ELLIPTIC DIFFERENTIAL OPERATORS
AND DIFFUSION PROCESSES

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This article develops survey lectures for general mathematical
audiences which the author delivered at the 27th Annual
Meeting of the Australian Mathematical Society at the
University of Queensland, 1983, and at the 10th Austrian
Congress of Mathematicians in Innsbruck, 1981. The central
theme of these lectures was the use of probabilistic methods
in the study of linear elliptic-parabolic differential
equations of second order.

The starting point will be an orientative discussion of
the role of Brownian motion in classical potential theory.
It will then be discussed that, given an elliptic-
parabolic differential operator L of a certain type,
there exists a uniquely determined diffusion process which
is linked with L formally in the same way in which
Brownian motion is linked with the Laplace operator. The
fundamental results of K. Itô, D.W.Stroock and S.R.S.
Varadhan will be in the centre of this part of the paper.
We will then proceed to the discussion of more refined
problems of the same type for differentiable manifolds.
A glimpse at stochastic Riemannian geometry will then close
our tour d'horizon.

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1. Laplace Operator and Brownian Motion

On $\mathbb{R}^p$, $p \geqslant 1$, equipped with the Euclidean norm $\|.\|$, we consider the *Brownian semigroup* $(g_t)_{t>o}$ where

(1) $$g_t(x) = (2\pi t)^{-p/2} e^{-\|x\|^2/2t} \qquad\qquad (x \in \mathbb{R}^p) \ .$$

$(g_t)_{t>o}$ is a semigroup in the sense that $g_s * g_t = g_{s+t}$ holds for all $s,t > o$.   It is advantageous to interpret the Brownian semigroup as a semigroup of positive linear operators, more precisely as a *semigroup of kernels* in the sense of measure theory (cf. [2], [16]). For this purpose we put

(2) $$P_t f = g_t * f \qquad\qquad (f \in \mathcal{B}^b),$$

where $\mathcal{B}^b = \mathcal{B}^b (\mathbb{R}^p)$ denotes the linear space of bounded, real-valued, Borel-measurable functions on $\mathbb{R}^p$. The semigroup property then appears in the form

(3) $$P_{s+t} = P_s P_t \qquad\qquad (s,t > o).$$

It is easily seen that $(P_t)_{t>o}$ is a *Feller semigroup* (or Feller-Dynkin semigroup in the terminology of Williams [23], p.115) which means that $(P_t)_{t>o}$ is a semigroup of kernels having the following three properties:

(F_1) $$\qquad P_t(C_o) \subset C_o \ ,$$

(F_2) $$\qquad \lim_{t\to o} \|P_t f - f\| = o \qquad\qquad (f \in C_o) \ ,$$

(F_3) $$\qquad P_t 1 = 1 \ .$$

Here $C_o = C_o(\mathbb{R}^p)$ denotes the space of continuous real-valued functions on $\mathbb{R}^p$ vanishing at infinity and $\|...\|$ stands for the sup-norm.

The semigroup $(P_t)_{t>o}$ is closely related to the Laplace operator

Δ. In fact, it is well known that C_c^2, the space of C^2-functions on $\mathbb{R}^p$ with compact support, is contained in the domain of the infinitesimal generator of the Brownian semigroup and that on C_c^2 the infinitesimal generator coincides with $\frac{1}{2}\Delta$ (cf. [5]). So we have

(4)
$$\lim_{t\to o} \frac{P_t f - f}{t} = \frac{1}{2}\Delta f \qquad (f \in C_c^2)$$

(even in the sense of uniform convergence). From the semigroup property (3) the following integrated form of (4) follows:

(5)
$$P_t f(x) = f(x) + \int_o^t P_s(\tfrac{1}{2}\Delta f)(x)ds \qquad (f \in C_c^2)$$

for all $x \in \mathbb{R}^p$.

