of weight monomial and describe a different kind of orthogonal tableau, which will be equinumerous (with respect to some fixed shape) with the original orthogonal tableaux.

The entries of an N-semistandard tableau T are denoted in the usual matrix fashion: $T(i, j)$ is the *jth* entry in the *ith* row. A semistandard tableau T is said to satisfy the *value* $2p$ *case of Condition B* if whenever $i + j = 2p$ and $T(i, 1) = 2p - 1$ and $T(j, 2) = T(j, 3) = \cdots = T(j, h - 1) = 2p - 1$ and

$T(j,h) = 2p$, then $T(j-1,h) = 2p-1$. In words, taking $2p = 6$ and $2p-1 = 5$ (which we will do throughout the paper for the sake of readibility): If there is a 5 in the first column at $(i,1)$ and a 6 in a later column at (j,h) such that their row coordinates add up to 6, and if every other entry (except possibly the first one) in the jth row preceding the 6 is a 5, then the 6 at (j,h) must have a 5 directly above it. For example, the tableau shown here satisfies the value 6 case of Condition B if and only if * = 5:

1	4	4	*	6
2	5	5	6	
3	7	8		
5				

We say that the entry $T(j,h) = 2p$ is *protected above* by the entry $T(j-1,h) = 2p-1$. If $j = 1$ then it is impossible for T to satisfy the value $2p$ case of Condition B since there will be no place to put a "protecting" $2p-1$. Note that the condition still makes sense when $h = 2$; i.e., it is not necessary to have at least one $2p-1$ in the same row as the $2p$.

A *fine* $2n$*-orthogonal* tableau is a $2n$-semistandard tableau which satisfies the value $2p$ cases of both Conditions A and B for each $1 \leqq p \leqq n$. (Condition A, also called the orthogonal condition, was defined near the end of Section 2.) A *fine* $(2n+1)$*-orthogonal* tableau satisfies in addition the value $2n+1$ case of Condition A. The tableau shown above is not a fine 8-orthogonal tableau even when $* = 5$ since the 4 at (1, 2) is not protected above by a 3. However, if the (1, 2) and (3, 1) entries were interchanged, the resulting tableau would be fine orthogonal. Knowing that the value $2p$ case of Condition A is satisfied makes the evaluation of the $2p$ case of Condition B more straightforward. Then the presence of a $2p$ in the first column would make a violation of Condition B impossible. Also note that if the value $2p-2$ case of Condition A is known to hold true, and if $T(i,1) = 2p-1$ while $T(j,h) = 2p$ with $i+j = 2p$, then it is necessarily true that $T(j,g) = 2p-1$ or $2p$ for $2 \leqq g < h$. In other words, there is a potential for the value $2p$ case of Condition B to fail only when Condition A barely holds true both for the values $2p-2$ and $2p$. Incidentally, semistandardness alone implies that if the leftmost $2p$ in the jth row has a $2p-1$ directly above it, then any other $2p$'s in the jth row will also have $2p-1$'s directly above.

The following result will not be used in this paper, but the proof should be read to help understand the above definitions.

PROPOSITION 4. *The number of N-orthogonal tableaux of shape λ is equal to the number of fine N-orthogonal tableaux of shape λ.*

Proof. By viewing N-semistandard tableaux as being built up by an increasing sequence of M-semistandard tableaux as M runs through the values $2,4,6,\ldots,N-2,N$ when N is even (or $2,4,6,\ldots,N-3,N-1,N$ when N is odd), we see that we can prove the proposition with induction on M if we can show that the number of ways of adjoining $2p - 1 = M - 1$ and $2p = M$ is the same at each stage for each definition. (When N is odd, the last stage going from $N - 1$ to N is the same in each case.) For the orthogonal tableaux we must satisfy both the $2p-1$ and $2p$ cases of Condition A, while for the fine orthogonal tableaux the $2p$ cases of both Conditions A and B must be satisfied. After adjoining some 5's and 6's, there are four classes of newly constructed semistandard tableaux which satisfy the 6-case of Condition A: those satisfying the 5-case of A but not the 6-case of B, vica versa, those satisfying both, and those satisfying neither. Obviously we will be done if we can show the first two classes are equinumerous. Here is how to bijectively convert a member of the second class into a member of the first class. In order to fail the 5-case of A while satisfying the 4-case of A, there must be $T(i, 1) = 5$ and $T(j, 2) = 5$ with $i + j = 6$. Change all of the 5's in the jth row to 6's, as shown here:

$$\begin{array}{ccccc} * & * & * & 5 & 6 \\ * & 5 & 5 & 6 & \\ * & + & + & & \\ 5 & & & & \end{array} \quad \longrightarrow \quad \begin{array}{ccccc} * & * & * & 5 & 6 \\ * & 6 & 6 & 6 & \\ * & + & + & & \\ 5 & & & & \end{array}$$

None of the entries below the 5's could have been 6's, since we are assuming the 6-case of A; hence the result is semistandard. Furthermore, the result violates the 6-case of B since obviously $T(j - 1, 2) < 5$. Going in the other direction, note that in order to fail the 6-case of B while satisfying the 5-case of A, there must be $T(i, 1) = 5$ and $T(j, 2) = 6$ with $i + j = 6$ and $T(j - 1, 2) < 5$. Convert all of the 6's in the jth row which do not have 5's above them into 5's. It is not hard to see that is a two-sided inverse to the other conversion. (Don't forget that we are not considering tableaux which fail both the 5-case of A and the 6-case of B.)

