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THE SIMILARITY PROBLEM FOR TENSOR PRODUCTS OF CERTAIN C*-ALGEBRAS

Florin Pop

We prove that every bounded representation of the tensor product of two C^*-algebras, one of which is nuclear and contains matrices of any order, is similar to a *-representation.

1. Introduction

A C^*-algebra $\mathcal{A}$ has the *similarity property* if every bounded representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is similar to a *-representation, that is, if there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}\pi S$ is a *-representation. This property was introduced by Kadison in [**4**], where he conjectured that all C^*-algebras have the similarity property. Haagerup [**3**] proved that a bounded representation is similar to a *-representation if and only if it is completely bounded, and also, that representations with a cyclic vector (or a finite cyclic set) are similar to *-representations. In addition, if π is completely bounded, then

$$\|\pi\|_{cb} = \inf\{\|S\|\,\|S^{-1}\|;\ S\pi S^{-1} \text{ is a *-representation}\}$$

and this infimum is attained ([**5**]).

Recently ([**6, 7, 8**]), Pisier introduced the notions of *similarity degree* and *length*, which have played a significant role in the study of the similarity problem.

The similarity degree $d(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is the smallest $\alpha \geqslant 0$ for which there is a constant $C_{\mathcal{A}}$ such that every bounded representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies $\|\pi\|_{cb} \leqslant C_{\mathcal{A}}\|\pi\|^{\alpha}$. The length $\ell(\mathcal{A})$ is the smallest integer d for which there is a constant K such that, for any n and any $X \in M_n(\mathcal{A})$, there is an integer $N = N(n, X)$, scalar matrices

$$\alpha_0 \in M_{n,N}(\mathbb{C}),\ \alpha_1 \in M_N(\mathbb{C}), \ldots, \alpha_{d-1} \in M_N(\mathbb{C}),\ \alpha_d \in M_{N,n}(\mathbb{C}),$$

and diagonal matrices $D_1, \ldots, D_d \in M_N(\mathcal{A})$ satisfying

$$\begin{cases} X = \alpha_0 D_1 \alpha_1 D_2 \ldots D_d \alpha_d \\ \prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leqslant K\|X\|. \end{cases}$$

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A C^*-algebra $\mathcal{A}$ has the similarity property if and only if $d(\mathcal{A}) < \infty$ and Pisier [**6**] proved the striking fact that $d(\mathcal{A}) = \ell(\mathcal{A})$.

Despite all the progress made so far, there are few concrete examples of C^*-algebras known to have the similarity property. We list them below, together with their respective lengths:

(i) If $\mathcal{A}$ is nuclear, then $\ell(\mathcal{A}) = 2$ ([**1**]).

(ii) If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then $\ell(\mathcal{A}) = 3$ ([**7**]).

(iii) $\ell(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) \leqslant 3$, $\mathcal{A}$ is arbitrary ([**3, 8**]).

(iv) If $\mathcal{M}$ is a type II_1 factor with property Γ, then $\ell(\mathcal{M}) = 3$ ([**2**]).

In this paper we add to the above list the following result: If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras such that $\mathcal{B}$ is nuclear and contains unital matrix algebras of any order, then $\ell(\mathcal{A} \bigotimes_{\min} \mathcal{B}) \leqslant 5$.

Throughout this paper we shall assume that all C^*-algebras and their Hilbert space representations are unital. We denote by $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \bigotimes_{\min} \mathcal{B}$ and $\mathcal{A} \bigotimes_{\max} \mathcal{B}$ the algebraic, the spatial (minimal), and the maximal tensor products of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.

