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REILLY INEQUALITIES OF ELLIPTIC OPERATORS ON CLOSED SUBMANIFOLDS

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Abstract

Using generalized position vector fields we obtain new upper bound estimates of the first nonzero eigenvalue of a kind of elliptic operator on closed submanifolds isometrically immersed in Riemannian manifolds of bounded sectional curvature. Applying these Reilly inequalities we improve a series of recent upper bound estimates of the first nonzero eigenvalue of the L_r operator on closed hypersurfaces in space forms.

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1. Introduction

Let M^n be a closed, connected and orientable n-dimensional Riemannian manifold isometrically immersed in the m-dimensional Euclidean space $\mathbb{E}^m$ $(m > n)$, let H denote the mean curvature of the immersion of M^n into $\mathbb{E}^m$ and let λ_1^Δ denote the first nonzero eigenvalue of the Laplacian on M^n. In 1977, Reilly [6] proved

$$\lambda_1^\Delta \le \frac{n}{\mathrm{vol}(M)} \int_M H^2 \, dM$$

and a generalized Reilly inequality

$$\lambda_1^\Delta \left(\int_M H_r \, dM \right)^2 \le n \, \mathrm{vol}(M) \int_M H_{r+1}^2 \, dM, \quad 0 \le r \le n-1,$$

where H_r denotes the r-mean curvature of M^n. In 2004, Alias and Malacarne [2] considered an L_r operator(see §4) on a closed, connected and orientable n-dimensional Riemannian manifold isometrically immersed in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$; if L_r is elliptic on M^n for some $0 \le r \le n-1$, they proved a Reilly

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inequality of $\lambda_1^{L_r}$ with H_r, H_s as follows

$$\lambda_1^{L_r}\left(\int_M H_s\,dM\right)^2 \leq c_r \int_M H_r\,dM \int_M H_{s+1}^2\,dM, \quad 0 \leq s \leq n-1 \tag{1.1}$$

where $c_r = (n-r)\binom{n}{r}$.

When the ambient space is the Euclidean sphere $\mathbb{S}^{n+1}(1)$, let X be the position vector of $M^n(\subset \mathbb{S}^{n+1}(1))$ in $\mathbb{E}^{n+2}$, and $\langle\ ,\ \rangle$ be the Euclidean metric on $\mathbb{E}^{n+2}$. Alias and Malacarne [2] obtained the Reilly inequality including $\langle X, \eta\rangle$

$$\lambda_1^{L_r}\left(\int_M H_s\langle X, \eta\rangle\,dM\right)^2 \leq c_r \int_M H_r\,dM \int_M H_{s+1}^2\,dM, \quad 0 \leq s \leq n-1 \tag{1.2}$$

where η is the gravity center vector of $M^n(\subset \mathbb{S}^{n+1}(1))$ in $\mathbb{E}^{n+2}$.

There is no similar result for the case of hyperbolic space $\mathbb{H}^{n+1}(-1)$. Naturally we hope to obtain such unified inequalities only with H_r, H_s for any simply connected space form $\mathbb{R}^{n+1}(c)$ of constant sectional curvature c.

On the other hand, when the ambient space is a Riemannian manifold $(\overline{M}^m, \overline{g})$ $(m > n)$ of sectional curvature bounded above by c, we define a tensor set on M^n:

$$\begin{aligned}\mathcal{A} = \{T \mid T &\text{ is a symmetric positive-definite (1.1)-tensor}\\ &\text{on } M^n \text{ such that } \operatorname{div}_M T = 0\}.\end{aligned}$$