When $N = 2n$ or $2n + 1$, the *weight monomial* of a fine N-orthogonal tableau is defined in exactly the same way as for a $2n$-symplectic tableau (Section 2). (Hence when $N = 2n + 1$, the letter $2n + 1$ is given weight 1.) The *orthogonal Schur function* $O_N(\lambda; x_1, \ldots, x_n)$ is defined to be the sum of the weight monomials of all fine N-orthogonal tableaux of shape λ. The following result is a corollary to the main result of this paper, Theorem 5, in the Nth fine situation. The quantities $h_\lambda(N, k)$ were defined at the end of Section 2.

COROLLARY 4. *Let $N = 2n(+1)$. Then the following Laurent polynomial identity holds:*

$$(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}(+1))^k = \sum h_\lambda(N, k) O_N(\lambda; x_1, \ldots, x_n),$$

where the sum is over all N-orthogonal shapes λ with $k - 2c$ squares, $c \geqq 0$.

In reference to some fixed general situation, a *general N-orthogonal tableau* is an N-semistandard tableau which satisfies the value q cases of both Conditions

A and B for each pair of letters $q-1 \longleftarrow q$ and which satisfies the value r case of Condition A for each unpaired letter r. Incidentally, when $N = 2n+1$ and the letter 1 is chosen to be the single unpaired letter, Condition B for fine N-orthogonal tableaux could be replaced with a different and slightly simpler condition. For each odd q, prohibit the following: $T(i,1) = q$ and $T(i-1,1) < q-1$ with $T(j,2) = q-1$ and $i+j = q$. Then it could be shown that the sum of the usual weight monomials over this family of tableaux would be $O_N(\lambda; x_1, \ldots, x_n)$. Recasting the general algorithm in the next section to take advantage of this would probably lead to a slightly simply simpler algorithm for this special case.

Although we will not use any of the following observations, it is interesting to note that the various conditions can be described in terms of dominoes, which occupy two squares, as follows. The condition for $2n$-symplectic tableaux can be restated as: If $T(i,u) = 2p$ and $T(j,v) = 2p-1$, then $i+j \leqq 2p$. It is sufficient to require this for $u = v = 1$; then a vertical domino of paired letters is prohibited from being too far down. Alternatively: If $T(i,u) \leqq 2p$ and $T(j,v) < 2p$, then $i+j \leqq 2p$. The condition for orthogonal tableaux can be restated as: If $T(i,u) = r$ and $T(j,v) = r$ (so necessarily $u \neq v$), then $i+j \leqq r$. It is sufficient to require this for $u = 1$ and $v = 2$; then a "broken" horizontal domino of equal entries is prohibited from being too far down. Alternatively: If $T(i,u) \leqq r$ and $T(j,v) \leqq r$ with $u \neq v$, then $i+j \leqq r$. The conditions for fine $2n$-orthogonal tableaux can be restated as: If $T(i,u) \leqq 2p$ and $T(j,v) \leqq 2p$ with $u \neq v$, then $i+j \leqq 2p$; and if $T(i,1) < 2p$ and $T(j,v) = 2p$ with $i+j = 2p$, then $T(j-1,v) = 2p-1$. Again this can be viewed as prohibiting certain kinds of broken horizontal dominos in certain positions.

5. The general orthogonal algorithm. In a fixed general situation, paired and unpaired letters from the alphabet $\{1, 2, \ldots, N\}$ are to be assigned weights x_i or x_i^{-1} and 1 respectively, as in the following example:

$$\begin{array}{ccccccccccccccc} 1 & < & 2 & \longleftarrow & 3 & < & 4 & < & 5 & \longleftarrow & 6 & < & 7 & < & 8 \\ 1 & & x_1 & & x_1^{-1} & & 1 & & x_2 & & x_2^{-1} & & 1 & & 1 \end{array}$$

The weight of a word or of a tableau formed from the alphabet is defined to be the product of the weights of the constituent letters. This agrees with the weights defined for fine orthogonal tableaux in Section 4.

Here is the main result of this paper.

THEOREM 5. *Suppose that m of the letters in the alphabet $\{1, 2, \ldots, N\}$ have been paired together, where $N = 2m + g$. The general orthogonal algorithm described below gives a bijection between the sets:*

$$\{1, 2, \ldots, N\}^k \leftrightarrow \cup \mathcal{P}(\lambda) \times \mathcal{Q}(\lambda, k),$$

where $\mathcal{P}(\lambda)$ is the set of all general N-orthogonal tableaux of shape λ, and $\mathcal{Q}(\lambda, k)$ and the union are as in Theorem 3. The bijection is weight preserving.

As with the coarse algorithm, for the general algorithm we only need to describe the insertion of an input letter into the current left tableaux, which is assumed to be general orthogonal with respect to some fixed general situation. In order to handle general orthogonal tableaux, the algorithm becomes more complicated as far as details are concerned. However, the spirit remains the same: When a violation occurs during the insertion process, two offending entries with inverse weights are annihilated. There are many possible ways in which a bump can create a violation of either Condition A or Condition B. During the "nitty-gritty" details of the proof, it will be most expedient to give up on the big view and to specify 15 subcases. However, the general algorithm is not that "ad hoc". (As before, for the sake of readibility we will take the symbols 5 and 6 to be synonymous with the generic paired letters $q-1 \longleftarrow q$ respectively.) A slightly inaccurate rough description is: Typically a 5 (or 6) will land in one of the first two columns, thereby causing a violation of one of the conditions together with a 6 (or respectively 5) in the other one of the first two columns. Then one "rewinds" one (sometimes two) steps and throws away the offending 5 or 6 before it lands while at the same time replacing the opposite entry 6 or 5 with an empty square Δ. Rather than drily listing 15 cases as in Table 2, the general algorithm will be stated in as close a fashion to the above "typical" pattern as possible. This will make it necessary to simultaneously use two "respectively" 's, so that four kinds of situations will be handled at once. The first pair of possibilities concerns the "landing" entry, which is a 5 (or 6). A complication arises concerning the second pair of possibilities, viz. that of the offending entry landing in the first or in the second column. Sometimes it is not the second column which is relevant, but instead a later column of the tableau which we will call the "key" column. Throughout we will assume that $i+j=q$; note that $i \geqq j$ in all meaningful situations. If $V(i,1)=V(j,2)=V(j,3)=\cdots=V(j,h-1)=5$ and $V(j,h)=6$, then the key column of V is the *hth* column. Otherwise, the key column is the second column. The second pair of possibilities consists of the first column and the key column; then opposite column refers to the other of these two possibilities. However, we will not call the second column the key column unless it is necessary to do so. If the $q=6$ case of Condition A is violated, then the lowest 5 or 6 entries in each of the first two columns of the tableau are said to participate in the violation. If the $q=6$ case of Condition B is violated, then the leftmost 6 in the tableau and all 5's in columns to the left of that 6 are said to participate in the violation. The notation conventions for U, c, V, d, and W are as in Section 3.