When speaking about the Brownian semigroup $(g_t)_{t>o}$ or $(P_t)_{t>o}$, we have passed from the Laplace equation to the heat equation

(6)
$$\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = o .$$

In fact, for $f \in \mathcal{B}^b$, the function

$$u(x,t) = P_t f(x)$$

solves the heat equation on the half-space $\mathbb{R}^p \times]o, +\infty[$. Consequently the property (F_2) expresses that u solves the *Cauchy problem* for $f \in C_c^2$ on this half-space. The function

$$(x,t) \to g_t(x)$$

given by (1) is nothing else than the *fundamental solution* of (6) with pole at the origin (o,o).

We shall now pass to the probabilistic meaning of the Brownian semigroup. In the language of a physicist, this means that we replace the macroscopic observation level by the microscopic one. We introduce

(7)
$$\Omega = C(\mathbb{R}_+, \mathbb{R}^p)$$

the set of all *continuous paths* in $\mathbb{R}^p$, i.e. of all continuous maps

$\omega : \mathbb{R}_+ \to \mathbb{R}^p$ where $\mathbb{R}_+$ is the closed half-line $[o, +\infty[$. We introduce

the *position map* for the time $t \in \mathbb{R}_+$

(8) $X_t : \Omega \to \mathbb{R}^p$

                                        $\omega \to \omega(t)$

as well as the σ-algebras

(9) $F = \sigma(X_s \; ; \; s \in \mathbb{R}_+)$

and

(10) $F_t = \sigma(X_s \; ; \; s \in [o,t])$

generated by all X_s respectively by all X_s with $o \leqslant s \leqslant t$. F is

the σ-algebra of all events (of interest to us); F_t is the σ-algebra

of *events up to time* t .

On F the *Wiener measure* P is available. It is the only

probability measure on F with the following three properties:

(w_1) $P\{X_o = o\} = 1$;

(w_2) $X_t - X_s$ is independent of F_s $(s < t)$;

(w_3)   the distribution of the increment $X_t - X_s$, $s < t$,

                 equals the p-dimensional Lebesque measure

                 λ^p with g_{t-s} as density, i.e.

                 $P\{X_t - X_s \in B\} = \int_B g_{t-s} d\lambda^p$

                 for every Borel set $B \subset \mathbb{R}^p$.

Translation $\omega \to x + \omega$ yields a measurable bijection of Ω onto

itself for every $x \in \mathbb{R}^p$. We denote by P^x the image of P under

this bijection. While P lives - according to (w_1) - on the set of

paths $\omega \in \Omega$ starting at the origin, P^x lives on $\{X_o = x\}$ the set of

paths $\omega \in \Omega$ starting at x . The family $(P^x)_{x \in \mathbb{R}^p}$ of probability

measures is called *Brownian motion*, it is the steering mechanism of a

stochastic process describing the movement of a Brownian particle.   It

is important to observe that the family (P^x) has the following

properties:

(D$_1$) $\qquad\qquad\qquad P^x\{X_o = x\} = 1 \qquad\qquad\qquad (x \in \mathbb{R}^p)$;

(D$_2$) $\qquad\qquad\qquad x \to P^x(A)$ is Borel measurable $\qquad (A \in F)$;

(D$_3$) $\qquad E^x(f \circ X_t \mid F_s) = E^{X_s}(f \circ X_{t-s}) \qquad\qquad P^x$-almost surely

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x \in \mathbb{R}^p, s < t, f \in B^b).$

As usual, we denote by E^x (resp. $E^x(. \mid .)$) the expectation (resp. the

conditional expectation) with respect to P^x. (D$_3$) is the famous (weak)

Markov property.   It interrelates the different probability measures

P^x in such a way that the Brownian particle 'has no memory': not the

full history of the particle counts for the prediction of its future;

for the prediction, only the present position of the particle (at time s)

is necessary.   (For details, see for example [2].)

An arbitrary family $(P^x)_{x \in \mathbb{R}^p}$ of probability measures on F

with the above properties (D$_1$)-(D$_3$) is called a *diffusion* (or diffusion

process) on $\mathbb{R}^p$.   So Brownian motion is a particular diffusion.