2. Preliminary Results

It is well known that, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $f : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $g : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ are commuting *-representations, then the map $\psi : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H})$ defined on elementary tensors by $\psi(a \otimes b) = f(a)g(b)$ is bounded with respect to the maximal C^*-norm on $\mathcal{A} \otimes \mathcal{B}$. If, however, ψ is bounded with respect to the spatial norm on $\mathcal{A} \otimes \mathcal{B}$, then it extends to a *-representation of $\mathcal{A} \bigotimes_{\min} \mathcal{B}$, so ψ is completely contractive. Therefore, boundedness alone with respect to the spatial norm implies automatic complete contractivity. The technical results in this section belong to this circle of ideas.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\mathcal{X} \subseteq \mathcal{A}$ is an operator system, that is, a closed, self-adjoint, unital vector subspace. Denote by $\mathcal{X} \bigotimes_{\min} \mathcal{B}$ the closure of $\mathcal{X} \otimes \mathcal{B}$ (elementary operators) in the spatial norm inherited from $\mathcal{A} \bigotimes_{\min} \mathcal{B}$. Let $\varphi : \mathcal{X} \to \mathcal{B}(\mathcal{H})$ be a unital completely positive map and let $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be a *-representation such that $\varphi(x)\pi(b) = \pi(b)\varphi(x)$ for every $x \in \mathcal{X}, b \in \mathcal{B}$. Suppose, in addition, that the map defined on $\mathcal{X} \otimes \mathcal{B}$ with values in $\mathcal{B}(\mathcal{H})$ taking $\sum_{i=1}^{n} x_i \otimes b_i$ to $\sum_{i=1}^{n} \varphi(x_i)\pi(b_i)$ is bounded with respect to the spatial norm on $\mathcal{X} \otimes \mathcal{B}$, so it extends to a bounded map ω on $\mathcal{X} \bigotimes_{\min} \mathcal{B}$. Under these hypotheses we have

Lemma 2.1. *If $\mathcal{B}$ is nuclear, then the map ω is completely positive on $\mathcal{X} \bigotimes_{\min} \mathcal{B}$.*

PROOF: The map taking $\sum_{i=1}^{n} x_i \otimes b_i$ to $\sum_{i=1}^{n} \varphi(x_i) \otimes b_i$ is completely positive from $\mathcal{X} \underset{\min}{\bigotimes} \mathcal{B}$ to $\mathcal{B}(\mathcal{H}) \underset{\min}{\bigotimes} \mathcal{B}$, and the map taking $\sum_{i=1}^{n} y_i \otimes b_i$ to $\sum_{i=1}^{n} y_i \otimes \pi(b_i)$ is completely positive from $\mathcal{B}(\mathcal{H}) \underset{\min}{\bigotimes} \mathcal{B}$ to $\mathcal{B}(\mathcal{H}) \underset{\min}{\bigotimes} \pi(\mathcal{B})$. Then the map taking $\sum_{i=1}^{n} x_i \otimes b_i$ to $\sum_{i=1}^{n} \varphi(x_i) \otimes \pi(b_i)$ is completely positive from $\mathcal{X} \underset{\min}{\bigotimes} \mathcal{B}$ to $\mathcal{B}(\mathcal{H}) \underset{\min}{\bigotimes} \pi(\mathcal{B})$, as the composition of the two previous maps. Since the ranges of φ and π commute, the latter map's range is included in $\pi(\mathcal{B})' \underset{\min}{\bigotimes} \pi(\mathcal{B})$. Note that, since $\mathcal{B}$ is nuclear, so is $\pi(\mathcal{B})$. The map from $\pi(\mathcal{B})' \otimes \pi(\mathcal{B})$ into $\mathcal{B}(\mathcal{H})$ taking $\sum_{i=1}^{n} y_i \otimes z_i$ to $\sum_{i=1}^{n} y_i z_i$ extends to a *-representation of $\pi(\mathcal{B})' \underset{\max}{\bigotimes} \pi(\mathcal{B})$ $= \pi(\mathcal{B})' \underset{\min}{\bigotimes} \pi(\mathcal{B})$. This shows that ω, as a composition of three completely positive maps, is completely positive on $\mathcal{X} \underset{\min}{\bigotimes} \mathcal{B}$. $\square$

PROPOSITION 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\mathcal{B}$ nuclear. Let $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a complete contraction and let $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be a *-representation such that $\varphi(a)\pi(b) = \pi(b)\varphi(a)$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$. If the map $\Theta : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H})$, defined on elementary tensors by $\Theta(a \otimes b) = \varphi(a)\pi(b)$, is bounded with respect to the spatial norm on $\mathcal{A} \otimes \mathcal{B}$, then it extends to a complete contraction on $\mathcal{A} \underset{\min}{\bigotimes} \mathcal{B}$.*