General Orthogonal Insertion Procedure. *Do ordinary Schensted bumping until an inserted element d lands in V/W and participates in a violation of Condition A or Condition B. (If this never happens then the usual output tableau Z becomes the new left tableau.) If $d=r$ is an unpaired letter then proceed as in the coarse insertion procedure. In all cases where a violation arises, a tableau X with an empty square Δ is to be formed from V or from U.*

Assume that $d = 5$ (*or* 6) *participates in a* $q = 6$ *violation of Condition A. If the special case* (3) *does not hold, then suppose that d lands at* $W(i+1,1)$ *in the first column* (*or at* $W(j+1,2)$ *in the second column*).

(1) *If there exists a* 6 (*or* 5 *respectively*) *in the key* (*respectively first*) *column of* V, *then replace it with* Δ.

(2) *Otherwise, if there is not an opposite entry in the opposite column, it must necessarily be true that* $d = 6$ *was bumped out of* U *by* $c = 5$. *Then in* U *replace the* 6 *in the second* (*respectively first*) *column with* Δ.

(3) *Suppose that* $d = 6$ *lands at* $W(j+1,2)$, *with* $W(j,2) = 5$, $W(i,1) = 6$, *and* $W(i-1,1) = 5$. *Then in* U *replace* $U(i,1) = 6$ *with* Δ.

Assume that only a violation of Condition B is participated in by $d = 5$ *or* 6.

(4) *If* $d = 6$ *lands at* $W(j,h)$, *then in* V *replace* $V(i,1) = 5$ *with* Δ.

(5) *If* $d = 5$ *lands at* $W(i,1)$, *then in* V *replace the leftmost* 6 *in the jth row with* Δ.

In all cases slide Δ *out of* X *to produce the new left tableau* Z.

Proofs of any implicit or explicit claims in the statement of the procedure are deferred to Lemma 7.1. With the exception of the special case (3) and the notion of key column, the specified action in each case is the obvious action to take if the philosophy of annihilation with preservation of weight stated above is kept in mind. (One should also keep in mind that the notion of "horizontal domino" suggests that the annihilated pair of entries should be from different columns, in contrast to the "vertical domino" symplectic situation, wherein the annihilated pair both come from the first column.) In the following pictures, $* \leqq 4$ and $+ \geqq 7$. Here is an illustration of one possibility for case (1), showing U, c, V, d, W, e, and X:

$$\begin{matrix} * & * & & * & * & & * & * & & * & * \\ & & c=5 & & & & & & & & \\ * & 6 & & * & 5 & & * & 5 & & * & 5 \\ & & & & & d=6 & & & & & \\ * & + & & * & + & & * & 6 & & * & + \\ & & & & & & & & e=+ & & \\ 5 & & & 5 & & & 5 & & & \Delta & \end{matrix}$$

The careful reader might object that Condition B is violated in U and that therefore the bumping should have stopped sooner. However the first sentence of the procedure specifies that bumping stops when a landing entry participates in a violation. Here $c = 5$ was bumped out of the (1, 2) location of T (for which Condition B was not violated); the violation of this condition was not participated in by the * which landed in U. If the 5 at (4, 1) in each tableau in the above example is replaced by a 6 and the 5 at (2, 2) in X is replaced by a 6, then the example will now illustrate one instance of case (2): There is now no 5 in the first column for the problem 6 to cancel with. Here the bumping is backed up two steps to U. Then the problem 5 is dropped before it can bump the 6 in the second column, and the 6 at (4, 1) is replaced by Δ.

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One is supposed to check for (3) first. Here is an example:

$$
\begin{array}{ll} \ast & \ast \\ \ast & 6 \\ 5 & + \\ 6 & \end{array}
\quad c = 5 \quad
\begin{array}{ll} \ast & \ast \\ \ast & 5 \\ 5 & + \\ 6 & \end{array}
\quad d = 6 \quad
\begin{array}{ll} \ast & \ast \\ \ast & 5 \\ 5 & 6 \\ 6 & \end{array}
\quad e = + \quad
\begin{array}{ll} \ast & \ast \\ \ast & 6 \\ 5 & + \\ \Delta & \end{array}
$$

From the stated heuristic principles, one would normally expect that the 6 in the second column would be dropped before landing, while the 5 in the first column was replaced by Δ. However, because of subtle interactions which occur between Conditions A and B when constructing an inverse to the insertion procedure, it is necessary to define this one case in this fashion. Both cases (4) and (5) are relatively straightforward; here is an example of (5), showing V, d, W, e, and X:

$$
\begin{array}{lll} \ast & \ast & \ast \\ \ast & 5 & 6 \\ \ast & + & \\ + & & \end{array}
\quad d = 5 \quad
\begin{array}{lll} \ast & \ast & \ast \\ \ast & 5 & 6 \\ \ast & + & \\ 5 & & \end{array}
\quad e = + \quad
\begin{array}{lll} \ast & \ast & \ast \\ \ast & 5 & \Delta \\ \ast & + & \\ + & & \end{array}
$$

The following boundary values are to be used to make sense of any references in this paper to locations just outside of a tableau T of shape λ; viz. $T(0, j) = 0$, $T(i, 0) = i - 1$, and $T(i, j) = \infty$ for (i, j) otherwise not in λ.

