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COHOMOLOGY OF SPLIT ALGEBRAS AND OF TRIVIAL EXTENSIONS*

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TO IDUN REITEN FOR HER 60TH BIRTHDAY

Abstract. We consider associative algebras Λ over a field provided with a direct sum decomposition of a two-sided ideal M and a sub-algebra A – examples are provided by trivial extensions or triangular type matrix algebras. In this relative and split setting we describe a long exact sequence computing the Hochschild cohomology of Λ. We study the connecting homomorphism using the cup-product and we infer several results, in particular the first Hochschild cohomology group of a trivial extension never vanishes.

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1. Introduction. In this paper we consider split algebras $\Lambda = A \oplus M$ where A is a subalgebra of Λ and M is a two-sided ideal. Our main purpose is to compute the Hochschild cohomology $H^*(\Lambda, \Lambda)$ using the cohomology theory of A and M.

Our motivations are three-fold. First, decompositions providing split algebras arise in various examples. Note for instance that trivial extensions and triangular matrix algebras (see below) are split algebras, their cohomology has been investigated recently by several authors [**3**, **8**, **12**]. Second it is known that degree one Hochschild cohomology provides insight to representation theory through universal covers, and its vanishing is

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related to the notion of simply connected algebras, [**17**]. D. Happel shows in [**10**] that a finite representation type algebra over an algebraically closed field of characteristic zero is simply connected if and only if the first Hochschild cohomology H^1 space of its Auslander algebra is zero. Moreover R. Buchweitz and S. Liu provided a proof of the same statement assuming only that the field is algebraically closed (Oberwolfach 2000). It has been suspected that for a finite dimensional algebra over an algebraically closed field, the vanishing of the first Hochschild cohomology space implies that its ordinary quiver has no oriented cycles. This was proved wrong and a family of counterexamples can be found in [**4**]. It is also conjectured that a tilted algebra is simply connected if and only if its first Hochschild cohomology space vanishes. This has been proved for tame tilted algebras, see [**2**]. In another direction, degree two cohomology concerns the deformation theory of algebras, see [**7**]. Finally split algebras are interesting to study in relation to Happel's question [**9**]: if the Hochschild cohomology vector spaces of a finite dimensional algebra vanish after some degree, is the algebra of finite homological dimension? The present paper is a first step for considering this question in a relative and split framework.

We describe now the contents of each section of the article.

In Section 2 we obtain a long exact sequence involving $H^*(\Lambda, \Lambda)$. If M is a projective left or right A-module the other terms of this sequence are direct sums of vector spaces $\mathrm{Ext}^q_{A-A}(M^{\otimes^p_A}, X)$ for $p+q=*$ or $p+q=*+1$ depending on whether $X=M$ or $X=A$. We study in detail the connecting homomorphism of this long exact sequence in order to obtain results on the dimensions of the Hochschild cohomology vector spaces of Λ.

More precisely we obtain for any Λ-bimodule X a double complex whose total cohomology is the Hochschild cohomology of Λ with coefficients in X. The first quadrant spectral sequence involved converges, the terms at the first level are unknown but can probably be approximated through new spectral sequences. Nevertheless, if M is projective as a left or right A-module, we show that the cohomology of the p-th column $\mathcal{C}^p(X)$ is $\mathrm{Ext}^*_{A-A}(M^{\otimes^p_A}, X)$.

In Section 3 we do not assume that M is projective on one side. When M is a square zero ideal and the bimodule X verifies $MX = XM = 0$ then the horizontal differentials of the double complex are 0. Consequently the Hochschild cohomology $H^*(\Lambda, X)$ is the direct sum of the cohomologies of the columns, namely

$$H^n(\Lambda, X) = \bigoplus_{p+q=n} H^q(\mathcal{C}^p(X)).$$

Of course the Λ-bimodule Λ does not verify the above hypothesis. Nevertheless the bimodules on both sides of the sequence $0 \to M \to \Lambda \to \Lambda/M \to 0$ do, and we consider the corresponding long exact sequence in Hochschild cohomology. An interesting result we obtain is that the connecting homomorphism

$$\delta : H^n(\Lambda, \Lambda/M) \to H^{n+1}(\Lambda, M)$$

is bigraded of bidegree $(1, 0)$, that is $\delta = \oplus_{p+q=n} \delta^{p,q}$, where

$$\delta^{p,q} : H^q(\mathcal{C}^p(\Lambda/M)) \to H^q(\mathcal{C}^{p+1}(M)).$$

Moreover in Section 4 we provide a precise description of $\delta^{p,q}$ involving only two terms, using the cup product of a cocycle with the identity endomorphism of M. This

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description enables us to determine whether $\delta^{p,q}$ annihilates or not in some interesting cases.

In Section 5 we consider a special case of split algebras: trivial extensions TA. They are of the form $A \oplus DA$ where DA is the dual A-bimodule of A endowed with the zero multiplicative structure. We show that its first Hochschild cohomology is a direct sum of four vector spaces (see Theorem 5.5). One of the factors is the center of A, in particular $H^1(TA, TA)$ never vanishes. More generally we show that $H^n(A, A) \oplus H_n(A, A)$ is a direct summand of $H^n(TA, TA)$. This result is a consequence of the fact that for these algebras the component $\delta^{0,q}$ of the connecting homomorphism is zero.

As a direct consequence of our computations we show that $H^1(TA, TA) = k \oplus H^1(A, A)$ for a *one-way* algebra A, see Definition 5.10. This generalizes previous results obtained in [**15**, **16**].

Finally we specialize to triangular matrix algebras and one-point extensions the results we have obtained for general split algebras. In this way we recover computations performed in [**6**, **8**, **9**, **12**].

2. Split algebras and the double complex. Let k be a field. As stated in the introduction a split algebra Λ is a k-algebra with a subalgebra A and a two-sided ideal M such that $\Lambda = A \oplus M$. In other words Λ consists of the following data: a k-algebra A and a multiplicative A-bimodule M with a product, that is an associative A-bimodule map $M \otimes_A M \to M$, $m \otimes m' \mapsto m.m'$. The algebra structure in $A \oplus M$ is given by

$$(a + m)(a' + m') = aa' + am' + ma' + m.m'.$$

Let X be a Λ-bimodule. As usual, the Hochschild cohomology vector spaces of Λ with coefficients in X are the cohomology groups of the following cochain complex, (see for instance [**1**, **5**, **14**, **18**])

$$0 \longrightarrow X \xrightarrow{d} \mathrm{Hom}_k(\Lambda, X) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Hom}_k(\Lambda^{\otimes n}, X) \xrightarrow{d} \cdots$$

where for $n \geq 1$

$$\begin{aligned} df(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 f(x_2 \otimes \cdots \otimes x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\ + (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1} \end{aligned}$$

and for $x \in X$ and $\lambda \in \Lambda$

$$(dx)(\lambda) = \lambda x - x\lambda.$$

Since $\Lambda = A \oplus M$, we have a decomposition of $\Lambda^{\otimes n}$ as a direct sum of vector spaces in terms of A and M: let $M^{p,q}$ be the sub-vector space spanned by $(p+q)$-tensors $x_1 \otimes \cdots \otimes x_{p+q}$ such that exactly p of the x_i's belong to M while the other x_i's belong to A. Clearly

$$\Lambda^{\otimes n} = \bigoplus_{p+q=n} M^{p,q}.$$

Moreover the Hochschild complex above organizes in a double complex whose (p, q)-spot is $\mathrm{Hom}_k(M^{p,q}, X)$. Indeed, the image of d restricted to $\mathrm{Hom}_k(M^{p,q}, X)$ is contained in $\mathrm{Hom}_k(M^{p+1,q}, X) \oplus \mathrm{Hom}_k(M^{p,q+1}, X)$. The horizontal and vertical components of d are denoted d_h and d_v respectively. The cohomology of the total complex is $H^*(\Lambda, X)$.