The Brownian semigroup $(P_t)_{t>o}$ can be recovered from Brownian

motion (cf. [2], p.399-401).   One has

(11) $\qquad\qquad\qquad P_t f(x) = E^x(f \circ X_t) \qquad\qquad$ for all $f \in B^b$,

$x \in \mathbb{R}^p$, $t > o$.   For an arbitrary diffusion the equalities (11) can be

used in order to define a family $(P_t)_{t>o}$ of operators on B^b. $(P_t)_{t>o}$

turns out to be a semigroup of Markovian kernels on $\mathbb{R}^p$, hence

satisfying $P_t 1 = 1$ for all $t > o$. $(P_t)_{t>o}$ is called the *transition*

semigroup of the given diffusion.   This semigroup is not a Feller

semigroup, in general. Property (F_1) is missing. Only a weakened form

of (F_2) - however equivalent to (F_2) in the presence of (F_1) (cf. [23],

p.115) - is available, namely $(P_t f)_{t>o}$ converges pointwise to f as

$t \to o$ (even for continous bounded functions f). Property (F_3) remains

valid.

The great advantage of interpreting the Brownian semigroup as the

transition semigroup of a diffusion process, namely of Brownian motion,

lies in the well-known possibility of a probabilistic, intuitively

simple approach to the main results of classical potential theory

(cf. [6] and [3]). The price that one has to pay for this advantage is

not negligible since one has to pass from the Laplace operator to the

fundamental solution of the heat equation in order to arrive at the

Brownian semigroup. For more general elliptic or even elliptic-parabolic

differential operators a remarkable amount of hard analysis can be

expected in order to arrive (eventually) at similar probabilistic

approaches to potential theoretic problems. The theory of harmonic

spaces develops a general machinery which allows to pursue this kind of

an approach (cf. [3], [4], [7]).

In what follows we intend to discuss procedures which, for certain

elliptic-parabolic differential operators L , will allow a probabilistic

approach to the corresponding potential theory by going straight-away

from L to an associated diffusion.

2. Elliptic-parabolic differential operators and associated diffusions

We will consider a linear differential operator L defined on all

of $\mathbb{R}^p$ of the form

(12) $$L = \frac{1}{2} \sum_{i,j=1}^{p} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{p} b_i(x) \frac{\partial}{\partial x_i}$$

which satisfies the following assumptions: all coefficients

$$x \to a_{ij}(x) \quad \text{and} \quad x \to b_i(x)$$

are *bounded* and *continuous* real-valued functions on $\mathbb{R}^p$. L is *semi-elliptic* (or elliptic-parabolic) in the sense that the matrix

$$a(x) = (a_{ij}(x))$$

is symmetric and positive semi-definite for all $x \in \mathbb{R}^p$. The operator L is called *elliptic* if $a(x)$ is even positive definite for all $x \in \mathbb{R}^p$

A diffusion $(P^x)_{x \in \mathbb{R}^p}$ on $\mathbb{R}^p$ will be called *associated to* L if its transition semigroup $(P_t)_{t>o}$ satisfies the same condition as Brownian motion does for the special operator $L = \frac{1}{2}\Delta$ (see (5)), namely:

$$(13) \qquad P_t f(x) = f(x) + \int_0^t P_s(Lf)(x)ds \qquad (f \in C_c^2, \ x \in \mathbb{R}^p).$$

According to the definition (11) of the transition semigroup and according to (D_1), this can be rewritten in the form

$$(14) \qquad E^x(H_t^f) = o \qquad (f \in C_c^2, \ x \in \mathbb{R}^p),$$

where

$$(15) \qquad H_t^f = f \circ X_t - f \circ X_o - \int_0^t Lf \circ X_s ds$$

is defined on the set Ω.

The probability measures P^x of a diffusion associated to L are interrelated by means of the Markov property (D_3) in a complicated way. The family $(H_t^f)_{t \geqslant o}$ of real random variables leads to the possibility of detangling these measures with the intention to prove the existence of diffusions associated to L later on. This possibility is due to the following fact which can be easily proved: $(H_t^f)_{t \geqslant o}$ is a *martingale with respect to each probability measure* P^x of a given diffusion associated with L. So, by definition,

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$$E^x(H_t^f \mid F_s) = H_s^f \qquad\qquad P^x\text{-almost surely}$$

whenever $o \leqslant s < t$, $x \in \mathbb{R}^p$ and $f \in C_c^2$.