(Since $I_n \in \mathcal{A}$ we know that $\mathcal{A} \neq \emptyset$.) Given any $T \in \mathcal{A}$, Grosjean [5] considered the extrinsic upper bounds for the first nonzero eigenvalue of the elliptic operators L_T defined on (M^n, g) (that is, in terms of the second fundamental form of an isometric immersion of (M^n, g) into $(\overline{M}^m, \overline{g})$) of the form

$$L_T u = \operatorname{div}_M(T\nabla^M u)$$

where $u \in C^\infty(M)$, div_M and ∇^M denote, respectively, the divergence and the gradient of the metric g on M^n. Let ϕ be an isometric immersion of (M^n, g) into $(\overline{M}^m, \overline{g})$, and λ_1^T be the first nonzero eigenvalue of the operator L_T, if $c \leq 0$ we assume that $(\overline{M}^m, \overline{g})$ is simply connected, and if $c > 0$ we assume that $\phi(M^n)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$. Then he obtained

$$\lambda_1^T \leq \frac{\sup_M |H_T|^2 + \sup_M c(\operatorname{tr} T)^2}{\inf_M \operatorname{tr}(T)}, \tag{1.3}$$

and

$$\lambda_1^T \leq \sup_M(|H_T||H| + c(\operatorname{tr} T)), \quad (\text{if } |H_T| = \text{constant}) \tag{1.4}$$

where $H_T(x) = \sum_{1\leq i\leq n} h(Te_i, e_i)$, $\{e_i\}_{1\leq i\leq n}$ is an orthonormal basis of the tangent space $T_x(M)$ and h is the second fundamental form of ϕ.

Inspired by the work of Alias and Malacarne [2] and Grosjean [5], we study these T–S type upper bound estimates of the first nonzero eigenvalue of the L_T operator, and prove the following results.

THEOREM 1.1. *Let ϕ be an isometric immersion of a closed, connected Riemannian manifold (M^n, g) $(n \geq 2)$ into a complete Riemannian manifold $(\overline{M}^m, \overline{g})$ $(m > n)$ of sectional curvature bounded above by $c(c > 0)$, we assume that $\phi(M^n)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$, then we have*

$$\lambda_1^T \leq \frac{\int_M \operatorname{tr} T \, dv_g}{V}\left[c + \frac{1}{V}\frac{1}{\inf_M(\operatorname{tr} S)^2}\int_M |H_S|^2 \, dv_g\right], \quad \text{for all } S \in \mathcal{A} \tag{1.5}$$

where V is the volume of $\phi(M^n)$. If equality holds, then $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$. If $(\overline{M}^m, \overline{g})$ is a constant curvature space of sectional curvature c and $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$, then equality holds.

THEOREM 1.2. *Let ϕ be an isometric immersion of a closed, connected Riemannian manifold (M^n, g) $(n \geq 2)$ into a complete Riemannian manifold $(\overline{M}^m, \overline{g})$ $(m > n)$ of sectional curvature bounded above by c, if $c \leq 0$ we assume that $(\overline{M}^m, \overline{g})$ is simply connected, and if $c > 0$ we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$. Then*

$$\lambda_1^T \leq \sup_M\left[c \operatorname{tr} T + \sup_M\left(\frac{|H_T|}{\operatorname{tr} S}\right)|H_S|\right] \quad \text{for all } S \in \mathcal{A}. \tag{1.6}$$

If equality holds, then $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$. If $(\overline{M}^m, \overline{g})$ is a constant curvature space of sectional curvature c and $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$, then equality holds.

REMARK 1.3. Theorems 1.1 and 1.2 generalize Grosjean's [5] work. In fact, letting $S = T$ and I respectively in (1.6), we obtain (1.3) and (1.4) easily.

Applying Theorems 1.1 and 1.2 to the L_r operator, we derive the H_r, H_s type upper bound estimates of its first nonzero eigenvalue of hypersurfaces isometrically immersed in space forms, which extend the corresponding results in [1, 2, 8].