6. Proof of the coarse bijection. In this section we return to the context of Sections 2 and 3: all letters are unpaired, and orthogonal tableaux are those for which Condition A holds for all $s \leqq N$.

LEMMA 6.1. *Given a semistandard tableau T, let r be the minimal value for which Condition A fails. Then there exist i' and j' such that $i' + j' = r + 1$ with $T(i', 1) = T(j', 2) = r$.*

Proof. There are no more than $r - 1$ letters in the first two columns which are $\leqq r - 1$, and at least $r + 1$ which are $\leqq r$. Hence there are at least two r's in the first two columns. But since T is semistandard, there can be no more than one r per column. So there is exactly one r in each of the first two columns, and exactly $r - 1$ entries $\leqq r - 1$.

LEMMA 6.2. *When applied to an N-orthogonal tableau T, the coarse insertion procedure makes sense and produces an N-orthogonal tableau Z.*

Proof. There are initially no violations of A in T, and violations of A can arise only with a newly landed entry taking part. So if r participates in a violation of A after landing in V/W, then r is the minimal value for which the condition is violated by W. By Lemma 6.1, there is another r in W as claimed in the description of the procedure. So the procedure makes sense. It is well known that bumping and sliding out preserve semistandardness, see e.g. Section 3 of [**10**]. Now check that Z is orthogonal. Any violation of A arising from bumping is avoided by the prescribed annihilation, and then bumping stops. At a fixed location, the only possible change due to sliding Δ out is an increase in the size of the entry there, which is an improvement as far as A is concerned. If no annihilations occur, then the definition of the procedure assures that the output satisfies A.

Proof of Theorem 3. The overall framework of the proof is induction on k. From the various definitions and Lemma 6.2, it is clear that the output of the algorithm is contained in the claimed set. (Note that an N-orthogonal tableau is necessarily of N-orthogonal shape, and that the parity of the number of squares in the left tableau is always the same as the parity of k.) Given an orthogonal tableau T of shape λ and an input letter b from the alphabet, the insertion procedure produces a new orthogonal tableau Z of shape μ together with knowledge of the change from λ to μ (or, just the memory of λ). Hence the proof of the bijection consists of constructing a two-sided inverse to the insertion procedure: Given any Z of shape μ and a λ which differs from μ by one square, we need to recover the corresponding tableau T and loose letter b unambiguously.

If λ has one more square than does μ, then adjoin Δ to Z at that location. We will *backslide* Δ to the northwest with a series of *reverse jdt steps*: Interchange Δ with the larger of the two entries which are above or to the left of Δ. (Pick the entry above if the two are equal.) At each step Condition A is tested for by temporarily assigning to Δ the maximum value of the two entries above or to the left of Δ, even if doing this violates column strictness. If μ has one more square than λ, then the usual Schensted inverse is performed: The entry in the square $\mu - \lambda$ is taken as a loose letter and is *unbumped* into the row above. This is repeated until an entry is unbumped out of the first row.

COARSE INSERTION INVERSE. *If $\mu \supset \lambda$, do the usual Schensted inverse to get T and b. If $\mu \subset \lambda$, then backslide Δ until it participates in a violation of Condition A, say in row i. Let $r - 1$ be the maximum value for which Condition A fails here, and set $j = r - i$. Replace Δ with r. To get T and b do the usual Schensted unbumping starting with a loose letter r being unbumped into the jth row.*

Assume $\lambda \supset \mu$. We first confirm that the tableau T produced is orthogonal. Condition A does not become violated during the backsliding before Δ stops, by the checks being performed. From the value of the violation of A, a value $\leqq r-1$ must have been temporarily assigned to Δ. But this was the maximum of the two entries to the left or above. So replacing Δ by r does not violate semistandardness in those directions. If the last switch with Δ was horizontal with something $\leqq r - 1$, then A would have already been violated to value $r - 1$, contradicting the orthogonality of Z. If the last switch with Δ was vertical with something $\leqq r$, then Δ should have stopped backsliding sooner. So semistandardness in the other two directions is all right. Since A was alright before the arrival of Δ, the failure of A at value $r-1$ is by only a deficit of one square. Therefore changing the value of the stopped Δ from the temporary value to r will cure the failure of the $(r-1)$-case of A. A failure at value r will not be created because the new r is the only r now in the first two columns: The maximality of the value of the failure of A rules out having another r in the column other than the one in which Δ stopped. Hence Condition A is satisfied for all values before unbumping begins. And unbumping can only help Condition A. So T is orthogonal.