PROOF: Consider the operator system of $\mathcal{A} \otimes M_2$

$$\mathcal{X} = \left\{ \begin{pmatrix} \lambda I & x \\ y & \mu I \end{pmatrix} ;\ x, y \in \mathcal{A} \right\}$$

and define $\Phi : \mathcal{X} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by

$$\Phi\left(\begin{pmatrix} \lambda I & x \\ y & \mu I \end{pmatrix} \right) = \begin{pmatrix} \lambda I & \varphi(x) \\ \varphi(y) & \mu I \end{pmatrix}.$$

It is well-known that φ is completely contractive if and only if Φ is completely positive. Define also $\widetilde{\pi} : \mathcal{B} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by

$$\widetilde{\pi}(b) = \begin{pmatrix} \pi(b) & 0 \\ 0 & \pi(b) \end{pmatrix}.$$

Finally, define

$$T : \mathcal{X} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

by $T(X \otimes b) = \Phi(X)\widetilde{\pi}(b)$. Notice that Φ and $\widetilde{\pi}$ commute, T is bounded with respect to the spatial norm inherited by $\mathcal{X} \otimes \mathcal{B}$ from $(\mathcal{A} \otimes M_2) \underset{\min}{\bigotimes} \mathcal{B}$, and its norm satisfies $\|T\| \leqslant 2\|\Theta\| + 2$.

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From Lemma 2.1, T is completely positive and, by Arveson's extension theorem, T has a unital completely positive extension to $(\mathcal{A} \otimes M_2) \bigotimes_{\min} \mathcal{B}$, denoted by $\widetilde{T}$. As a unital completely positive map on a C^*-algebra, $\widetilde{T}$ is completely contractive. The map

$$j : \mathcal{A} \bigotimes_{\min} \mathcal{B} \to (\mathcal{A} \otimes M_2) \bigotimes_{\min} \mathcal{B}$$

defined on elementary tensors by

$$j(a \otimes b) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes b = \begin{pmatrix} 0 & a \otimes b \\ 0 & 0 \end{pmatrix}$$

is completely isometric and

$$\widetilde{T}\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes b\right) = T\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes b\right) = \begin{pmatrix} 0 & \Theta(a \otimes b) \\ 0 & 0 \end{pmatrix}.$$

We conclude that Θ is a complete contraction. ∎

Corollary 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\mathcal{B}$ nuclear. If π is a bounded representation of $\mathcal{A} \bigotimes_{\min} \mathcal{B}$ such that $\pi|_{\mathcal{A}}$ is completely bounded and $\pi|_{\mathcal{B}}$ is self-adjoint, then π is completely bounded and $\|\pi\|_{cb} \leqslant \|\pi|_{\mathcal{A}}\|_{cb}$.*

3. The Main Result

We are ready to prove the main result of this paper.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras such that $\mathcal{B}$ is nuclear and contains unital matrix algebras of any order. If π is a bounded representation of $\mathcal{A} \bigotimes_{\min} \mathcal{B}$, then π is completely bounded and $\|\pi\|_{cb} \leqslant \|\pi\|^5$.*

Proof: There exists $S \in \mathcal{B}(\mathcal{H})$ invertible such that $\|S\| \cdot \|S^{-1}\| \leqslant \|\pi\|^2$ and $\rho = S\pi S^{-1}$ is self-adjoint on $\mathcal{B}$ [3]. Since ρ is unital and $\mathcal{B}$ contains matrices of any order, then so does $\rho(\mathcal{B})$. Since $\rho(\mathcal{A})$ and $\rho(\mathcal{B})$ commute, we have $\|\rho|_{\mathcal{A}} \otimes \mathrm{Id}_{M_n}\| \leqslant \|\rho\|$, which shows that $\rho|_{\mathcal{A}}$ is completely bounded and $\|\rho|_{\mathcal{A}}\|_{cb} \leqslant \|\rho\| \leqslant \|\pi\|^3$. It follows from Corollary 2.3 that ρ is completely bounded and $\|\rho\|_{cb} \leqslant \|\pi\|^3$. This shows that $\pi = S^{-1}\rho S$ is completely bounded and

$$\|\pi\|_{cb} \leqslant \|S\| \cdot \|S^{-1}\| \cdot \|\rho\|_{cb} \leqslant \|\pi\|^5.$$

∎

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Department of Mathematics and Computer Science
Wagner College
Staten Island, N.Y. 10301
United States of America
e-mail: fpop@wagner.edu