This crucial observation leads to the following

DEFINITION. A probability measure P on F is called a *solution of the martingale problem* for L and for a point $x \in \mathbb{R}^p$ if

(M_1) $\qquad\qquad (H_t^f)_{t \geqslant o}$ is a martingale with respect to P $\qquad (f \in C_c^2)$

and if

(M_2) $\qquad\qquad P\{X_o = x\} = 1.$

For reasons of illustration let us return for a moment to the situation of §1, namely to $L = \frac{1}{2}\Delta = \frac{1}{2}\frac{d^2}{dx^2}$ in dimension $p = 1$. For the two functions $f = id$ and $= id^2$, that is $f(x) = x$ and $= x^2$, one obtains for any probability measure P on F satisfying $P\{X_o = o\} = 1$ that H_t^{id} and $H_t^{id^2}$ coincide with

$$X_t \quad \text{and} \quad X_t^2 - t \qquad\qquad P\text{-almost surely},$$

respectively. According to (W_1) the (1-dimensional) Wiener measure is such a measure P. It is a classical result of P. Lévy (cf. [10], p.75) that $(X_t)_{t \geqslant o}$ and $(X_t^2 - t)_{t \geqslant o}$ are martingales for a probability measure P on F satisfying $P\{X_o = o\} = 1$ if and only if P is the Wiener measure. From this it can be derived that the Wiener measure is the only solution of the martingale problem for $x = o$ (given the operator $L = \frac{1}{2}\Delta$ in dimension $p = 1$). The proof has to take care of the fact that the two functions id and id^2 do not have a compact support.

It is the preceding illustration which motivates the following definition - now again in the general situation.

DEFINITION. The *martingale problem* (for the differential operator L) is said to be *well posed* if, for every $x \in \mathbb{R}^p$, it has exactly one

solution.

By means of compactness arguments involving probability measures, one is led to the first result of fundamental importance due to Stroock and Varadhan [20], [21]. (For quick references see the presentation of Priouret [18].)

PROPOSITION 1. *Assume that the martingale problem for L is well posed. Then the family of its solutions $(P^x)_{x \in \mathbb{R}^p}$ is the only diffusion associated to L. Furthermore, the corresponding transition semigroup $(P_t)_{t>o}$ is a Feller semigroup.*

One is thus led to the problem of deciding when the martingale problem is well posed. Essentially, two types of answers are known:

THEOREM 1. *In each of the following two cases the martingale problem is well posed:*

(I)   *the matrix field $x \to a(x)$ is of the form*

$$a(x) = \sigma(x)\sigma^T(x) \qquad\qquad (x \in \mathbb{R}^p)$$

where the matrix field $x \to \sigma(x)$ as well as the vector field $x \to (b_1(x),\ldots,b_p(x))$ satisfy the global Lipschitz condition on $\mathbb{R}^p$.
(SV) *The differential operator L is elliptic.*

Condition (I) is due to K. Itô [11]. The proof uses the technique of stochastic differential equations. (SV) is due to Stroock and Varadhan [20], [21]. The proof makes essential use of the martingale formulation by means of an approximation of the situation (SV) by that of (I). Furthermore, it uses the so-called Cameron-Martin formula which allows the passage from the case where the *drift*, which by definition is the vector field

(16) $x \to b(x) = (b_1(x),\ldots,b_p(x)),$

vanishes to the general case by means of a martingale argument. For the technical details the reader is referred to Priouret [18].

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A Feller semigroup associated to L can always be interpreted as the transition semigroup of a diffusion (cf. [23]). This fact leads to a purely non-probabilistic version of Theorem 1 in connection with Proposition 1.

COROLLARY. *Under each of the assumptions* (I) *or* (SV) *there exists exactly one Feller semigroup* $(P_t)_{t>o}$ *for which condition* (13) *is satisfied.*

No purely analytic proof of this result seems to be known. In conclusion we can say that by means of the martingale problem it is possible to pass directly from the differential operator L to an associated diffusion process which, in addition, turns out to be uniquely determined by each of the conditions (I) and (SV).

