MODERN DYNAMICAL SYSTEMS THEORY

L. MARKUS

University of Minnesota, Minneapolis, Minn. 55455, U.S.A., and
University of Warwick, Coventry, Warwickshire CV 4 7AU, U.K.

Abstract. In order to analyse generic or typical properties of dynamical systems we consider the space $\mathscr{V}$ of all C^1-vector fields on a fixed differentiable manifold M. In the C^1-metric, assuming M is compact, $\mathscr{V}$ is a complete metric space and a generic subset is an open dense subset or an intersection of a countable collection of such open dense subsets of $\mathscr{V}$. Some generic properties (i.e. specifying generic subsets) in $\mathscr{V}$ are described. For instance, generic dynamic systems have isolated critical points and periodic orbits each of which is hyperbolic. If M is a symplectic manifold we can introduce the space $\mathscr{H}$ of all Hamiltonian systems and study corresponding generic properties.

1. Copernican Astronomy as a Natural System

Nicolaus Copernicus designed his model of the Solar system to achieve the greatest simplicity, within his physical and philosophical axiomatic framework. He sought to eliminate the unnecessary hypotheses of the Ancients concerning coincidences and correlations of planetary orbits by explaining these orbits in terms of his heliocentric kinematics. He wrote, "I correlate all the movements of the other planets and their spheres or orbital circles with the mobility of the Earth."

Much current research in the mathematical theory of dynamical systems proceeds in this same spirit – to eliminate the special situations depending on coincidences and correlations and to concentrate on the mathematically typical or generic cases. Of course, what is considered typical depends on the range of possibilities permitted. For instance, Copernicus allowed only uniform circular motions; but within the framework of Kepler–Newton geometry all circular orbits would be discarded as ungeneric ellipses.

The Copernican model of the Solar system consists on the central star Sol surrounded by a family of planets in concentric circular orbits. Of course, Copernicus perturbed the basic circular orbits by epicycles, and Kepler and Newton later introduced more intricate elliptical perturbations, but the fundamental conceptual picture remains that of the Sun encircled by its family of planets.

A different conceptual framework for the Solar system views this astronomical system as basically a double star, with components of Sol and Jupiter moving in almost circular orbits around their common center of mass. More detailed structure includes Saturn as a third star (or proto-star gas ball), and further the gas balls Uranus and Neptune, each moving in ever large circular orbits about the system mass center. Moreover each of these stars carries its own planetary system – namely Mercury, Venus, Earth and Mars about Sol, the Jovian satellites about Jupiter, and the other corresponding satellite systems accordingly placed.

Whether we accept the Copernican model or the multi-star model of the Solar system is a philosophical choice rather than a scientific decision in our current state

Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems, 11–18.

https://doi.org/10.1017/S0074180900070315 Published online by Cambridge University Press

of astronomical knowledge. In fact, the two physical models are kinematically, even dynamically equivalent – they differ only in the psychological emphasis arising from the renaming of Jupiter as a star rather than a planet. In some future era when there exists a comprehensive theory of planetary and stellar evolution and structure, it may be possible to select one of these two physical models as the prefered and established description. In terms of such an evolutionary stellar theory one of the above models might appear as 'more natural, simpler, and more typical' within the framework of the theory. If the Copernican model is vindicated then the evolutionary theory would have explained why there should be two types of solar planets, inner terrestrial planets and outer gaseous giants. If the multi-star model becomes accepted, then the evolutionary theory would be expected to predict that most stars develop planetary systems in a natural way. But until a coherent astronomical theory of planetary systems is developed, there is no way of selecting one of these models as simpler or more natural than the other.

The purpose of this philosophical discussion on the methodology of mathematical astronomy has been to underline the importance of the concept of natural or generic properties of dynamical systems. Accordingly much of the research of the past decade in the mathematical theory of ordinary differential equations has concentrated on the discovery and examination of such generic properties, with the attempt to discard ungeneric systems displaying any special 'coincidences and correlations'.

2. Fundamentals of Global Differentiable Dynamics

In order to present an exposition of the modern theory of generic dynamical systems, as conducted by many pure and applied mathematicians during the past decade, we shall briefly review some of the fundamental concepts of global differential geometry and analysis.