Next show that this procedure is a two-sided inverse to the "forward" insertion procedure. It is well known that sliding out and backsliding are inverses, as well as bumping and unbumping [**10**]. Hence we need only confirm that the transitions from bumping to sliding out or vica versa occur at the same time, regardless of which procedure is done first. First check that this is true "locally". In the forward procedure, when r is replaced with Δ, and $r-1$ violation of A would exist if Δ were temporarily assigned a value. So an A violation will be encountered here during backsliding. On the other hand suppose that the inverse procedure is followed by the forward procedure. Note that if the r that is unbumped into the jth row by the inverse is instead bumped into the $(j+1)st$ row, then an r violation of A occurs. So here the forward procedure will stop and rewind one step and replace an r with Δ, thereby matching up with the inverse picture at this point. So the forward and inverse operations involving the r entries and Δ are locally two-sided inverses to each other. If the forward procedure is applied to the T and b produced from Z, no premature $< r$ violations of A occur because we are just recreating the $< r$ portion of Z. Hence the claimed inverse is a pre-inverse. Now start with an orthogonal T and a letter b. Even with Δ temporarily assigned values, Condition A will be no worse off for values $\geqq r$ as Δ is slid out than it was in T. Hence the backsliding does not stop prematurely, and we have a post-inverse as well. For the $\mu \supset \lambda$ case the arguments for the unbumping portion of the $\lambda \supset \mu$ case can be used again.

7. Proof of the general bijection. The treatment of unpaired letters within the general algorithm is identical to the treatment of letters in the coarse algorithm. Furthermore, it is easy to see that pairs of letters $q-1 \longleftarrow q$ do not interact in any significant fashion with unpaired letters r. (In other words, the letter r does not care whether $q-1$ and q are paired or not.) Therefore here we will prove just the $N = 2n$ fine special case of the general case. Adding g unpaired letters anywhere (as long as paired letters remain adjacent) will not harm anything, and the proof of the coarse case can be referred to for this portion of a general alphabet. The proof of the even fine case roughly follows the proof of the coarse case; we will omit some routine verifications or explanations, especially if they are the same or similar to verifications in the coarse case.

LEMMA 7.1. *When applied to a general N-orthogonal tableau, the general insertion procedure makes sense and produces a general N-orthogonal tableau.*

The procedure preserves weight.

Proof. First we confirm that the procedure handles all possibilities and makes sense. Suppose that d lands into a violation of A in W. Then this must be the minimal violation of A. Counting d, there must now be a total of at least three 5's and 6's in the first two columns of W, by reasoning similar to that of Lemma 6.1. Note that the non-trivial sense of "key column" can arise in (1) only if $d = 6$ lands in the first column after being bumped by a 5. But then that $5 = c$ participated in a violation of B in V before the bumped 6 landed in W, if there was not a "protecting" 5 above the 6 in the key column. Since this would stop the procedure one step earlier, we see that here there must be a protecting 5 in a non-trivial key column, and (1) applies. So if neither (3) nor (1) holds, we are concerned with just the first two columns in (2); i.e., non-trivial key columns will never occur in (2). Now in (2) there must be an entry equal to d in the other of the first two columns. By semistandardness and the nature of bumping, there cannot be a 6 in the same column with a newly landed $d = 5$. Hence we must have $d = 6$ landing in a column which has a 5 in it, and with another 6 in the other of the first two columns. If a 6 lands directly below a 5, then it must have been bumped out by a 5. So $c = 5$ with there being another 6 in the other column as stated in (2). This same reasoning applies to (3), so we see there that $c = 5$ also. In (4), it is necessary for $W(i, 1) = 5$ in order to have a violation of B. But $i \geqq j$ implies that $V(i, 1) = W(i, 1) = 5$. In (5) it is possible (when $i = j$) for the leftmost 6 in the jth row of V to be different from the leftmost 6 in the jth row of W, but we only need to know that one exists. There are no other cases to consider for d landing into a violation of B: A short argument using semistandardness, the non-existence of earlier violations of B, and the definition of bumping rules out having a 5 land into a "connecting" role in a column g with $1 < g < h$. The wording "If only a violation of B ..." assures unambiguity in the definition of the procedure. Since each of the possibilities discards one 5 and one 6, it is clear that weight is always preserved.

Now verify that a fine $2n$-orthogonal tableau is produced, if we start with one. If no annihilations occur, the output tableau will satisfy A. Can bumping indirectly create a violation of B? This can momentarily happen in the fashion described in the first example after the statement of the insertion rule, but notice that the violation will always immediately disappear, as in that example. Hence an annihilation free insertion produces a fine orthogonal tableau. Case by case, it is clear that violations of either condition participated in by a newly arrived entry are avoided by backing up to U or V and annihilating the offending entries as specified. It is easy to see that the actions specified to avoid violations of either $q = 6$ condition will not cause a secondary $q = 6$ violation of A, and that actions specified to avoid a B violation will not cause another $q = 6$ violation of B. However, it is conceivable that an action specified to avoid an A violation could cause a secondary B violation. Much of the complexity of the algorithm is caused by this consideration. Case by case, it can be checked that this does not happen. The most interesting case is when a 6 lands too far down in the first column with a 5 above it. If there is a protecting 5 in the key column, it is to be erased, temporarily leaving its 6 unprotected. However, the violation of B disappears after Δ is slid out. So at this point it is clear that the $q \leqq 6$ cases of the two conditions are satisfied. Sliding Δ out only helps higher cases of A. Suppose that sliding Δ out cases a $q > 6$ violation of B. Put 9's and 10's into a configuration which violates B. At least one of these participating entries must be in a position which was occupied by an entry < 9 before the slide out began. But this would contradict the $q = 8$ case A for the original tableau T. Hence an insertion involving an annihilation produces a fine orthogonal tableau.