PROPOSITION 2.1. *The vertical differentials d_v of the above double complex depend neither on the product of M nor on the actions of M on X.*

Proof. Let $f \in \mathrm{Hom}_k(M^{p,q}, X)$, in other words $f : \Lambda^{\otimes(p+q)} \to X$ vanishes on $(p+q)$-tensors which have not exactly p components of M and q components of A. We evaluate $d_v f$ on a tensor $(x_1 \otimes \cdots \otimes x_{p+q+1}) \in M^{p,q+1}$. We shall see that the terms where the product of M or the action of M on X appear are zero. If $x_1 \in M$ then $x_1 f(x_2 \otimes \cdots \otimes x_{p+q+1}) = 0$ since $(x_2 \otimes \cdots \otimes x_{p+q+1}) \notin M^{p,q}$, regardless the action of M on X. Similarly $f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{p+q+1}) = 0$ if x_i and x_{i+1} belong to M since the tensor belongs to $M^{p-1,q+1}$ regardless the value of $x_i x_{i+1}$. The behaviour of the last term of the coboundary formula is analogous to the first one. $\square$

In order to determine the vertical cohomology, we first note that the cohomology of the 0-th column is the Hochschild cohomology $H^*(A, X) = \mathrm{Ext}^*_{A-A}(A, X)$, where X is considered as an A-bimodule by restriction of scalars.

We now simplify the notation by omitting the tensor product sign for tensor products over the ground field k; tensor signs between vectors are replaced by commas.

The following result is announced in [**6**], but the proof provided there is incomplete.

THEOREM 2.2. *The cohomology of the column $p = 1$ is* $\mathrm{Ext}^*_{A-A}(M, X)$.

Proof. We will first provide a free resolution of M as an A-bimodule. Consider the bar resolution of M as a left A-module (see [**18**, 8.6.12]),

$$\cdots \to AAM \to AM \to 0$$

and the Hochschild resolution of A as an A-bimodule

$$\cdots \to AAA \to AA \to 0.$$

Tensoring them over A provides the complex

$$\cdots \to AMAA \oplus AAMA \to AMA \to 0.$$

The cycles in each degree of the Hochschild resolution are projective left A-modules since the resolution splits as a sequence of left A-modules. The Künneth formula ensures that the last complex has zero homology in positive degrees and $M \otimes_A A = M$ in degree zero. Next we apply the functor $\mathrm{Hom}_{A-A}(-, X)$ and we use the identification $\mathrm{Hom}_{A-A}(AZA, X) = \mathrm{Hom}_k(Z, X)$ in order to verify that the coboundaries provide the first column of the double complex. $\square$

REMARK 2.3. A direct computation shows that $\mathrm{Hom}_{A-A}(M^{\otimes_A p}, X)$ is the zero degree cohomology of the p-th column. In order to generalize this result to the other vertical cohomology groups for $p \geq 2$ we need to assume additional hypothesis on M as follows.

THEOREM 2.4. *Let A be a k-algebra, M be an A-bimodule which is right or left projective and X be an $A \oplus M$-bimodule. The cohomology of the p-th column in degree q is* $\mathrm{Ext}^q_{A-A}(M^{\otimes^p_A}, X)$.

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The proof of the above theorem is given at the end of this section, as a consequence of the next result.

PROPOSITION 2.5. *Let C, B and A be k-algebras, ${}_CN_B$ be a $C-B$-bimodule and ${}_BM_A$ be a $B-A$-bimodule. The homology of the following complex is equal to* $\mathrm{Tor}_*^B(N, M)$

$$\cdots \xrightarrow{b'} CCNMA \oplus CNBMA \oplus CNMAA \xrightarrow{b'} CNMA \longrightarrow 0,$$

where the term in degree n is a free $C-A$-bimodule of the form CZA and

$$Z = \bigoplus_{\substack{i,j,k\geq 0;\\ i+j+k=n}} C^k N B^j M A^i.$$

Note that the boundary formula is provided by the standard resolution of an algebra as a bimodule:

$$b'(c, x_1, \dots, x_n, a) = (cx_1, \dots, x_n, a) + \sum_{i=1}^{n-1}(-1)^i(c, x_1, \dots, x_ix_{i+1}, \dots, x_n, a)$$
$$+(-1)^n(c, x_1, \dots, x_na).$$

By assumption products of type nm are zero when $n \in N$ and $m \in M$; the bimodule action gives products in case of elements of the form bm, nb, cn, etc. Each summand of the formula must have an element of C and an element of A on each side, otherwise its value is 0.

REMARK 2.6. Before proving Proposition 2.5 we note that for a $C-A$-bimodule X, the functor $\mathrm{Hom}_{C-A}(-, X)$ applied to the above complex provides the second column $\mathcal{C}^2(X)$ in case $A = B = C$ and $N = M$, since by virtue of Proposition 2.1 we can assume that the product of M and the actions of M on X are trivial.

Proof. (Proposition 2.5) We consider the bar resolution of M as a left B-module

$$\cdots \longrightarrow BBM \longrightarrow BM \longrightarrow 0$$

and we apply the functor $N \otimes_B -$

$$\cdots \longrightarrow N \otimes_B BBM \longrightarrow N \otimes_B BM \longrightarrow 0,$$

obtaining in this way the standard complex which is used to compute $\mathrm{Tor}_*^B(N, M)$

$$\cdots \longrightarrow NBM \longrightarrow NM \longrightarrow 0. \tag{1}$$

Next we use the Hochschild resolution of C as a C-bimodule

$$\cdots \longrightarrow CCC \longrightarrow CC \longrightarrow 0.$$

Its homology is non-zero only in degree zero, with value C. Tensoring the above resolution with (1) over C gives

$$\cdots \longrightarrow CNBM \oplus CCNM \longrightarrow CNM \longrightarrow 0.$$

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We assert that the homology of this complex is still $\mathrm{Tor}_*^B(N, M)$. Indeed we can use again the Künneth formula since the set of cycles of the bar resolution of C (which splits as a sequence of right C-modules) is a projective right module, and the homology is zero except in degree zero with value C. Hence the homology of the above complex is the tensor product of the homologies, that is $C \otimes_C \mathrm{Tor}_*^B(N, M) = \mathrm{Tor}_*^B(N, M)$.

Finally we consider the Hochschild resolution of A and we tensor it over A with the above complex. As before, the resulting homology is $\mathrm{Tor}_*^B(N, M) \otimes_A A = \mathrm{Tor}_*^B(N, M)$. A non difficult computation shows that the resulting boundaries coincide with those described in the statement. □

PROPOSITION 2.7. *Let A, B, C, D be k-algebras and ${}_DU_C$, ${}_CN_B$, ${}_BM_A$ be bimodules. Assume* $\mathrm{Tor}_*^B(N, M) = 0$ *in positive degrees. Then* $\mathrm{Tor}_*^C(U, N \otimes_B M)$ *is the homology of the following complex:*

$$\cdots \longrightarrow DDUNMA \oplus DUCNMA \oplus DUNBMA \oplus DUNMAA \longrightarrow DUNMA \longrightarrow 0$$

with n-th term DZA where

$$Z = \bigoplus_{i+j+k+l=n} D^l U C^k N B^j M A^i.$$

Proof. By hypothesis the complex of the preceding proposition has homology only in degree zero, with value $N \otimes_B M$. Since the modules are C-free on the left, this complex is a resolution of the left C-module $N \otimes_B M$. Applying the functor $U \otimes_C -$ provides a complex whose homology is $\mathrm{Tor}_*^C(U, N \otimes_B M)$.