We consider dynamical systems described by vector differential equations of first order, say

$$\mathrm{d}x^i/\mathrm{d}t = v^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n,$$

since any higher order differential equations can be reduced to this form. For simplicity of exposition we take time-independent or autonomous differential systems. But we emphasize that there is no supposition that these dynamical systems are conservative, or even arise from any Newtonian mechanical problem. At the end of this paper we shall mention some quite recent results that pertain to conservative Hamiltonian dynamics, but at present we make no such assumptions.

In the theory of local dynamical systems we study differential equations in an open subset of the real number space R^n; whereas in global dynamics the space is a general differentiable manifold M^n. We specify a global dynamical system as a tangent vector field v on M^n; and in any local chart $(x^1, \ldots, x^n)$ on M^n we denote the dynamical system v by its components $v^i(x^1, \ldots, x^n)$, say

$$(v) \qquad \mathrm{d}x^i/\mathrm{d}t = v^i(x^1, \ldots, x^n), \quad i = 1, 2, \ldots, n,$$

or

$(v) \qquad \dot{x} = v(x).$

There are two basic motivations for studying differential systems on general differentiable manifolds.

(i) *Mathematical motivation.* It is of interest to study differential systems within the most general context for which the concepts of differentiation and mathematical analysis are meaningful. Thus we take the ambiant space M^n to be of arbitrary finite dimension n, and locally differentiably equivalent to R^n. That is, M^n is a differentiable n-manifold (separable, metrizable C^∞-manifold without boundary), for instance R^n, or the n-sphere S^n, or the n-torus T^n.

(ii) *Physical motivation.* Physical dynamical systems are often described by first-order vector differential equations involving the displacements, velocities, angles, and other generalized coordinates. If the generalized coordinates are unrestricted real variables, then the space in which the system evolves in time is some real vector space R^n. However, frequently the generalized coordinates are restricted by constraint or energetic equalities, or account for some angular periodicities of the physical configuration, and in these cases the space of the system is some differentiable manifold M^m.

For instance, an ordinary planar pendulum has a configuration space of a circle S^1, and a velocity-phase space of a product cylinder $S^1 \times R^1$. A spherical pendulum has a configuration space of a sphere S^2, and a velocity-phase space that is the tangent bundle TS^2. The configuration space of a rigid rotor is the rotation matrix group $SO(3)$, which is diffeomorphic to the real projective space P^3, and the velocity-phase space is the 6-manifold TP^3 (which incidently is the product $P^3 \times R^3$).

Besides the greater generality and applicability, the main advantages of the global viewpoint for dynamics are:

(i) The notation and methodology of global differential geometry (involving manifolds, vector fields, trajectory curves, etc.) are highly suited to the requirements of the problems of dynamics. Old problems can be carefully phrased and solved, and new problems and concepts are suggested. For example, a careful discussion of the spherical pendulum requires knowledge that the tangent bundle TS^2 is not the product $S^2 \times R^2$. New concepts of structural stability and genericity arise naturally when various dynamical systems are compared globally.

(ii) The global viewpoint emphasizes the unified family of all the trajectories as a portrait of a given dynamical system, rather than singling out special trajectories by their initial data. This is particularly important in physical systems where we wish to classify and compare the diverse modes of asymptotic behavior of the trajectories of the system.

Let M^n be a differentiable manifold and let v be a tangent C^r-vector field on M^n. Then in overlapping charts $(x^1, \ldots, x^n)$ and $(\bar{x}^1, \ldots, \bar{x}^n)$ on an open set of M^n, the components of v are,

$(v) \qquad \dot{x}^i = v^i(x) \quad \text{and} \quad \dot{\bar{x}}^i = \bar{v}^i(\bar{x}),$

https://doi.org/10.1017/S0074180900070315 Published online by Cambridge University Press

where the contravariant vector transformation law holds for the C^r-functions ($r=1, 2, \ldots, \infty$) v^i and $\bar{v}^i$,

$$\bar{v}^i = (\partial \bar{x}^i / \partial x^j)\, v^j, \quad (\text{sum on } j), \quad i = 1, \ldots, n.$$

A solution or trajectory of this vector field or differential system v is a C^1-curve,

$$I \to M^n : t \to P_t, \quad (I \text{ open interval in } R),$$

whose tangent vector at each point coincides with the vector of the field v.