Proof of Theorem 5. Only additional considerations for the fine case will be mentioned; i.e., parts of the proof of Theorem 3 will be used implicitly. Now Δ is to be backslid until either it participates in an even violation of Condition A with its temporarily assigned value, or until a violation of Condition B occurs for a value $q \geqq \Delta + 2$. (Use the value temporarily assigned to Δ.) If an even case of Condition A is violated, let $q - 2 = 4$ be the maximum value for which it fails. If Condition B is violated, let $q = 6$ be the value for which it fails. In other words, Δ has the temporary value 4 when it stops in either case. Let X be the tableau at hand when Δ stops, and let $\alpha - 1$ (respectively $\beta - 1$) be the number of entries in the first (second) column of X other than Δ which are $\leqq 4$. Let $X(\beta, h-1)$ be the rightmost entry (possibly Δ) in the βth row of X which is $\leqq 5$. The usage of the letters V and X does not coincide with the usage in the forward procedure.

FINE INSERTION INVERSE. *If $\mu \supset \lambda$, do the usual Schensted inverse to get T and b. If $\mu \subset \lambda$, then backslide Δ until a non-fine orthogonal tableau X is obtained as described above, with Δ stopping at $X(s,t)$. Replace Δ in X with the letter indicated in Table* 1 *to obtain a tableau V. (There all* 5 *or* 6 *entries in the first two columns of X before the replacement of Δ are listed in the columns headed "1" and "2". In case* (12) *the new* 6 *is to be interchanged with the rightmost* 5 (*if any*) *in the same row; in cases* (7), (9), *and* (13) *the new* 6 *is to be interchanged with the* 5 *below it.) Then using the d and y listed in the table, get T and b from the ordinary Schensted inverse: Start by unbumping d into the yth row of V.*

The inverse procedure could be described more concisely; it is stated in this fashion to facilitate checking that it is a two-sided inverse for all possible cases coming from both the forward and reverse direction.

As before, the $\mu \supset \lambda$ case can be handled with the arguments for the unbumping portion of the $\lambda \supset \mu$ case. So take $\lambda \supset \mu$. For now assume that the details for the region of 5 or 6's work. Apply the inverse to a fine orthogonal Z and for now assume that the inverse always makes sense. So we get a tableau T and a letter b. By the definition of the inverse, there will be no $q > 6$ violations of B in T arising from backsliding. Unbumping cannot cause a $q \leqq 4$ violation of B unless a q violation of A existed in Z. So T is fine orthogonal. The same observation implies that the forward procedure when applied to the T and b

TABLE 1
Inverse Cases

Condition Violated	t	s	1	2	Other Conditions	Δ	d	y	Case Number
A	1	α		None or 5	$X(\beta-1,h)\leq 4$ $X(\beta,h)>6$	5	6	$\beta-1$	(1)
A	1	α		5	$X(\beta-1,h)\leq 4$ $X(\beta,h)=6$	6	5	$\beta-1$	(2)
A	1	α		5	$X(\beta-1,h)\geq 5$ $\alpha>b$	5	6	β	(3)
A	1	α		5	$X(\beta-1,h)\geq 5$ $\alpha=\beta$	5	6	β	(4)
A	1	α		6	$\alpha>\beta$	6	5	$\beta-1$	(5)
A	1	α		6	$\alpha=\beta$	6	5	$\beta-1$	(6)
A	1	α	5			6*	5	$\beta-1$	(7)
A	1	α	6			5	6	$\beta-1$	(8)
A	2	β		5		6*	5	$\alpha-1$	(9)
A	2	β		6		5	6	$\alpha-1$	(10)
A	2	β	5			5	6	α	(11)
A	2	β	None or 6			6*	5	$\alpha-1$	(12)
B	1	α	5	5 or 6		6*	5	$\beta-1$	(13)
B	2	β	5	6		5	6	α	(14)
B	h	$\beta-1$	5	5		5	6	α	(15)

gotten from Z will not prematurely stop because of any $q \leqq 4$ violations of B. Hence the claimed inverse is a pre-inverse. Apply the insertion procedure to a fine orthogonal T and a letter b to get a fine orthogonal Z; then apply the inverse. It can be seen that the fine orthogonality of T together with the way in which Δ is temporarily assigned values imply that the backsliding will not stop prematurely due to a $q > \Delta$ value B violation. So we have a post-inverse.

Now for the 5 or 6 region details. The inverse procedure is unambiguous (the cases are disjoint) and nothing needs to be proved to make sense. Does it handle all possible cases for X? (Note that if Δ slides all of the way to (1, 1), it will be temporarily assigned the value 0 because of the boundary value conventions stated at the end of Section 5. This creates a value 0 violation of Condition A, which is then handled routinely.) Suppose Δ stops due to a $q-2=4$ violation of A, with $q-2$ maximal. Then Δ is in the first or second column, and the maximality of $q-2$ implies that there is at most one 5 or 6 in the first two columns. Having no other such entries is handled in cases (1) and (12). An other such entry is either in the second or the first column, and it is either a 5 or a 6. It is obvious that within this classification the subcases are exhaustive. (In case (1) the "Other Condition" holds automatically if there is no 5 in the second column.) Now suppose that Δ stops due to a $q=6$ violation of B. By the backsliding stopping criteria, Δ has a temporary value $\leqq 4$ and hence the entries