We consider as in the previous proposition a resolution of the algebra D as a D-bimodule. The Künneth formula shows that tensoring this resolution over D with the complex obtained above provides a new complex whose homology is $\mathrm{Tor}_*^C(U, N \otimes_B M)$. □

REMARK 2.8. If $\mathrm{Tor}_*^C(U, N \otimes_B M)$ is zero in positive degrees, the complex of Proposition 2.7 becomes a projective resolution of the $D-A$-bimodule $U \otimes_C N \otimes_B M$. Applying the functor $\mathrm{Hom}_{D-A}(-, {}_DX_A)$ to this projective resolution gives a cochain complex whose homology is $\mathrm{Ext}_{D-A}^*(U \otimes_C N \otimes_B M, X)$.

Proof. (Theorem 2.4) Since M is right or left projective, then $\mathrm{Tor}_*^A(M, M)$ is zero in positive degrees. The complex of Proposition 2.5, in the case $N = M$, $A = B = C$, is a projective resolution of the A-bimodule $M \otimes_A M$. We have already noticed that applying the functor $\mathrm{Hom}_{A-A}(-, X)$ to this resolution yields precisely the second column of the double complex and, consequently, its cohomology is $\mathrm{Ext}_{A-A}^*(M \otimes_A M, X)$.

The same procedure applies to Proposition 2.7, and we obtain that the third columns has cohomology $\mathrm{Ext}_{A-A}^*(M^{\otimes_A^3}, X)$. By induction the end of the proof is now obvious. □

3. The connecting homomorphism. We return to the double complex which we use to compute the Hochschild cohomology of an arbitrary split algebra $\Lambda = A \oplus M$ with coefficients in a Λ-bimodule X. The filtration of the total complex arising from the columns provides a first quadrant spectral sequence, hence converging to $H^*(\Lambda, X)$ (see for instance [**13**, **18**]). In the preceding section we have computed the first level vector spaces $E_1^{*,*}$, assuming M is A-projective on one side. However the differential at the first level appears hard to compute even with these hypothesis on M.

We focus on a special case of interest for specific computations that we will perform in the next section.

THEOREM 3.1. *Let A be a k-algebra, M be an A-bimodule, $\Lambda = A \oplus M$ be the corresponding split algebra with $M^2 = 0$, and let X be a Λ-bimodule verifying $MX = XM = 0$ (in other words X is an A-bimodule with actions trivially extended to Λ). Then the horizontal coboundaries of the double complex are zero. As a consequence*

$$H^n(\Lambda, X) = \bigoplus_{p+q=n} H^q(\mathcal{C}^p(X)),$$

where $\mathcal{C}^p(X)$ denotes the p-th column.

Proof. Let $\varphi : M^{p,q} \to X$ be a cochain in other words $\varphi : \Lambda^{\otimes(p+q)} \to X$ is a cochain that vanishes on each component $M^{p',q'}$ of $\Lambda^{\otimes(p+q)}$ different from $M^{p,q}$.

By definition $d_h\varphi = d\varphi|_{M^{p+1,q}}$ therefore

$$\begin{aligned} d_h\varphi(x_1, \dots, x_{p+q+1}) = x_1\varphi(x_2, \dots, x_{p+q+1}) + \sum_{i=1}^{p+q}(-1)^i\varphi(x_1, \dots, x_ix_{i+1}, \dots, x_{p+q+1}) \\ + (-1)^{p+q+1}\varphi(x_1, \dots, x_{p+q})x_{p+q+1}. \end{aligned}$$

The first term is zero, indeed if $x_1 \in M$ we have $MX = 0$, while if $x_1 \in A$ then $(x_2, \dots, x_{p+q+1}) \in M^{p+1,q-1}$ and φ is zero when evaluated on it. The last term is zero for the same reasons. Each middle term vanishes since either both x_i and x_{i+1} belong to M (hence $x_ix_{i+1} = 0$) or $(x_1, \dots, x_ix_{i+1}, \dots, x_{p+q+1}) \in M^{p+1,q-1}$. □

The above decomposition and the results of the previous section yield the following:

COROLLARY 3.2. *Let A be a k-algebra and M be an A-bimodule projective on one side. Let Λ be the split algebra $A \oplus M$ with $M^2 = 0$, and let X be a Λ-bimodule such that $MX = XM = 0$. Then*

$$H^n(\Lambda, X) = \bigoplus_{p+q=n} \operatorname{Ext}^q_{A-A}(M^{\otimes_A^p}, X),$$

where $M^{\otimes_A^0} = A$.

We now consider for a split algebra $\Lambda = A \oplus M$ the following exact sequence of Λ-bimodules

$$0 \longrightarrow M \longrightarrow \Lambda \xrightarrow{\pi} \Lambda/M = A \longrightarrow 0.$$

Note that M is a Λ-bimodule since M is a two-sided ideal of Λ. Of course Λ/M is in fact A considered as a Λ-bimodule with zero actions of M on both sides. This exact sequence of coefficients provides a long exact sequence in Hochschild cohomology

$$\begin{aligned} 0 \longrightarrow H^0(\Lambda, M) \longrightarrow H^0(\Lambda, \Lambda) \xrightarrow{\pi^0} H^0(\Lambda, A) \xrightarrow{\delta^0} \\ H^1(\Lambda, M) \longrightarrow H^1(\Lambda, \Lambda) \xrightarrow{\pi^1} H^1(\Lambda, A) \xrightarrow{\delta^1} \\ \dots \\ H^n(\Lambda, M) \longrightarrow H^n(\Lambda, \Lambda) \xrightarrow{\pi^n} H^n(\Lambda, A) \xrightarrow{\delta^n} \end{aligned}$$

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Our next purpose is to describe the connecting homomorphism δ^n in order to combine this information with knowledge on $H^*(\Lambda, A)$ and $H^*(\Lambda, M)$. This will provide information on $H^*(\Lambda, \Lambda)$ which is our main purpose. We begin by studying δ^0.

PROPOSITION 3.3. *Let A be a k-algebra, M be an A-bimodule and let $\Lambda = A \oplus M$ be the corresponding split algebra. The above connecting homomorphism δ^0 vanishes if and only if the center A^A of A has symmetric action on M (that is $am = ma$ for every $a \in A^A$ and $m \in M$).*

Proof. The center Λ^Λ of $\Lambda = A \oplus M$ is as follows:

$$\Lambda^\Lambda = [A^A \cap A^M] \oplus [M^M \cap M^A],$$

where A^M are the elements of A acting symmetrically on M, while M^M is the center of the multiplicative bimodule M and $M^A = H^0(A, M) = \{m \in M \mid am = ma \text{ for every } a \in A\}$.

In the long exact sequence above

$$0 \longrightarrow M^\Lambda \longrightarrow \Lambda^\Lambda \xrightarrow{\pi^0} A^\Lambda = A^A \xrightarrow{\delta^0} H^1(\Lambda, M) \longrightarrow \cdots$$

we have $\mathrm{Im}\,\pi^0 = A^A \cap A^M$. Hence $\mathrm{Ker}\delta^0 = A^A \cap A^M$ so $\delta^0 = 0$ if and only if $A^A \cap A^M = A^A$ which is equivalent to $A^A \subset A^M$. □

EXAMPLE 3.4. Let f be an automorphism of A and let $M = {}^fA$ be the A-bimodule A with left action twisted by f. Then $\delta^0 = 0$ if and only if f is the identity on central elements of A. Indeed, let Fix_f be the subalgebra of elements fixed by f. For $M = {}^fA$ we have $A^M = A^A \cap \mathrm{Fix} f$.

EXAMPLE 3.5. In case Λ is the trivial extension TA of A we have $M = DA$ and the center of A acts symmetrically on M. Consequently $\delta^0 = 0$.

We next prove that each connecting homomorphism has bidegree $(1, 0)$.