The usual local existence, uniqueness, regularity results are valid. That is, for each initial point $P_0 \in M^n$ there exists a unique trajectory P_t of v through P_0 at $t=0$ (and defined on some maximal time duration I). If M^n is compact (and we shall assume this henceforth for simplicity of exposition), the maximal interval I is all R. In this case the solutions of v define a C^r-flow or action of R on M^n. That is, there is a C^r-map,

$$\Phi : R \times M^n \to M^n : (t, P_0) \to P_t,$$

and for fixed $t \in R$,

$$\Phi_t : P_0 \to P_t,$$

is a C^r-diffeomorphism of M^n onto itself, and the group property holds for all times t_1, $t_2 \in R$,

$$\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1 + t_2}, \qquad \Phi_0 = \text{Identity}.$$

Every trajectory of v on M^n is either

(i) a point P_0,

(ii) a C^r-diffeomorphic image of a circle S^1,

(iii) a bijective regular C^r-differentiable image of a line R^1.

The case (i) is a critical point where v vanishes at P_0, case (ii) corresponds to a periodic solution or closed orbit, but case (iii) can lead to curves that are not topological lines, say for an irrational flow on the torus T^n.

An invariant set Σ of v on M^n is a subset that is the union of whole trajectories of v. That is, a subset $\Sigma \subset M^n$ is invariant for the flow of v in case each trajectory initiating in Σ remains forever in Σ. Of course, a critical point or a closed orbit is necessarily an invariant set. Also, for each initial point P_0 in the compact manifold M^n, the past (negative) and future (positive) limit set of the trajectory P_t

$$\alpha(P_0) = \bigcap_{\tau < 0} \overline{\bigcup_{t < \tau} P_t} \quad \text{and} \quad \omega(P_0) = \bigcap_{\tau > 0} \overline{\bigcup_{t > \tau} P_t},$$

are each compact connected invariant sets. Clearly $\alpha(P_0)$ and $\omega(P_0)$ depend only on the trajectory P_t and not on the initial point P_0. If $P_0 \in \omega(P_0)$ then P_t is called future recurrent or Poisson stable, and P_0 is recurrent if it is both past and future recurrent. A rather weaker property is regional recurrence of P_0 – namely each neighborhood U of P_0 in M^n has a trajectory U_t that meets U for some arbitrarily large past and future times. The set Ω of all regionally recurrent points, usually called the nonwandering set, is a compact invariant set containing all critical points, periodic orbits, and recurrent trajectories of the dynamical system v.

An invariant set Σ for the flow v in M^n is called future stable in case: for each neighborhood W of Σ in M^n there exists a subneighborhood $W_1 \subset W$ such that $P_0 \in W_1$ implies that $P_t \in W$ for all future times $t > 0$. If in addition $\omega(P_0) \subset \Sigma$ then the set Σ is future asymptotically stable (and analogues statements hold for past times).

We illustrate these concepts of recurrence and stability by some examples of invariant sets for flows in vector spaces and cylinders.

Example 1. $\dot{x} = Ax$ for $x \in R^n$ and A real constant matrix. If the matrix A is nonsingular, with complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ not zero, then the origin $x = 0$ is the unique critical point.

If no eigenvalue of A is pure imaginary, that is $\operatorname{Re} \lambda_j \neq 0$, then the origin $x = 0$ is called a hyperbolic critical point. Define the attractor (stability) set of all points $P \in R^n$ for which $\omega(P) = 0$, and similarly the repellor (instability) set by $\alpha(P) = 0$. Then the attractor set is a linear subspace whose dimension equals the number of eigenvalues whose real parts are negative. Also $x = 0$ is asymptotically stable just in case the attractor space is all R^n.

Now consider a C^1-differential system v on a differentiable manifold M^n. Let P_0 be a critical point for v and take a local chart (x) centered at P_0 to write the differential system

$$(v) \qquad \dot{x} = Ax + \cdots.$$

We define P_0 to be a hyperbolic critical point of v in case no eigenvalue of A is pure imaginary. In this case we define the attractor and repellor sets and prove that these are each C^1-differentiable submanifolds of M^n, in fact they are each regular bijective images of vector spaces of the appropriate dimensions.