TABLE 2
Forward Cases

Condition Violated	d	Landed in Column	Other 5/6's in 1st Col.	Other 5/6's in 2nd Col.	Other Conditions	Inverse Case
A	5	2	5 & 6			(7)
A	6	2	5	5		(3)
A	6	2	6	5	$h' - 1 > 2$	(2)
A	6	2	6	5	$h' - 1 = 2$	(5)
A	6	2	5 & 6			(8)
A	6	2	5 & 6	5		(13)
A	5	1		5 & 6		(9)
A	6	1	5	5	$W(\beta, h') > 6$ $\alpha > \beta$	(11)
A	6	1	5	5	$W(\beta, h') > 6$ $\alpha = \beta$	(4)
A	6	1	5	5	$W(\beta, h') = 6$	(15)
A	6	1		5 & 6		(10)
A	6	1	5	5 & 6		(14)
B	6	h'	5	None or 5		(1)
B	5	1		5 or 6	$\alpha > \beta$	(12)
B	5	1		6	$\alpha = \beta$	(6)

above and to the left of Δ are $\leqq 4$. Note that Δ cannot be above a connecting 5: semistandardness and the definition of reverse *jdt* steps imply that a 5 cannot be pulled down into a connecting role; and Δ would stop due to the same B violation before it could slide into such a position from the right. Hence either Δ is in the first column above a 5 (case (13)) or in a later column above the 6 participating in the violation. The latter situation is split into subcases (14) and (15), depending upon whether Δ is in the second column or is further to the right. In all three cases it is obvious that the other 5's or 6's in the first two columns are necessarily as claimed. So all possibilities for X are handled.

If Δ stopped backsliding because of A, then replacing Δ with a 5 or 6 will cure the $q-2=4$ violation of A, or else Δ would have stopped sooner. For either an A or a B stoppage, replacing Δ with a 5 or 6 will not cause a $q=6$ violation of A for the same reason. Case by case, it is clear that no B violations exist after the inverse procedure and that the specified interchanges insure semistandardness. Hence the inverse produces fine orthogonal tableaux.

Table 2 indicates how the claimed inverse is to be applied to situations arising from the forward procedure. Here Schensted bumping stops when the newly arrived entry d in W participates in a violation of A or B. The "Other 5/6's" columns refer to W. And α, β, and h' are as α, β, and h were defined for X before. For a $q=6$ violation of A to occur, either a 5 or a 6 lands in the second or first column of W. After d lands, there must be a total of at least three 5 or

6's in the first two columns. As noted in the proof of Lemma 7.1, a B violation can only arise when a 5 lands in the first column or the participating 6 lands in the second or a higher column. So all possibilities are treated.

We will not give all of the verifications that the claimed inverse is actually a two-sided inverse. One of the easy cases is (10). Here there is a 6 in the second column below Δ and $\alpha \geqq \beta+2$. Replace Δ with a 5 and use a 6 to unbump the largest entry $p \leqq 4$ in the first column of V. Now go forward. Letting p bump the 6 back out causes a $q = 6$ violation of A in W when the 6 lands in the first column. The key column is the second, and following (1) of the general insertion procedure returns us to where we started. Cases (5) and (6) help to illustrate why there are so many cases. In both cases one has Δ in the first column and a 6 in the second. In both cases for the forward situation one has a 6 in the first column and a 6 in the second column, with a 5 arriving in row β. When $\alpha > \beta$ in case (5), a violation of A occurs during the forward procedure. However, if $\alpha = \beta$, then a violation of B occurs during the forward procedure before the violation of A can arise. One of the trickier cases is (2). If $\alpha = \beta$ then it can be seen that such an inverse problem can never arise, because Z would have had a $q = 6$ violation of B to start with. So assume $\alpha > \beta$. Then replacing Δ with a 6 yields a semistandard tableau; unbump a 5 into row $\beta - 1$. Going forward, we see that (2) of the general insertion procedure returns us to where we started. A similar situation occurs in (4), the hardest case. Even if $X(\beta-1, h) = 5$, it cannot be the case that $X(\beta, h) = 6$, or else Z would have had a $q = 6$ violation of B to start with. Going forward, bump $d = 6$ into row $\beta + 1$, where it will land in the first column and cause a $q = 6$ violation of A. Since $X(\beta, h) > 6$, the key column is the second column. Hence the 5 in the second column is replaced with Δ. We are not back to where we started, since Δ was one square to the left. However, it easy to see that Δ will now slide out to the same location where it started when we began the inverse backsliding procedure. A similar situation occurs in (12), wherein replacing Δ with a 6 necessitates the special horizontal interchange operation. In three cases it is necessary to vertically interchange the 6 which replaces Δ with a 5 directly beneath it. In all four cases, the special interchange operation is needed because Δ slightly "overshoots" during backsliding. One of vertical interchange cases is (13), where one should take care to note that the special case (3) of the original general insertion procedure comes into play in the forward direction.

8. Representation theory interpretation. The relevant Lie group for this paper is O_N, the "full" orthogonal subgroup of $GL_N(\mathbf{C})$, not the special orthogonal subgroup SO_N. When studying tensor representations, O_N is the more natural group to consider [**6**]. According to [**14**, Theorem 5.7.G], the distinct irreducible tensor representations of O_N are indexed by N-orthogonal partitions. Let $O_N(\lambda)$ denote such a representation. Let O_N^+ set-wise consist of the same group elements as does SO_N: The different notation is used to indicate that we regard this as a subset of elements of O_N, and not as a subgroup. (This avoids the undesirable splitting of O_{2n} characters indexed by partitions with exactly n rows when restricting to SO_N. Any representation of O_N indexed by a shape λ with more than $[N/2]$ rows when restricted to SO_N has the same character as its "associate" representation which is indexed by a shape λ' which has no more than $[N/2]$ rows. Using this fact is actually disadvantageous here, giving another reason to use O_N^+ and not SO_N.) An element g of O_{2n}^+ has $2n$ eigenvalues occurring in inverse pairs: $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$; an element g of O_{2n+1}^+ has $2n$ such eigenvalues together with the value 1 occurring once. The character of $O_N(\lambda)$ at an element g in O_N^+ is a function only of these eigenvalues of g; therefore denote this character by $O_N^+(\lambda; x_1, \ldots, x_n)$. According to Theorem 6.1 of [**6**], this character can be combinatorially described as $O_N(\lambda; x_1, \ldots, x_n)$, the orthogonal Schur function defined in Section 4 of this paper with fine orthogonal tableaux. More simply, by Theorem 3.1 of [**6**], the dimension of the representation $O_N(\lambda)$ is equal to the number of orthogonal tableaux of shape λ.