PROPOSITION 3.6. *Let $\Lambda = A \oplus M$ be a split algebra with $M^2 = 0$. The connecting homomorphism*

$$\delta^n : H^n(\Lambda, \Lambda/M) \to H^{n+1}(\Lambda, M)$$

has bidegree $(1, 0)$ *with respect to the decomposition provided in Theorem 3.1.*

Proof. We denote by $H^q(\mathcal{C}^p(X))$ the cohomology of the p-th column in degree q, where X is a Λ-bimodule. Since we proved that the horizontal differentials are zero when $MX = XM = 0$, we have for such an X

$$H^n(\Lambda, X) = \bigoplus_{p+q=n} H^q(\mathcal{C}^p(X)).$$

Both M and Λ/M verify the above assumption on X. Then

$$\delta^n : \bigoplus_{p+q=n} H^q(\mathcal{C}^p(\Lambda/M)) \to \left(\bigoplus_{p+q=n} H^q(\mathcal{C}^{p+1}(M)) \right) \oplus H^{n+1}(\mathcal{C}^0(M)).$$

We assert that the image of $\delta^n|_{H^q(\mathcal{C}^p(\Lambda/M))}$ is contained in $H^q(\mathcal{C}^{p+1}(M))$, hence $\delta^n = \oplus_{p+q=n}\delta^{p,q}$ where

$$\delta^{p,q} : H^q(\mathcal{C}^p(\Lambda/M)) \to H^q(\mathcal{C}^{p+1}(M)).$$

In order to prove the assertion let $\varphi : M^{p,q} \to A$ be a cocycle of the Hochschild complex of Λ/M. We use the given inclusion $\Lambda/M \subset \Lambda = A \oplus M$ to obtain $\bar{\varphi} : M^{p,q} \to \Lambda$, taking into account that A inside Λ has a non trivial action of M on it. The image of the coboundary of $\bar{\varphi}$ in the Hochschild complex of Λ is contained in M and provides a well defined element in $H^{p+q+1}(\Lambda, M)$ by general arguments. Considering the double complex for Λ, we have two components $d\bar{\varphi} = d_v\bar{\varphi} + d_h\bar{\varphi}$. In fact $\delta\varphi = d\bar{\varphi}$. Now we will prove that $d\bar{\varphi}$ has zero values on every component of $\Lambda^{\otimes n+1}$ except maybe on $M^{p+1,q}$. In order to prove that $d_v\bar{\varphi} = 0$, let $(x_1, \dots, x_{p+q+1}) \in M^{p,q+1}$, then

$$d_v\bar{\varphi}(x_1, \dots, x_{p+q+1}) = x_1\bar{\varphi}(x_2, \dots, x_{p+q+1}) + \sum_{i=1}^{p+q}(-1)^i\bar{\varphi}(x_1, \dots, x_i x_{i+1}, \dots, x_{p+q+1})$$
$$+ (-1)^{p+q+1}\bar{\varphi}(x_1, \dots, x_{p+q})x_{p+q+1}.$$

We observe that the middle terms remain unchanged for φ or $\bar{\varphi}$, namely

$$\sum_{i=1}^{p+q}(-1)^i\bar{\varphi}(x_1, \dots, x_i x_{i+1}, \dots, x_{p+q+1}) = \sum_{i=1}^{p+q}(-1)^i\varphi(x_1, \dots, x_i x_{i+1}, \dots, x_{p+q+1}).$$

Concerning the first and the last terms we first assume that both x_1 and x_{p+q+1} belong to A, then all the terms of the sum are in A and coincide with the terms of $d_v\varphi$. Since $\varphi : M^{p,q} \to A$ is a cocycle, we obtain that the value of the above expression is zero.

If $x_1 \in M$ and $x_{p+q+1} \in A$ then $(x_2, \dots, x_{p+q+1}) \in M^{p-1,q+1}$, hence φ is zero on it. The last term remains in A, and all the terms of $d_v\bar{\varphi}$ evaluated on the tensor $(x_1, \dots, x_{p+q+1})$ coincide with the terms of $d_v\varphi$ (the first one vanishes in both cases). Since φ is a cocycle, we infer that $d_v\bar{\varphi}(x_1, \dots, x_{p+q+1}) = 0$ also in this case. The remaining cases $x_1, x_{p+q+1} \in M$, or $x_1 \in A$, $x_{p+q+1} \in M$ can be studied in an analogous way.

We conclude that $d\varphi = d_h\bar{\varphi} \in \mathrm{Hom}(M^{p+1,q}, M)$. $\square$

4. Operations. We introduce operations in the double complex $\mathrm{Hom}_{A-A}(M^{*,*}, X)$ of the previous sections in order to describe the (p, q)-component $\delta^{p,q}$ of the connecting homomorphism

$$\delta^{p,q} : H^q(\mathcal{C}^p(\Lambda/M)) \to H^q(\mathcal{C}^{p+1}(M)).$$

Recall that the following is an operation on Hochschild cohomology of bimodules over a k-algebra A (see **[5, 7]**). Let X and Y be Λ-bimodules, $f : \Lambda^{\otimes_n} \to X$ and $g : \Lambda^{\otimes_m} \to Y$ be Hochschild cochains. The *cup product* (see **[7]**) $f \smile g$ is defined as the composition

$$\Lambda^{\otimes n+m} \cong \Lambda^{\otimes n} \otimes \Lambda^{\otimes m} \xrightarrow{f\otimes g} X \otimes Y \longrightarrow X \otimes_\Lambda Y.$$

One has $d(f \smile g) = df \smile g + (-1)^n f \smile dg$, so the product is well defined in cohomology:

$$H^n(\Lambda, X) \otimes H^m(\Lambda, Y) \to H^{n+m}(\Lambda, X \otimes_\Lambda Y).$$

In case $\Lambda = A \oplus M$ is a split algebra this operation goes clearly through the double complex, that is, if $f : M^{p,q} \to X$ and $g : M^{p',q'} \to Y$ are cochains, then

$$f \smile g : M^{p,q} \otimes M^{p',q'} \to X \otimes_\Lambda Y$$

is the product cochain. Note that $M^{p,q} \otimes M^{p',q'}$ is naturally a direct summand of $M^{p+p',q+q'}$, the value of $f \smile g$ on the complement is zero.

THEOREM 4.1. *Let $\Lambda = A \oplus M$ be a split algebra with $M^2 = 0$, and let*

$$0 \to M \to \Lambda \to \Lambda/M \to 0$$

be the corresponding short exact sequence. The (p, q)-component $\delta^{p,q}$ of the connecting homomorphism δ in the long exact Hochschild cohomology sequence of Λ is given by

$$\delta^{p,q}\varphi = 1_M \smile \varphi + (-1)^{p+q+1}\varphi \smile 1_M.$$

REMARK 4.2. In the statement of this theorem, $\varphi : M^{p,q} \to \Lambda/M$ is an arbitrary cocycle and $1_M : M \to M$ is the identity morphism which is indeed a 1-cocycle; it belongs to the (1, 0)-spot of the double complex and corresponds to the projection $A \oplus M \to M$ in the usual Hochschild complex of M. Note that if $M \neq 0$ this projection is a non-zero element in $H^1(\Lambda, M)$.

Note also that we have $\Lambda/M \otimes_\Lambda M = \Lambda/M \otimes_A M = A \otimes_A M = M$ as well as $M \otimes_\Lambda \Lambda/M = M$.