Example 2. $\dot{x} = Ax$ and $\dot{\theta} = 1$, where $x \in R^{n-1}$ and $\theta \in S^1$. Here the manifold $M^n = R^{n-1} \times S^1$ is a cylinder. There are no critical points but the circle, $x = 0$, $0 \leqslant \theta < 1$, is a periodic orbit σ. The Poincaré map around this periodic orbit σ is given by $x_0 \to e^A x_0$, and its eigenvalues $\mu_1, \ldots, \mu_{n-1}$ are the (nontrivial) characteristic multipliers, and $\mu_n = 1$.

If none of the characteristic multipliers has a modulus of unity, $|\mu_j| \neq 1$ for all $1 \leqslant j < n-1$, then the periodic orbit is defined to be hyperbolic. Again we define the attractor set by $\omega(P) \subset \sigma$ and the repellor set by $\alpha(P) \subset \sigma$, and it is clear that these are each cylinders of dimensions specified by the moduli of the characteristic multipliers.

Now consider a C^1-differential system v on a differentiable manifold M^n. Let σ be a periodic orbit of v with Poincaré map of a transversal section yielding the characteristic multipliers $\mu_1, \ldots, \mu_{n-1}$. If all $|\mu_j| \neq 1$ then σ is a hyperbolic periodic orbit. Again the attractor and repellor sets are C^1-submanifolds which are regular bijective images of either cylinders or generalized nonorientable Mobius bands, depending on the moduli and arguments of the complex characteristic multipliers.

3. Generic Properties of Dynamical Systems

Let M^n be a compact differentiable manifold. Denote by $\mathscr{V}$ the set of all dynamical systems, that is, C^1-vector fields on M^n. In order to make precise our ideas of perturbation and approximation of dynamical systems in $\mathscr{V}$ we specify the C^1-topology in $\mathscr{V}$. Namely, two vector fields u and v in $\mathscr{V}$ are nearby one another in case their vector components, and also the corresponding first partial derivatives, are nearly equal,

$$|u^i(x)-v^i(x)|<\varepsilon,$$

and

$$\left|\frac{\partial u^i}{\partial x^j}-\frac{\partial v^i}{\partial x^j}\right|<\varepsilon, \quad i,j=1,\ldots,n.$$

Here the components are expressed in some fixed finite collection of local charts covering M^n. Define the metric distance $\|u-v\|_1$ between the vector fields u and v in $\mathscr{V}$ as the infimum of all such bounds $\varepsilon>0$ for which the above inequalities hold globally on M^n. Then it is known that $\mathscr{V}$ is a complete metric space, in fact a Banach space as defined by that distinguished Polish mathematician of the past generation.

For the study of perturbations of differential systems the open sets of $\mathscr{V}$ are important. Recall that a set $\mathscr{O}$ is open in $\mathscr{V}$ in case any small perturbation of a member of $\mathscr{O}$ always yields a member of $\mathscr{O}$. For approximation theory of differential systems the dense sets of $\mathscr{V}$ are important. Namely, a set $\mathscr{D}$ is dense in $\mathscr{V}$ in case each member of $\mathscr{V}$ can be approximated by members of $\mathscr{D}$.

We define a generic set $\mathscr{G}$ of $\mathscr{V}$, or refer to a generic property specifying $\mathscr{G}$, when $\mathscr{G}$ is a countable intersection of open and dense subsets of $\mathscr{V}$. It is a standard theorem of Baire that a generic set of a complete metric space $\mathscr{V}$ is dense in $\mathscr{V}$. We think of a generic subset $\mathscr{G}$ of $\mathscr{V}$ as comprising almost all the members of $\mathscr{V}$, with the complement $\mathscr{V}-\mathscr{G}$ being a negligible collection of differential systems. With this terminology we can now state some results concerning the generic properties of dynamical systems.

Definition. Let $\mathscr{V}$ be the metric space of all C^1-vector fields on a compact differentiable manifold M^n.