3. Diffusions on manifolds

Up to now the coefficients of the differential operator L were assumed to be bounded (and continuous). The treatment of the continuous, but unbounded case leads in a natural way to the problem of studying diffusions associated to a semi-elliptic differential operator on a differentiable manifold. In fact, the natural idea of treating the unbounded case by restricting the differential operator L to open subsets U of $\mathbb{R}^p$ on which L coincides with a differential operator L_U on $\mathbb{R}^p$ satisfying the conditions of §2, by restricting the diffusion associated to L_U to the set U in an appropriate way, and by glueing together the pieces can be carried out successfully (cf. [18]).

So let us pass immediately to the case of a C^∞-*manifold* M *of dimension* p. We assume that M is connected and has a countable base. A *semi-elliptic* (respectively *elliptic*) differential operator on M will be a map

$$L \; : \; C^2(M) \to C(M)$$

which in each chart $x = (x_1,\ldots,x_p)$ can be written in the form

$$L = \frac{1}{2} \sum_{i,j=1}^{p} a_{ij}(x) \; \frac{\partial^2}{\partial x_i \partial x_j} \; + \; \sum_{i=1}^{p} b_i(x) \; \frac{\partial}{\partial x_i} \; + c$$

where the coefficients are continuous and the matrix $(a_{ij}(x))$ is positive semi-definite (respectively definite) for all admissible points x. For reasons of simplicity we will assume $L1 = 0;$ that is, $c = 0$ as we did before. Contributions to the general case where $c = L1 \leqslant 0$ can be found in [1].

The probabilistic setting of §1 has now to be modified (even for $M = \mathbb{R}^p$) in order to study a phenomenon which unbounded coefficients cause in general, namely the *explosion* phenomenon. It is due to a possibly finite *lifetime* of the moving particle described by a diffusion.

In order to take care of this phenomenon we consider the one-point compactification $M_\delta = M \cup \{\delta\}$ of M (with δ as an isolated point if M is compact) and the space $C_\delta(M)$ of all continuous maps $\omega : \mathbb{R}_+ \to M_\delta$ which respect δ in the sense of a "cimetary": $\omega(t_0) = \delta$ for some $t_0 \in \mathbb{R}_+$ implies $\omega(t) = \delta$ for all $t \geqslant t_0$. The number

(17) $\zeta(\omega) = \inf\{t \in \mathbb{R}_+ : \omega(t) = \delta\}$

is then called the *lifetime* of the path $\omega \in C_\delta(M)$.

The appropriate probabilistic setting is now the following: we choose $\Omega = C_\delta(M)$ and define $(X_t)_{t \geqslant 0}$, F and $(F_t)_{t \geqslant 0}$ formally in the same way as before. A *diffusion* on M is then a family $(P^x)_{x \in M}$ of probability measures on Ω with the properties (D_1)-(D_3) reformulated in the appropriate way. The only essential difference to §1 is the definition of B^b : this has to be now the space of all bounded Borel-measurable functions on M extended to M_δ by assigning the value 0

to δ. The definition (11) of the transition semigroup $(P_t)_{t\geqslant o}$ yields
then

$$P_t 1(x) = P^x\{X_t \in M\} = P^x\{\zeta > t\} \qquad\qquad (x \in M)$$

and hence

(18) $P_t 1 \leqslant 1$.

So $(P_t)_{t\geqslant o}$ is now a semigroup of *sub-Markov kernels on M*. All P_t
are Markov kernels on M , that is they satisfy

(19) $P_t 1(x) = 1$ $(x \in M,\ t \in I\!R_+)$

if and only if

(19') $\zeta = +\infty$ P^x-almost surely

for all $x \in M$. For $M = I\!R^p$ this corresponds to the situation which
we studied before. We do not encounter the phenomenon of an explosion
there.

As before, one can study diffusions associated to L and the
martingale problem. Via localization and the technique of glueing pieces
together, one is then led to (cf. [18], p.111).