Proof. We lift the cocycle φ to $\bar{\varphi} : M^{p,q} \to \Lambda$ as in the previous section. Since $\delta^{p,q}\varphi = d_h\bar{\varphi}$, we consider $(x_1, \ldots, x_{p+q+1}) \in M^{p+1,q}$. In the coboundary formula the middle terms are all zero. If x_i and x_{i+1} belong to M then $x_i x_{i+1} = 0$ since $M^2 = 0$. Otherwise x_i or x_{i+1}, or both of them lie in A, hence $(x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+q+1})$ belongs to $M^{p+1,q-1}$ and φ is zero evaluated on this tensor. We have proved that

$$\delta^{p,q}\varphi(x_1, \ldots, x_{p+q+1}) = x_1\varphi(x_2, \ldots, x_{p+q+1}) + (-1)^{p+q+1}\varphi(x_1, \ldots, x_{p+q})x_{p+q+1},$$

which corresponds to the formula involving the cup product with the identity endomorphism of M. $\square$

EXAMPLE 4.3. We describe the connecting homomorphism component

$$\delta^{p,0} : \mathrm{Hom}_{A-A}(M^{\otimes_A p}, A) \to \mathrm{Hom}_{A-A}(M^{\otimes_A p+1}, M).$$

Let $\varphi \in \mathrm{Hom}_{A-A}(M^{\otimes_A p}, A)$ be a cocycle, then

$$(\delta^{p,0}\varphi)(m_1, \ldots, m_{p+1}) = m_1\varphi(m_2, \ldots, m_{p+1}) + (-1)^{p+1}\varphi(m_1, \ldots, m_p)m_{p+1}.$$

For $p = 0$ we have

$$\delta^{0,0} : \mathrm{Hom}_{A-A}(A, A) \to \mathrm{Hom}_{A-A}(M, M).$$

Recall that $A \otimes_A M$ and $M \otimes_A A$ are identified with M. Then $\delta^{0,0}\varphi(m) = m\varphi(1) - \varphi(1)m$. Since the center A^A is identified with $\mathrm{Hom}_{A-A}(A, A)$, then $\mathrm{Ker}\delta^{0,0} = A^A \cap A^M$, in other words the kernel of δ^0 is the set of central elements of A which act symmetrically on M, as in Proposition 3.3. Note also that $\delta^0 = \delta^{0,0}$.

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5. Trivial extensions.

DEFINITION 5.1. The *trivial extension* TA of an algebra A is the split algebra obtained by using the A-bimodule $DA = \mathrm{Hom}_k(A, k)$ endowed with the zero multiplicative structure.

We recall that for trivial extentions the connecting homomorphism δ^0 of the long exact cohomology sequence is zero, see Example 3.5. Our next purpose is to compute the first Hochschild cohomology vector space of a trivial extension. For this we study the first connecting homomorphism δ^1. Since $\delta^0 = 0$ the long exact cohomology sequence for TA gives the following exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(TA, DA) &\longrightarrow H^1(TA, TA) \longrightarrow H^1(TA, A) \xrightarrow{\delta^1} \\ H^2(TA, DA) &\longrightarrow H^2(TA, TA) \longrightarrow H^2(TA, A) \xrightarrow{\delta^2} \\ \ldots \end{aligned}$$

Using Theorem 3.1 for $X = TA/DA = A$ or $X = DA$ we have

$$H^n(TA, X) = \bigoplus_{p+q=n} H^q(\mathcal{C}^p(X)).$$

Moreover $\delta^n = \oplus_{p+q=n} \delta^{p,q}$ where

$$\delta^{p,q} : H^q(\mathcal{C}^p(A)) \longrightarrow H^q(\mathcal{C}^{p+1}(DA))$$

REMARK 5.2. The following facts hold without any projectivity hypothesis on the A-bimodule DA

1. $H^*(\mathcal{C}^0(X)) = H^*(A, X)$,
2. $H^*(\mathcal{C}^1(X)) = \mathrm{Ext}^*_{A-A}(DA, X)$ (*compare Theorem* 2.2),
3. $H^0(\mathcal{C}^p(X)) = \mathrm{Hom}_{A-A}(DA^{\otimes_A p}, X)$ (*compare Remark* 2.3).

Using them we get that for $n = 1$

$$\delta^1 : \mathrm{Hom}_{A-A}(DA, A) \oplus H^1(A, A) \xrightarrow{\begin{pmatrix} \delta^{1,0} & 0 \\ 0 & \delta^{0,1} \\ 0 & 0 \end{pmatrix}} \begin{aligned} &\mathrm{Hom}_{A-A}(DA \otimes_A DA, DA) \\ &\oplus \mathrm{Ext}^1_{A-A}(DA, DA) \oplus H^2(A, DA) \end{aligned}$$

PROPOSITION 5.3. *The connecting morphisms $\delta^{0,1}$ and $\delta^{1,0}$ verify:*

(i) $\delta^{0,1} = 0$*;*

(ii) *under appropriate identifications* $\delta^{1,0}\varphi = \varphi + \varphi^*$.

REMARK 5.4. The first item of this proposition will be generalized in Proposition 5.9.

Proof.

(i) Let φ be a vertical cocycle at the (0, 1)-spot of the double complex of A, namely $\varphi : A \to A$ is a usual derivation of the algebra A. We know that $\delta^{0,1}\varphi = 1_{DA} \smile \varphi + \varphi \smile 1_{DA}$ (see Theorem 4.1):

$$\begin{aligned} \delta^{0,1}\varphi : (DA)A \oplus A(DA) &\longrightarrow DA \\ (f, a) + (b, g) &\mapsto f\varphi(a) + \varphi(b)g. \end{aligned}$$

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We assert that $\delta^{0,1}\varphi$ is actually a vertical coboundary in the double complex of DA, namely $\delta^{0,1}\varphi = d_v\varphi^*$. Indeed

$$(d_v\varphi^*)(f, a) = -\varphi^*(fa) + \varphi^*(f)a.$$

For every $x \in A$ we have

$$\begin{aligned} -(\varphi^*(fa))(x) + (\varphi^*(f)a)(x) &= \\ -(fa)(\varphi(x)) + \varphi^*(f)(ax) &= \\ -f(a\varphi(x)) + f(\varphi(ax)) &= \\ f(-a\varphi(x) + \varphi(ax)) &= f(\varphi(a)x). \end{aligned}$$

The last equality holds since φ is a derivation. Finally we obtain

$$(d_v\varphi^*)(f, a) = f\varphi(a).$$

Similarly we prove that $(d_v\varphi^*)\,(b, g)$ equals $\varphi(b)g$.

(ii) Let $\varphi \in \mathrm{Hom}_{A-A}(DA, A)$ or by adjointness let $\beta : DA \otimes_{A-A} DA \to k$ be a bilinear form, given by $\beta(f, g) = g(\varphi(f))$. We know that $\delta^{1,0}\varphi = 1_{DA} \smile \varphi + \varphi \smile 1_{DA}$, more precisely $(\delta^{1,0}\varphi)(f, g) = f\varphi(g) + \varphi(f)g$. Now each $\psi \in \mathrm{Hom}_{A-A}(DA \otimes_A DA, DA)$ is also identified with a bilinear form $\beta : DA \otimes_{A-A} DA \to k$, namely $\beta(f, g) = \psi(f, g)(1)$. Through this identification, we have that $\delta^{1,0}\beta = \beta + \beta^t$, where $\beta^t(f, g) = \beta(g, f)$. Indeed

$$(\delta^{1,0}\varphi)(f, g)(1) = (f\varphi(g))(1) + (\varphi(f)g)(1) = f(\varphi(g)) + g(\varphi(f)). \qquad \square$$

We consider the set of skew-symmetric bilinear forms β over DA such that $\beta(fa, g) = \beta(f, ag)$ and we denote this set $\mathrm{Alt}_A(DA)$. In the proof of the next Theorem we will show that $\mathrm{Alt}_A(DA) = \mathrm{Ker}\delta^{1,0}$. This vector space coincides with $\mathcal{E}(DA)$ as considered by M. Saorin in [**16**].