By definition, the group O_N acts on the space $V = \mathbf{C}^N$ of column vectors. This representation is denoted $O_N(1)$ and has character $(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}(+1))$. The following result can be proved by applying the techniques of Sections 8 and 9 of [**7**] to the one determinant expression for $O_N^+(\lambda; x_1, \ldots, x_n)$ given in Appendix A2 of [**6**].

PROPOSITION 8. *The result of multiplying an irreducible orthogonal character by the character of the defining representation is:*

$$O_N^+(1; x) O_N^+(\lambda; x) = \sum O_N^+(\mu; x),$$

where the sum is over all N-orthogonal shapes μ which can be obtained from λ by adjoining or removing one square.

Iterating this result $k-1$ times gives another proof of Corollary 4. With any proof, Corollary 4 describes the decomposition of the representation $\otimes^k V$ of O_N into irreducibles. Since there is (at least in theory) a basis for the representation space of $O_N(\lambda)$ whose elements are indexed by fine N-orthogonal tableaux, we see that the main result of this paper, Theorem 5, implies that our general orthogonal algorithm does indeed model tensor representations of O_N in the same sense that the Schensted and Berele algorithms model tensor representations of GL_N and Sp_{2n} respectively.

9. Comments and open problems. The existence of such an algorithm for orthogonal representations is yet another manifestation of the remarkable and mysterious correspondence between jeu de tacquin and Schensted-like procedures on the one hand and representation theory on the other. We do not know of any explanation at a deeper level. Such algorithms seem to inexorably arise; this algorithm did so in spite of some adversity in the form of the complicated notion of fine orthogonal tableau. In light of Berele's algorithm and the duality of tensor representations of the orthogonal and symplectic groups (e.g. [**8**, Proposition 1]), this algorithm is pretty much what this author had expected all along, especially with the following heuristic picture [**8**, Section 4] in mind. The bilinear forms defining Sp_{2n} and O_N are respectively elements of $\wedge^2 V^*$ and S^2V^*. Corresponding to these forms are elements of $\otimes^2 V$ which are fixed under the actions of Sp_{2n} and O_N. Picture them as vertical and horizontal dominoes respectively. Part of the description of the restrictions of the λth tensor representation of GL_N to Sp_{2n} and O_N uses pavings of subshapes of λ with vertical and horizontal dominos respectively. The key step in Berele's algorithm can be viewed as annihilating an inverse pair of letters when they form a vertical domino which been has pushed too far down in the tableau. The annihilation step in the orthogonal algorithm presented here can be viewed as annihilating an inverse pair of letters when they form a (possibly broken) horizontal domino which has been pushed too far down. Thus, this algorithm seems more natural than Sundaram's [**11**], which retained the vertical annihilation of Berele's algorithm. Her algorithm worked only for the odd cases $SO_{2n+1}(= O^+_{2n+1}$, but with a different set of irreducible representations). The tableaux she employed were not semistandard and therefore do not seem as closely related to the Young symmeterizer construction of irreducible tensor representations.

Sundaram's thesis [**12**] is based upon Berele's algorithm, and contains several nice results concerning the algorithm and tensor representations of Sp_{2n}. These include: a combinatorial description of the branching coefficients for the restriction from GL_{2n} to Sp_{2n} mentioned above, a combinatorial proof [**13**] of the Cauchy series identity for Sp_{2n}, and a characterization of the usual Knuth relations in this context. Hopefully these results can be extended to the orthogonal case via the algorithm given here. Finding the analog of the Knuth relations, viz. characterizing which input words are mapped to the same left tableau by the algorithm, is an open problem in both cases.

The paper [**7**] extended Berele's algorithm to handle an input alphabet consisting of both paired and unpaired letters. However, the unpaired letters corresponded to free eigenvalues (as opposed to unity eigenvalues as is the case here). The purpose of doing this was to study tensor representations of "intermediate" symplectic groups (especially Sp_{2n+1}), defined by possibly degenerate skew symmetric bilinear forms. It is natural to hope that similar results can be found in the orthogonal context.

Although Weyl first classifies and describes tensor representations of O_N in terms of N-orthogonal partitions in Chapter V of [**14**], he soon switches in Chapter VII to the more familiar requirement of having no more than n rows in the partition (where $N = 2n$ or $2n+1$), together with the notion of associate representation. All other authors since, including the influential Littlewood, seem to do the same. We believe that the original orthogonal partition indexing of irreducible tensor representations is more natural because of the following fact [**14**, Theorem 5.7.A]: The subspace of trace free tensors of symmetry type λ is null unless the sum of the first two column lengths of λ is $\leqq N$. Perhaps using this original indexing will also be helpful elsewhere, say for example while studying the Brauer centralizer algebra for the orthogonal $\otimes^k V$ context (e.g. [**5**]). Note that the lengths of the $(n+1)st$ through Nth "extra" rows in N-orthogonal partitions are 0 or 1. These are the residue classes of $\mathbf{Z}$ modulo 2, which is the length of a horizontal domino.

Addendum. Defining Condition B as "... then either $T(j-1,h) = 2p-1$ or the value $2p$ case of Condition A is violated" and requiring $V(i+1,1) > 6$ and $V(j+1,2) > 6$ in the definition of key column would have been slightly better. The first change would prevent simultaneous violations of A and B during insertion, which are now handled by the wording "only a violation of B" in the algorithm.

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