THEOREM 2. *There exists exactly one diffusion process on M
associated to L if L satisfies one of the following two conditions:*
(I') L *has coefficients in* $C^2(M)$
(S'V') L *is elliptic.*

It is important to observe that both conditions are chart
invariant. For $M = I\!R^p$, C^2-coefficients of L imply that condition
(I) is locally satisfied. The matrix field a is then of the form
$a(x) = \sigma(x)\sigma^T(x)$, $x \in I\!R^p$, where the new matrix field $x \to \sigma(x)$ is
locally Lipschitz (cf. [18], p.82).

In the situation of Theorem 2 the unique diffusion associated with
L has a transition semigroup $(P_t)_{t>o}$ which is not anymore uniquely
determined by the one condition

(20) $$P_t f = f + \int_0^t P_s (Lf) ds \qquad\qquad (f \in C_c^2)$$

However, (20) determines $(P_t)_{t>o}$ uniquely if there is no explosion (cf. [1]). Furthermore, $(P_t)_{t>o}$ is not a Feller semigroup in general, even if there is no explosion. So it is natural to ask, when there is no explosion and when the transition semigroup is of Feller type.

Several answers to these questions are known. We restrict our discussion to two particular situations in which interesting answers are available.

The *first situation* is that of $M = \mathbb{R}^p$. Several conditions are known which assure an infinite lifetime of the diffusion. One of the simplest and most important is the following (cf. [21], p. 255 for further details).

PROPOSITION 2. *There is no explosion if the differential operator L is elliptic and if for some constant $K > o$*

(21) $$\|a(x)\| \leqslant K(1 + \|x\|^2)$$

and

(22) $$< x,b(x) > \; \leqslant K(1 + \|x\|^2)$$

hold for all $x \in \mathbb{R}^p$.

Similar, but somewhat stronger growth conditions are necessary to assure, in addition, that the transition semigroup of the diffusion associated to L is a Feller semigroup. Condition (22) has to be replaced by

$$\left| < x,b(x)> \right| \; \leqslant K(1 + \|x\|^2) \qquad\qquad (x \in \mathbb{R}^p);$$

also a stronger version of (21) is needed. For details the reader is referred to Leha-Ritter [13]. Conditions close to being necessary and sufficient can be found in Azencott [1].

The *second situation* concerns the case where M is a *Riemannian* (C∞-)*manifold* and where

(23) $L = \frac{1}{2} \Delta_M$

is - up to the factor $\frac{1}{2}$ - the *Laplace-Beltrami operator* Δ_M of M.

The unique diffusion associated to L is called *Brownian motion on* M.

It can also be obtained by the global solutions of a stochastic

differential equation (cf. [24], p.46 and the acknowledgement mentioned

there).

The manifold M is called *stochastically complete* if Brownian

motion on M does not explode. The following result is due to Yau [25]

who gave an analytical proof. For a probabilistic proof see Ichihara [9].

THEOREM 3. *Every complete Riemannian manifold M with Ricci*

curvature bounded from below is stochastically complete. Furthermore,

the transition semigroup of Brownian motion on M is then a Feller

semigroup.

Here completeness of M refers to metric completeness with respect

to the global metric derived from the Riemannian metric. The curvature

condition is fulfilled if the sectional curvature of M is bounded from

below. Related results have been obtained by Azencott [1], Debiard-

Gaveau-Mazet [8] and Pinsky [17].

4. An outlook on stochastic Riemannian geometry

In the preceding situation (M Riemannian manifold and $L = \frac{1}{2} \Delta_M$)

the transition semigroup $(P_t)_{t>o}$ of Brownian motion on M has a

density $p_t(x,y)$ *with respect to the natural measure* λ_M on M which

depends smoothly on $(t,x,y) \in]o,+\infty[\times M \times M$ (cf. [1], [15]). So we have

for all $f \in \mathcal{B}^b$

(24) $P_t f(x) = \int p_t(x,y) f(y) \lambda_M(dy)$ $(t > o)$.