We use a star symbol in order to denote the dual of a vector space, while the notation D is kept for the dual of a vector space endowed with a bimodule structure.

THEOREM 5.5. *Let TA be the trivial extension of a finite-dimensional algebra A. Then*

$$H^1(TA, TA) = A^A \oplus H_1(A, A)^* \oplus H^1(A, A) \oplus \mathrm{Alt}_A(DA).$$

Before proving this result we note that since the center of a k-algebra is not zero we get the following result.

COROLLARY 5.6. *Let TA be the trivial extension of a finite-dimensional algebra A. Then the first Hochschild cohomology group of TA does not vanish.*

Proof. (Theorem 5.5.) From the long exact sequence and the description of δ^1, we have

$$H^1(TA, TA) = H^1(TA, DA) \oplus \mathrm{Ker}\delta^1.$$

We have that $\mathrm{Ker}\delta^1 = H^1(A, A) \oplus \mathrm{Ker}\delta^{1,0}$ and

$$\mathrm{Ker}\delta^{1,0} = \{\varphi \in \mathrm{Hom}_{A-A}(DA, A) \mid \varphi + \varphi^* = 0\}.$$

Using adjointness we have

$$\mathrm{Hom}_{A-A}(DA, A) = (DA \otimes_{A-A} DA)^*.$$

So we have proved the following

$$\mathrm{Ker}\delta^{1,0} = \mathrm{Alt}_A(DA).$$

From Remark 5.2

$$H^1(TA, DA) = \mathrm{Hom}_{A-A}(DA, DA) \oplus \mathrm{Ext}^1_{A-A}(A, DA).$$

Actually

$$\mathrm{Hom}_{A-A}(DA, DA) = \mathrm{Hom}_{A-A}(A, A) = A^A.$$

Finally

$$H^1(A, DA) = H_1(A, A)^*$$

since for a finite dimensional algebra A and a finite dimensional A-bimodule N the following fact holds: $H_n(A, N)^* = H^n(A, DN)$ (see for instance [**5**]). Note also that N and DDN are bimodules which are canonically isomorphic by the evaluation map. □

THEOREM 5.7. *Let A be an arbitrary algebra (not necessarily finite-dimensional). Then*

$$H^1(TA, TA) = A^A \oplus H^1(A, TA) \oplus \mathrm{Alt}_A(DA).$$

Proof. In the proof of the above theorem note that $H^1(A, A) \oplus H^1(A, DA) = H^1(A, TA)$. □

THEOREM 5.8. *Let A be a finite dimensional k-algebra. Then the vector space $H^n(A, A) \oplus H_n(A, A)$ is a direct summand of $H^n(TA, TA)$.*

In order to prove this theorem, we provide the following result generalizing Proposition 5.3. Recall that $\delta^{0,q} : H^q(A, A) \to \mathrm{Ext}^q_{A-A}(DA, DA)$ is a component of the connecting homomorphism $\delta^q : H^q(TA, A) \to H^{q+1}(TA, DA)$.

PROPOSITION 5.9. *We have $\delta^{0,q} = 0$ for all $q \geq 1$.*

Proof. Let $\phi \in \mathrm{Hom}(A^{\otimes q}, A)$ be a Hochschild cocycle at the $(0, q)$-spot of the double complex. Next we provide $\phi' \in \mathrm{Hom}((DA)^{1,q-1}, DA)$ such that $d_v\phi' = \delta^{0,q}\phi$, by the following formula

$$\phi'(a_1, \dots, a_n, f, b_1, \dots, b_m)(x) = \epsilon(n, q) f(\phi(b_1, \dots, b_m, x, a_1, \dots, a_n))$$

where $n + m + 1 = q$ and

$$\epsilon(n, q) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ (-1)^{q+1} & \text{if } n \text{ is even.} \end{cases}$$

Observe that $\delta^{0,q}(\phi) \in \mathrm{Hom}((DA)^{1,q-1}, DA)$ and $\delta^{0,q}(\phi) = 1 \smile \phi + (-1)^{q+1}\phi \smile 1$. So the following three cases arise:

$$
\begin{aligned}
&(\delta^{0,q}\phi)(f, b_1, \dots, b_q)(x) = f(\phi(b_1, \dots, b_q)x)\\
&(\delta^{0,q}\phi)(a_1, \dots, a_i, f, b_1, \dots, b_j)(x) = 0 \qquad \text{for } i \neq 0, j \neq 0, i+j = q\\
&(\delta^{0,q}\phi)(a_1, \dots, a_q, f)(x) = (-1)^{q+1} f(x\phi(a_1, \dots, a_q)).
\end{aligned}
$$

The verification that $d_v\phi' = \delta^{0,q}\phi$ is left to the reader. □

Proof. (Theorem 5.8.) By the previous result $H^q(A, A)$ is contained in the image of the morphism $H^q(TA, TA) \to H^q(TA, A)$. Concerning the homology factor note, as we remarked before, that $H^q(\mathcal{C}^0(DA)) = H^q(A, DA)$. We know from Proposition 3.6 that $H^q(A, DA) \cap \mathrm{Im}\delta^{q-1} = 0$ therefore $H_q(A, A) = H^q(A, DA) \subset H^q(TA, TA)$. □

We generalize now a result obtained in [**15**, **16**]. In [**15**] it is shown that for a triangular schurian algebra A we have $H^1(TA, TA) = k \oplus H^1(A, A)$. Actually the same equality holds for triangular algebras, or for 2-nilpotent algebras whose quiver do not contain oriented cycles of length ≤ 2, see [**16**].

Our next purpose is to use our previous computations on H^1 of a trivial extension in order to show that this result holds for one-way algebra, a family of algebras that we define below and which include the algebras considered above. Note that the proof in [**16**] of the above equality under the mentioned hypothesis also works for one-way algebras.

DEFINITION 5.10. A *one-way* algebra is a finite dimensional algebra endowed with a complete set S of orthogonal idempotents such that.

1. For $e \neq f$ in S, if $eAf \neq 0$ then $fAe = 0$.
2. For all $e \in S$, we have $\dim_k(eAe) = 1$.
3. S, has more than one element (that is A is not k) and A is an indecomposable algebra (that is the graph with set of vertices S and an edge between e and f in case eAf or fAe is not zero is a connected graph).

THEOREM 5.11. *Let A be a finite dimensional one-way algebra, and let TA be its trivial extension. Then*

$$H^1(TA, TA) = k \oplus H^1(A, A).$$

In order to prove this formula we use Theorem 5.5, and two results as follows.

LEMMA 5.12. *Let A be a one-way algebra. Then*

$$\mathrm{Hom}_{A-A}(DA, A) = 0.$$

Proof. Take $\varphi \in \mathrm{Hom}_{A-A}(DA, A)$ and $e, f \in S$ distinct. Since $\varphi(D(eAf)) \subset fAe$, the form φ has to vanish on $D(eAf)$. Now $\varphi(D(eAe)) \subset eAe$ so $\mathrm{Im}\varphi \subset \oplus_{e\in S} eAe$. But since the algebra is indecomposable and different from k there exist $f \neq e$ such that $eAf \neq 0$. Hence there is no non-zero two-sided ideal contained in the vector space $\oplus_{e\in S} eAe$. □

LEMMA 5.13. *Let A be a one-way algebra for a system S of idempotents. Then $H_1(A, A) = 0$.*

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Proof. Let $E = \times_{e\in S} ke$ be the subalgebra of A generated by S. Note that $A \otimes_E A$ is a projective A-bimodule since

$$A \otimes_E A = \bigoplus_{e\in S} Ae \otimes eA$$

and each summand is a projective A-bimodule using the fact that

$$A \otimes A = \bigoplus_{e,f\in S} Ae \otimes fA.$$

Note also that the decomposition

$$A \otimes_E A \otimes_E A = \bigoplus_{e,f\in S} Af \otimes fAe \otimes eA$$

shows that this bimodule is projective as an A-bimodule.