2. Preliminaries

Let ϕ be an isometric immersion of a compact, connected Riemannian manifold (M^n, g) $(n \geq 2)$ into a Riemannian manifold $(\overline{M}^m, \overline{g})$ $(m > n)$ of sectional curvature bounded above by c. If $c \leq 0$ we assume that $(\overline{M}^m, \overline{g})$ is simply connected and if $c > 0$ we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$, and denote by ∇^M, $\overline{\nabla}$ the gradients taken in (M^n, g), $(\overline{M}^m, \overline{g})$, respectively. Using the fact that $\operatorname{div}_M T = 0$, we know that L_T is a self-adjoint and elliptic second-order differential operator on M^n with an equivalent form $L_T u = \operatorname{trace}(T \operatorname{Hess} u)$,

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it has discrete eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots$ where

$$\lambda_1^{L_T} = \inf\left\{\frac{-\int_M f L_T(f)\, dv_g}{\int_M f^2\, dv_g},\ f \in C^\infty(M),\ \int_M f\, dv_g = 0\right\}$$

is the first nonzero eigenvalue, and

$$\lambda_i^{L_T} = \inf\left\{\frac{-\int_M f L_T(f)\, dv_g}{\int_M f^2\, dv_g},\ f \in C^\infty(M) \text{ and } \int_M f\, dv_g = 0,\ \int_M f f_j\, dv_g = 0,\right.$$
$$\left.\text{where } L_T f_j = -\lambda_j{}^{L_T} f_j,\ f_j \in C^\infty(M),\ j = 1, \ldots, i-1\right\}$$

is the ith nonzero eigenvalue ($i = 2, \ldots, n$).

Let $o \in \overline{M}^m$ and let $\exp_o$ be the exponential map at this point, let $\{x_A\}_{1\leq A\leq m}$ be the normal coordinates centered in o, with respect to some orthonormal basis in $T_o(\overline{M}^m)$, and $s(\cdot) = d(\cdot, o)$ be the distance function from o in $\overline{M}^m$; if $c > 0$ we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/2\sqrt{c}$. Let $S_c(s)$, $\theta_c(s)$ be functions defined by

$$S_c(s) = \begin{cases} \dfrac{1}{\sqrt{c}} \sin \sqrt{c}s, & c > 0 \\ s, & c = 0 \\ \dfrac{1}{\sqrt{-c}} \sinh \sqrt{-c}s, & c < 0, \end{cases}$$

and $\theta_c(s) = (d/d_s)S_c(s)$. Obviously

$$\theta_c^2(s) + cS_c^2(s) = 1 \quad \text{and} \quad \theta'(s) = -cS_c(s). \tag{2.1}$$

Define the generalized position vector field X of M^n in $\overline{M}^m$, with respect to o, by $X = S_c(s)\overline{\nabla}s$, it is easy to see that its coordinates in the normal local frame are $\{(S_c(s)/s)x_A\}_{1\leq A\leq m}$.

REMARK 2.1. In the case $c = 0$, $X = S_c(s)\overline{\nabla}s$ is just the position vector field in m-Euclidean space $\mathbb{E}^m$.

LEMMA 2.2. *For $x \in \overline{M}^m$, and in the case $c > 0$, $x \in B(o, \pi/2\sqrt{c})$. Then for any $u \in T_x(\overline{M}^m)$, we have*

$$\sum_{A=1}^m [\overline{g}_x(\overline{\nabla}x_A, u)]^2 \leq \frac{s^2}{S_c^2}\overline{g}_x(u, u) + \left(1 - \frac{s^2}{S_c^2}\right)[\overline{g}_x(u, \overline{\nabla}s)]^2 \tag{2.2}$$

and equality holds when $(\overline{M}^m, \overline{g})$ is a constant curvature space with sectional curvature c.

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PROOF. Let $\exp_o \widetilde{x} = x$, $\widetilde{x} \in T_o(\overline{M}^m)$, then the map $(d\exp_o)_{\widetilde{x}} : T_o(\overline{M}^m) \to T_x(\overline{M}^m)$ is a linear isomorphism. Let $\gamma : [0, s] \to \overline{M}^m$ be a normalized geodesic with $\gamma(0) = o$, $\gamma(s) = x$, $\gamma'(0) = \widetilde{x}/|\widetilde{x}|$, where $|\widetilde{x}| = s = [\sum_{A=1}^m x_A^2]^{1/2}$, let $v = u - \overline{g}_x(u, \overline{\nabla} s)\, \overline{\nabla} s \in T_x(\overline{M}^m)$, then v is orthogonal to $\overline{\nabla} s$.