For fixed $y \in M$ the function $(t,x) \to p_t(x,y)$ is the minimal fundamental

solution of the heat equation

(25) $$\frac{\partial}{\partial t} - \frac{1}{2} \Delta_M = o$$

on $]o,+\infty[\times M$ with pole at (o,y). For $M = \mathbb{R}^p$ we are then back in the framework of §1 where

$$p_t(x,y) = g_t(x-y).$$

For the study of geometric properties of M it is important to know estimates for p_t. The so-called comparison method of Malliavin [14] and his school uses Brownian paths and stochastic differential equations to obtain such estimates. We close this article with one remarkable result due to Debiard-Gaveau-Mazet [8].

For fixed dimension p, let M_a denote the canonical Riemannian manifold of constant sectional curvature a (and dimension p). Two geodesic balls with centres $x_o \in M$ and $x_o^a \in M_a$ of the same radius $\rho > o$ can be identified by means of the normal charts and a linear isometry of the tangent spaces at x_o and x_o^a, for ρ sufficiently small (cf. [14], p.38). In the case of a Hadamard manifold even global identification of M and M_a is possible due to the Cartan-Hadamard-theorem. We recall that M is called a *Hadamard manifold* [22] if M is (metrically) complete, simply connected and of sectional curvature $K \leqslant o$.

Each manifold M_a is such a Hadamard manifold provided that $a \leqslant o$. The corresponding densities p_t^a are known. They can be calculated by means of recursion formulas (cf. [8], p.397). In the case $a = o$ we have $M_o = \mathbb{R}^p$ and hence $p_t^o(x,y) = g_t(x-y)$. Also in the general case $p_t^a(x,y)$ depends on x,y only via the distance $r = d(x,y)$ in the global metric.

The announced result of Debiard-Gaveau-Mazet [8] is now the following.

THEOREM 4. *Let M be a Hadamard manifold which is stochastically complete. Suppose that its sectional curvature K is bounded from above by a constant a ≤ o or from below by a constant b. Then (after global identification of M and M_a) one has*

$$p_t(x,y) \leqslant p_t^a(x,y) \qquad\qquad (x,y \in M)$$

and

$$p_t^b(x,y) \leqslant p_t(x,y) \qquad\qquad (x,y \in M)$$

respectively

In particular, one always has

$$p_t(x,y) \leqslant \left(\frac{1}{2\pi t}\right)^{p/2} e^{-r^2/2t} \qquad\qquad \text{with } r = d(x,y)$$

since $p_t^o(x,y) = g_t(x-y)$. It should be observed that, according to Theorem 3, lower boundedness of K implies stochastical completeness of M.

The results of Theorem 4 can be used in order to obtain comparison results about Green functions and the first eigenvalue of the differential operator $- \Delta_M$ (cf. [8]).

What is remarkable in the proofs of these results is the constant use of Brownian paths. They thus become the object of differential-geometric studies. This is due to the fact that the density p_t can be written in the form

$$p_t(x,y) = \lim_{\varepsilon \to o} \frac{1}{\lambda_M(B_\varepsilon(y))} \; P_t 1_{B_\varepsilon(y)}(x)$$

$$= \lim_{\varepsilon \to o} \frac{1}{\lambda_M(B_\varepsilon(y))} P^x\{X_t \in B_\varepsilon(y)\}$$

where $B_\varepsilon(y)$ denotes the geodesic ball of radius $\varepsilon > o$ with centre y.

Here again, we encounter the general message of this lecture: probabilistic methods, above all the theory of stochastic processes, allow deep insight in purely analytic problems.

References

[1]   R. Azencott, "Behaviour of diffusion semi-groups at infinity",
        Bull. Soc. Math. France 102, (1974) 193-240.

[2]   H. Bauer, *Probability theory and elements of measure theory*.
        (Academic Press, London-NewYork-Toronto-Sydney-San Francisco
        (1981).)

[3]   H. Bauer, Harmonic spaces and associated Markov processes. In:
        Potential Theory, (Centro Internazionale Matematico Estivo
        (C.I.M.E.), Stresa 1969. Edit. Cremonese, Rome (1970)).

[4]   H. Bauer, *Harmonische Räume*. (Jahrbuch Überblicke Mathematik 1981,
        9-35. Bibliographisches Institut, Mannheim (1981).)