Consider now the projective resolution of A as an A-bimodule

$$\cdots \to A \otimes_E A \otimes_E A \to A \otimes_E A \to A \to 0$$

where the boundary formula is provided by the standard Hochschild resolution of A as an A-bimodule. The homotopy contraction showing the exactness is defined as usual, by inserting 1 at the beginning of each tensor.

Applying the functor $-\otimes_{A^e} A$ one gets, after decomposing in terms of the orthogonal idempotents

$$\cdots \to \bigoplus_{e,f,g\in S} eAf \otimes fAg \otimes gAe \to \bigoplus_{e,f\in S} eAf \otimes fAe \to \bigoplus_{e\in S} eAe \to 0.$$

But $\oplus_{e,f\in S}\, eAf \otimes fAe = \oplus_{e\in S}\, eAe \otimes eAe$, since $eAf \neq 0$ implies $fAe = 0$ if $f \neq e$. Also, if e is any primitive idempotent, eAe is isomorphic to k. The boundary map, using this isomorphism, is null on $\oplus_{e\in S}\, eAe \otimes eAe$.

As before, the following term $\oplus_{e,f,g\in S}\, eAf \otimes fAg \otimes gAe$ of the complex may be written as:

$$\left(\bigoplus_{e\in S} eAe \otimes eAe \otimes eAe\right) \oplus \left(\bigoplus_{e\neq f\in S} eAf \otimes fAe \otimes eAe\right)$$
$$\oplus \left(\bigoplus_{e\neq f\in S} eAe \otimes eAf \otimes fAe\right) \oplus \left(\bigoplus_{e\neq f\neq g\in S} eAf \otimes fAg \otimes gAe\right).$$

The second and third summands are zero, and the restriction of the boundary map to the first one, composed with the isomorphism $eAe \cong k$ is the identity. So, already restricted to this first summand, the boundary map is surjective. Then $H_1(A, A) = 0$. $\square$

Proof. (Theorem 5.11.) We recall the decomposition of Theorem 5.5:

$$H^1(TA, TA) = A^A \oplus H_1(A, A)^* \oplus H^1(A, A) \oplus \mathrm{Alt}_A(DA).$$

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The hypothesis on A implies that the center A^A of A is the field k. From Lemma 5.12 we get $\mathrm{Alt}_A(DA)=0$, since $\mathrm{Alt}_A(DA)\subset \mathrm{Hom}_{A-A}(DA,A)$. The previous theorem shows that $H_1(A,A)=0$. □

The aim of the last part of this section is to show that the connecting homomorphisms of the long exact sequence on Hochschild cohomology are not all zero in general.

We consider split algebras with $M=A$ and $M^2=0$. These algebras are isomorphic to $A[x]/\langle x^2\rangle \simeq A\otimes k[\epsilon]$, where $k[\epsilon]=k[x]/\langle x^2\rangle$ is the algebra of dual numbers. We denote them by $A[\epsilon]$. Recall that an algebra A is *symmetric* if A is isomorphic to DA as an A-bimodule. In this case the trivial extension TA of A coincides with the split algebra $A[\epsilon]$.

It is well known (see for instance [**5**, **18** Proposition 9.4.1]) that if A and B are k-algebras (one of them finite dimensional) we have

$$H^n(A\otimes B, A\otimes B)=\bigoplus_{p+q=n} H^p(A,A)\otimes H^q(B,B).$$

It is also well known that if k is of characteristic different from 2 then

$$\dim_k H^*(k[\epsilon],k[\epsilon])=\begin{cases}2 & \text{if } *=0\\ 1 & \text{if } *>0.\end{cases}$$

If char $k=2$ then $\dim_k H^n(k[\epsilon],k[\epsilon])=2$ for all n. For a k-algebra A we infer that in characteristic different from 2

$$H^n(A[\epsilon],A[\epsilon])=H^n(A)\oplus\left(\bigoplus_{i=0}^{n} H^i(A)\right),$$

while in characteristic 2

$$H^n(A[\epsilon],A[\epsilon])=\bigoplus_{i=0}^{n}(H^i(A)\oplus H^i(A)).$$

Let $\Lambda=A\oplus M$ be a split algebra with $M^2=0$. Assume that all the connecting homomorphisms are zero. Then

$$H^n(\Lambda,\Lambda)=\left(\bigoplus_{p+q=n} H^q(\mathcal{C}^p(M))\right)\oplus\left(\bigoplus_{p+q=n} H^q(\mathcal{C}^p(A))\right).$$

In case M is projective on one side and all connecting homomorphisms are zero, we get

$$H^n(\Lambda,\Lambda)=\left(\oplus_{p+q=n}\mathrm{Ext}^q_{A-A}\left(M^{\otimes_A^p},M\right)\right)\oplus\left(\oplus_{p+q=n}\mathrm{Ext}^q_{A-A}\left(M^{\otimes_A^p},A\right)\right).$$

In case $M=A$, and still assuming that all connecting homomorphisms are zero, we get

$$H^n(A[\epsilon],A[\epsilon])=\bigoplus_{i=0}^{n}(H^i(A)\oplus H^i(A))$$

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which holds only in characteristic two. Hence the connecting homomorphisms are not zero in general.

REMARK 5.14. For trivial extensions one can describe the component

$$\delta^{p,0} : \mathrm{Hom}_{A-A}(DA^{\otimes_A p}, A) \to \mathrm{Hom}_{A-A}(DA^{\otimes_A p+1}, DA)$$

of the connecting homomorphism as follows, generalizing the second item of Proposition 5.3. The cyclic group of order $p+1$ acts on $\mathrm{Hom}_{A-A}(DA^{\otimes_A p+1}, DA)$ via

$$(t\varphi)(f_1 \otimes \cdots \otimes f_{p+1}) = \varphi(f_2 \otimes \cdots f_{p+1} \otimes f_1).$$

Identifying by adjunction the source with the target of $\delta^{p,0}$ we obtain

$$\delta^{p,0}\varphi = t\varphi + (-1)^{p+1}\varphi.$$

6. Triangular matrix algebras and one-point extensions. Recall that a *triangular matrix algebra* $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ consists of two algebras A and B and a $B-A$ bimodule M, where the product is obtained by matrix multiplication. Note that in case B is the ground field k such algebras are called *one-point extensions* of A. Our next purpose is to specialize to these algebras the results we have obtained for split algebras in order to recover results of C. Cibils, S. Michelena and M. I. Platzeck in [**6**, **12**], and of D. Happel for one-point extensions [**9**], see also [**3**, **8**].

REMARK 6.1. Triangular matrix algebras are split algebras with zero bimodule product. Indeed consider the algebra $A \times B$ and the trivially extended $A \times B$-bimodule M with structure given by $(a, b)m = bm$ and $m(a, b) = ma$. The split algebra $(A \times B) \otimes M$ with $M^2 = 0$ is exactly the algebra $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$.

Let $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ be a triangular matrix algebra. Consider the exact sequence of T-bimodules

$$0 \to M \to T \to A \times B \to 0$$

and the corresponding long exact sequence in Hochschild cohomology.

$$\begin{array}{l} 0 \longrightarrow H^0(T, M) \longrightarrow H^0(T, T) \longrightarrow H^0(T, A \times B) \xrightarrow{\delta^0} \\ \quad H^1(T, M) \longrightarrow \quad \cdots \\ \qquad\qquad\qquad\qquad \cdots \quad \longrightarrow H^{n-1}(T, A \times B) \xrightarrow{\delta^{n-1}} \\ \quad H^n(T, M) \longrightarrow H^n(T, T) \longrightarrow H^n(T, A \times B) \xrightarrow{\delta^n} \\ \quad H^{n+1}(T, M) \longrightarrow \quad \cdots \end{array}$$

We will use a suitable version of Corollary 3.2 in order to describe $H^n(T, M)$ and $H^n(T, A \times B)$. The following fact will enable us to perform a Tor computation for recovering Cibils and Michelena–Platzeck Theorem.