We use the notation $\widetilde{v} = [(d\exp_o)_{\widetilde{x}}]^{-1} v \in T_o(\overline{M}^m)$; by the standard Jacobi field estimate [5, 9], we have $|\widetilde{v}| \le s|v|/S_c(s)$, and equality holds when $(\overline{M}^m, \overline{g})$ is a constant curvature space.

Using $\overline{g}(\overline{\nabla} x_A, v) = v(x_A) = [(d\exp_o)_{\widetilde{x}}^{-1} v](x_A)$, we obtain

$$\sum_{A=1}^m \overline{g}(\overline{\nabla} x_A, v)^2 = |(d\exp_o)_{\widetilde{x}}^{-1} v|^2 = |\widetilde{v}|^2 \le \frac{s^2}{S_c^2(s)} |v|^2. \tag{2.3}$$

On the other hand,

$$\begin{aligned}\sum_{A=1}^m \overline{g}_x(\overline{\nabla} x_A, u)\overline{g}_x(\overline{\nabla} x_A, \overline{\nabla} s) &= \overline{g}_x((d\exp_o)_{\widetilde{x}}^{-1} u, (d\exp_o)_{\widetilde{x}}^{-1}(\overline{\nabla} s)) \\ &= \overline{g}_x((d\exp_o)_{\widetilde{x}}^{-1}(\overline{g}_x(u, \overline{\nabla} s)\overline{\nabla} s), (d\exp_o)_{\widetilde{x}}^{-1}(\overline{\nabla} s)) \\ &= \overline{g}_x(u, \overline{\nabla} s)\overline{g}_x((d\exp_o)_{\widetilde{x}}^{-1}(\overline{\nabla} s), (d\exp_o)_{\widetilde{x}}^{-1}(\overline{\nabla} s)) \\ &= \overline{g}_x(u, \overline{\nabla} s)\end{aligned}$$

so

$$\sum_{A=1}^m [\overline{g}_x(\overline{\nabla} x_A, u - \overline{g}_x(u, \overline{\nabla} s)\overline{\nabla} s)]^2 = \sum_{A=1}^m \overline{g}_x(\overline{\nabla} x_A, u)^2 - \overline{g}_x(u, \overline{\nabla} s)^2.$$

By (2.3) and the above formula, we have

$$\begin{aligned}\sum_{A=1}^m [\overline{g}_x(\overline{\nabla} x_A, u)]^2 &\le \frac{s^2}{S_c^2(s)} |u - \overline{g}_x(u, \overline{\nabla} s)\overline{\nabla} s|^2 + [\overline{g}_x(u, \overline{\nabla} s)]^2 \\ &= \frac{s^2}{S_c^2(s)} (|u|^2 - [\overline{g}_x(u, \overline{\nabla} s)]^2) + [\overline{g}_x(u, \overline{\nabla} s)]^2 \\ &= \frac{s^2}{S_c^2(s)} |u|^2 + \left(1 - \frac{s^2}{S_c^2(s)}\right)[\overline{g}_x(u, \overline{\nabla} s)]^2. \qquad \square\end{aligned}$$

We can easily see that all of the inequalities above are in fact equalities if $(\overline{M}^m, \overline{g})$ is of constant sectional curvature c.

Let $X^\top$, $X^\perp$ be the tangential and the normal projection of X respectively on the tangent bundle and the normal bundle of M^n. Grosjean [5] proved an important inequality

$$\operatorname{div}_M(T X^\top) \ge (\operatorname{tr}(T))\theta_c(s) + \overline{g}(X, H_T) \tag{2.4}$$

where the equality holds if T is the identity and $(\overline{M}^m, \overline{g})$ has a constant sectional curvature equal to c.

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Now we improve and simplify the proof process of (2.4), and obtain the fact that the equality holds if $(\overline{M}^m, \overline{g})$ has a constant sectional curvature equal to c, that is, the condition that T is the identity can be omitted.