[5]   C. Berg and G. Forst, *Potential theory on locally compact abelian
        groups*. (Ergebnisse der Math. 87 Springer-Verlag, Berlin-
        Heidelberg-New York (1975).)

[6]   K.L. Chung, *Lectures from Markov processes to Brownian motion*.
        (Grundlehren d. math. Wiss. 249, Springer-Verlag, New York-
        Heidelberg-Berlin (1982).)

[7]   C. Constantinescu and A. Cornea, *Potential theory on harmonic
        spaces*. (Grundlehren d. math. Wiss. 158, Springer-Verlag,
        Berlin-Heidelberg-New York (1972).)

[8]   A. Debiard, B. Gaveau et E. Mazet, "Théorèmes de comparaison en
        géométrie riemannienne". *Publ. RIMS, Kyoto Univ*. 12, (1976)
        391-425.

[9]   K. Ichihara , "Curvature, geodesics and the Brownian motion on a
        Riemannian manifold". Preprint, Dept. of Applied Sciences,
        Faculty of Engineering, Kyushu Univ.

[10] N. Ikeda and S. Watanabe, *Stochastic differential equations and
        diffusion processes*. (North Holland Mathematical Library 24
        North Holland Publ. Comp., Amsterdam-Oxford-New York, and
        Kodansha Ltd., Tokyo (1981).)

[11] K. Itô, "On stochastic differential equations". *Mem. Amer. Math.
        Soc*. 4 (1951).

[12]   I. Johnstone, *A probabilistic study of linear elliptic-parabolic equations of second order*.   (Notes on Pure Math. 12 Aust. Nat. Univ., Canberra (1979).)

[13]   G. Leha and G. Ritter, "On diffusion processes and their semigroups in Hilbert spaces with application to interacting stochastic systems". *Ann. of Probability* (to appear).

[14]   P. Malliavin, *Géométrie differentielle stochastique*. (Les Presses de l'Université de Montréal, Montréal (1978).)

[15]   H.P. McKean, *Stochastic integrals*.   (Academic Press, New York-London (1969).)

[16]   P.A. Meyer, *Probability and potentials*. (Blaisdell Publ. Co., Waltham, Mass.-Toronto-London (1966).)

[17]   M. Pinsky, *Stochastic Riemannian geometry. In: Probabilistic analysis and related topics, Vol. I,* ((ed. by A.T. Barucha-Reid), 199-236. Academic Press, New York-San Francisco-London (1978).)

[18]   P. Priouret, *Processus de diffusion et equations différentielles stochastiques. In: Ecole d'été de probabilités de Saint Flour III - 1973.* ((ed. par P.A. Meyer, P. Priouret, F. Spitzer). Lecture Notes in Math. 390, 37-113 (1974).)

[19]   P. Priouret, "Processus de diffusion". *Gazette des mathématiciens Soc. math. de France* 4 (1975) 71-87.

[20]   D.W. Stroock and S.R.S. Varadhan, "Diffusion processes with continuous coefficients I,II. *Comm. Pure Appl. Math.* 22 (1969) 345-400 and 479-530.

[21]   D.W. Stroock and S.R.S. Varadhan, *Multidimensional diffusion processes*. (Grundlehren der math. Wiss. 233, Springer-Verlag Berlin-Heidelberg-New York (1979).)

[22]   R. Walter, "Konvexität in riemannschen Mannigfaltigkeiten". *Jber. d. Dt. Math.-Verein.* 83 (1981) 1-31.

[23]   D. Williams, *Diffusions, Markov processes, and martingales, Vol. 1* (J. Wiley & Sons, Chichester-New York-Brisbane-Toronto (1979).)

[24]   D. Williams, *"To begin at the beginning:..." In: Stochastic Integrals,* (Proc. LMS Durham Symposium 1980 (ed. by D. Williams). Lecture Notes in Math. 851, 1-55 (1981).)

[25]   S.T. Yau, "On the heat kernel of a complete Riemannian manifold". *J. Math. Pures Appl.* 57, (1973) 191-201.

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