Define the subset $\mathscr{G}_1$ of $\mathscr{V}$ to consist of all differential systems v in $\mathscr{V}$ for which: $v\in\mathscr{G}_1$ has only a finite number of critical points and each of these is hyperbolic.

Define the subset $\mathscr{G}_2$ of $\mathscr{V}$ to consist of all differential systems v in $\mathscr{V}$ for which: $v\in\mathscr{G}_2$ has, for each integer $N\geqslant 1$, only a finite number of periodic orbits with period less than N, and each of these orbits is hyperbolic.

Theorem. $\mathscr{G}_1$ is open and dense in $\mathscr{V}$. $\mathscr{G}_2$ is generic in $\mathscr{V}$.

We remark that it is then immediate that $\mathscr{G}_1$, as well as the intersection $\mathscr{G}_1\cap\mathscr{G}_2$, is generic in $\mathscr{V}$. In this sense almost all differential systems in $\mathscr{V}$ have isolated and hyperbolic critical points and periodic orbits.

4. Generic Properties of Hamiltonian Dynamical Systems

Locally we specify a Hamiltonian differential system,

$$\frac{dx^i}{dt}=\frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x^i}, \quad i=1,\ldots,n,$$

by a real differentiable Hamiltonian function $H(x^1,\ldots,x^n, y_1,\ldots,y_n)$ in a real number space R^{2n}. A differentiable coordinate transformation $(x, y)\to(\bar{x}, \bar{y})$ allows us to write the differential system in the Hamiltonian format,

$$\frac{d\bar{x}^i}{dt}=\frac{\partial H}{\partial \bar{y}_i}, \qquad \frac{d\bar{y}_i}{dt}=-\frac{\partial H}{\partial \bar{x}^i}, \quad i=1,\ldots,n,$$

where $H(\bar{x}, \bar{y})=H(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$, in case the transformation is canonical (or symplectic). That is, the Jacobian matrix $T=\partial(\bar{x}, \bar{y})/\partial(x, y)$ satisfies everywhere the symplectic condition,

$$TJT^t=J, \quad \text{with} \quad J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

This discussion shows that a global theory of nonlinear autonomous Hamiltonian systems can be formulated on a differentiable manifold M^{2n} which is covered by a family of canonical charts interrelated by canonical coordinate transformations. Such a manifold M^{2n}, with the family of canonical charts, is called a symplectic manifold. Each real differentiable function H on a symplectic manifold M^{2n} specifies a C^1-vector field on M^{2n} which has the required Hamiltonian format in each canonical chart (x, y) of M^{2n}.

Let M^{2n} be a compact symplectic manifold and denote by $\mathscr{H}$ the set of all real differentiable functions on M^{2n}. Use the C^∞-topology on $\mathscr{H}$, wherein two Hamiltonian functions H_1 and H_2 are nearby in case the values of a finite set of partial derivatives are approximately equal everywhere on M^{2n} (normalize each H in $\mathscr{H}$ to have a minimum value of zero on M^{2n}, since augmenting H by a constant does not modify the corresponding Hamiltonian differential system). In this case $\mathscr{H}$ is a complete metric space and we can seek generic Hamiltonian systems in $\mathscr{H}$. Note that $\mathscr{H}$ itself is negligible in the space $\mathscr{V}$ of all vector fields on M^{2n}, but we agree to compare vector fields only within the fixed space $\mathscr{H}$ in order to find generic sets of $\mathscr{H}$.

The mathematical theory of generic Hamiltonian systems of $\mathscr{H}$ on a compact symplectic manifold M^{2n} is only in the preliminary stages of study and research. Some of the major mathematical discoveries of the past few decades can be formulated in these terms, but many problems remain unsolved. We close this short review by listing several properties which are valid for a generic set of Hamiltonian in $\mathscr{H}$.

(1) Only finitely many critical points, each with nonzero eigenvalues. But there exists a critical point with all eigenvalues pure imaginary, and rationally independent in the usual sense.

https://doi.org/10.1017/S0074180900070315 Published online by Cambridge University Press

(2) Noncountably many periodic orbits. But only a countable number of degenerate periodic orbits (having more than two characteristic multipliers of unity).

(3) Noncountably many almost periodic orbits each dense in some toral submanifold.

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