LEMMA 6.2. *If M is projective as a left B-module, the trivially extended $A \times B$-bimodule M is a projective left $A \times B$-module.*

Proof. Note that B is projective as a left $A \times B$-module, consequently the same holds for a direct summand of a free B-module. $\square$

The next result simplifies considerably this description.

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LEMMA 6.3. *Let M be a $B-A$-bimodule trivially extended to an $A\times B$-bimodule. Then for $p\geq 2$ we have*

$$M^{\otimes^p_{(A\times B)}}=0.$$

Proof. For $m\in M$ and $n\in M$ we have

$$m\otimes n=m(1,0)\otimes n=m\otimes(1,0)n=m\otimes 0=0. \qquad \square$$

THEOREM 6.4. (*See* [**6**, **12**]) *Let A and B be k-algebras, M be a $B-A$-bimodule and let $T=\begin{pmatrix} A & 0\\ M & B\end{pmatrix}=(A\times B)\oplus M$ be the triangular matrix algebra or equivalently the corresponding split algebra. Then there is a long exact sequence in Hochschild cohomology.*

$$\begin{aligned}
&0\longrightarrow 0\longrightarrow H^0(T,T)\longrightarrow H^0(A,A)\oplus H^0(B,B)\xrightarrow{\delta^{0,0}}\\
&\mathrm{Hom}_{B-A}(M,M)\longrightarrow\cdots\\
&\cdots\qquad\longrightarrow H^{n-1}(A,A)\oplus H^{n-1}(B,B)\xrightarrow{\delta^{n-1,0}}\\
&\mathrm{Ext}^{n-1}_{B-A}(M,M)\longrightarrow H^n(T,T)\longrightarrow H^n(A,A)\oplus H^n(B,B)\xrightarrow{\delta^{n,0}}\\
&\mathrm{Ext}^{n}_{B-A}(M,M)\longrightarrow\cdots
\end{aligned}$$

Proof. Theorem 3.1 provides the following decompositions

$$H^n(T,A\times B)=\bigoplus_{p+q=n}H^q(\mathcal{C}^p(A\times B))$$

$$H^{n+1}(T,M)=H^{n+1}(A\times B,M)\oplus\bigoplus_{p+q=n}H^q(\mathcal{C}^{p+1}(M))$$

and Proposition 3.6 shows that the connecting homomorphism δ^n is bigraded of bidegree $(1,0)$, that is, $\delta^n=\bigoplus_{p+q=n}\delta^{p,q}$.

In Section 2 we have proved that the cohomology in degree q of the column $\mathcal{C}^p(X)$ for a T-bimodule X is $\mathrm{Ext}^q_{(A\times B)-(A\times B)}(M^{\otimes^p_{(A\times B)}},X)$ whenever the $A\times B$-bimodule M is projective on one side. It is clear from the proofs of Section 2 that this condition can be relaxed, namely it is enough to require the vanishing of the Tor vector spaces between tensor powers of the bimodule and the bimodule itself – we thank Manuel Saorin for stressing this fact. In our situation the Lemma above shows that $Tor^*_{A\times B}(M^{\otimes^p_{(A\times B)}},M)=0$ for $p\geq 2$. In order to show that $Tor^*_{A\times B}(M,M)=0$, consider a projective resolution of M as a left B-module and extend the action to $A\times B$ letting A act by zero. As in the proof of Lemma 6.3 tensoring the above projective resolution by M over $A\times B$ provides a zero complex.

These considerations show that the cohomology of the columns can be replaced by Ext vector spaces between tensor powers of M. Many of them vanish using again the Lemma above, finally we obtain the following for the connecting homomorphism:

$$\begin{array}{ccc}
 & & H^{n+1}(A\times B,M)\\
 & & \oplus\\
H^n(A\times B,A\times B) & \xrightarrow{\delta^{n,0}} & \mathrm{Ext}^n_{(A\times B)-(A\times B)}(M,M)\\
\oplus & & \oplus\\
\mathrm{Ext}^{n-1}_{(A\times B)-(A\times B)}(M,A\times B) & \xrightarrow{\delta^{n-1,0}} & 0
\end{array}$$

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In fact

$$\mathrm{Ext}^{n-1}_{(A\times B)-(A\times B)}(M, A\times B) = 0 \quad \text{and}$$
$$H^{n+1}(A\times B, M) = 0.$$

In order to prove this last assertion, let $e=(1,0)$ and $f=(0,1)$ be the idempotents of the algebra $A\times B$. Note that an $A\times B$-bimodule Y is the direct sum of four bimodules which can be presented at the vertices of a square:

$$\begin{matrix} eYf & fYf \\ eYe & fYe. \end{matrix}$$

For instance eYf is an $A-B$-bimodule and eYe is an A-bimodule. We have that

$$\mathrm{Ext}^{*}_{(A\times B)-(A\times B)}(Y,Z) = \mathrm{Ext}^{*}_{A-B}(eYf, eZf) \oplus \mathrm{Ext}^{*}_{B-B}(fYf, fZf)$$
$$\oplus\, \mathrm{Ext}^{*}_{A-A}(eYe, eZe) \oplus \mathrm{Ext}^{*}_{B-A}(fYe, fZe).$$

Since the three components eMf, fMf and eMe are zero, we obtain

$$\mathrm{Ext}^{n-1}_{(A\times B)-(A\times B)}(M, A\times B) = \mathrm{Ext}^{n-1}_{(A\times B)-(A\times B)}(M, f(A\times B)e),$$

note that $f(A\times B)e = 0$. Similarly we obtain

$$H^{n+1}(A\times B, M) = 0 \text{ since } H^{n+1}(A\times B, M) = \mathrm{Ext}^{n+1}_{(A\times B)-(A\times B)}(A\times B, M).$$

Moreover the same type of arguments shows that

$$H^{n}(A\times B, A\times B) = H^{n}(A,A)\oplus H^{n}(B,B) \text{ and}$$
$$\mathrm{Ext}^{n}_{(A\times B)-(A\times B)}(M,M) = \mathrm{Ext}^{n}_{B-A}(M,M). \qquad \square$$

REMARK 6.5. The same result can be derived from the spectral sequence arising from the double complex. Indeed only the first two columns are non-zero at the first level, and the vector spaces have to be decomposed as we did above.

REMARK 6.6. If $B=k$ and M is any right A-module, we obtain Happel's long exact sequence [**9**]:

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow H^{0}(T,T) \longrightarrow H^{0}(A,A)\oplus k \longrightarrow \\ \mathrm{End}_{A}M \longrightarrow H^{1}(T,T) \longrightarrow H^{1}(A,A) \longrightarrow \\ \mathrm{Ext}^{1}_{A}(M,M) \longrightarrow H^{2}(T,T) \longrightarrow H^{2}(A,A) \longrightarrow \\ \mathrm{Ext}^{2}_{A}(M,M) \longrightarrow \quad \cdots \end{array}$$

PROPOSITION 6.7. *The connecting homomorphism of the cohomology long exact sequence for a triangular matrix algebra is given by*

$$\delta^{n} f = 1_{M} \smile f \ \textit{for} f \in H^{n}(A,A) \ \textit{and}$$
$$\delta^{n} g = (-1)^{n+1} g \smile 1_{M} \ \textit{for } g \in H^{n}(B,B).$$

The proof follows from the general description of $\delta^{p,q}$ given in Theorem 4.1.

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