LEMMA 2.3. *For all symmetric divergence-free positive-definite* (1.1)*-tensors T on M^n, we have the inequality (2.4), and the equality holds if $(\overline{M}^m, \overline{g})$ has a constant sectional curvature equal to c.*

PROOF. For $x \in \overline{M}$, let $\{e_i\}_{1\le i\le n}$ be an arbitrary local orthonormal frame at x, by using the standard Jacobi field estimates (see [9, Lemma 2.9, p. 153]), we have for all vectors v orthogonal to $\overline{\nabla} s$ at x, the inequality

$$\overline{g}_x(\overline{\nabla}_v \overline{\nabla} s, v) \ge \frac{\theta_c}{S_c}|v|_x^2$$

and equality holds if $\overline{M}$ has a constant sectional curvature equal to c.

Similar to the method applied in the proof of Lemma 2.2, for any $u \in T_x(\overline{M})$, let $v = u - \overline{g}_x(u, \overline{\nabla} s)\overline{\nabla} s$, by direct calculation we can obtain

$$\overline{g}_x(\overline{\nabla}_u \overline{\nabla} s, u) \ge \frac{\theta_c}{S_c}\{|u|_x^2 - [\overline{g}_x(u, \overline{\nabla} s)]^2\}.$$

So it follows that

$$\begin{aligned}\sum_{i=1}^n \overline{g}_x(\overline{\nabla}_{\sqrt{T}e_i}\overline{\nabla} s, \sqrt{T}e_i) &\ge \frac{\theta_c}{S_c}\sum_{i=1}^n\{\overline{g}_x(\sqrt{T}e_i, \sqrt{T}e_i) - [\overline{g}_x(\sqrt{T}e_i, \overline{\nabla} s)]^2\}\\ &= \frac{\theta_c}{S_c}\sum_{i=1}^n\{\overline{g}_x(Te_i, e_i) - [\overline{g}_x(\sqrt{T}(\overline{\nabla} s)^T, e_i)]^2\}\\ &= \frac{\theta_c}{S_c}[\mathrm{tr}\, T - \overline{g}_x(\sqrt{T}(\overline{\nabla} s)^T, \sqrt{T}(\overline{\nabla} s)^T)]\\ &= \frac{\theta_c}{S_c}[\mathrm{tr}\, T - \overline{g}_x(T(\overline{\nabla} s)^T, (\overline{\nabla} s)^T)].\end{aligned}$$

By [5, Equations (14) and (15)],

$$\begin{aligned}\mathrm{div}_M\, TX^\top &= \overline{g}_x(X, H_T) + \theta_c\overline{g}_x(T(\overline{\nabla} s)^T, (\overline{\nabla} s)^T) + S_c\sum_{i=1}^n \overline{g}_x(\overline{\nabla}_{\sqrt{T}e_i}\overline{\nabla} s, \sqrt{T}e_i)\\ &\ge \overline{g}_x(X, H_T) + \theta_c(\mathrm{tr}\, T)\end{aligned}$$

and the equality holds if $(\overline{M}^m, \overline{g})$ has a constant sectional curvature equal to c. □

COROLLARY 2.4. *Let $f(s)$ be a positive and $C^k (k \ge 1)$ function, where $s(\cdot) = d(\cdot, o)$ is the distance function in $\overline{M}^m$, then*

$$\int_M \frac{f'(s)}{S_c(s)} g_x(TX^\top, X^\top)\, dv \le \int_M f(s)|H_T||X^\perp|\, dv - \int_M (\mathrm{tr}\, T)\theta_c f(s)\, dv$$

and equality holds if $(\overline{M}^m, \overline{g})$ is a constant curvature space.

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PROOF. Using

$$\operatorname{div}_M(f(s)TX^\top) = f(s)\operatorname{div}_M TX^\top + g_x(TX^\top, \nabla^M f(s)), \quad \nabla^M f(s) = \frac{f'(s)}{S_c(s)}X^\top,$$

the proof follows easily from the inequality (2.4), the divergence theorem, and the compactness of M^n. □

3. Proofs of the theorems

PROOF OF THEOREM 1.1. For any $p \in \phi(M^n) \subset \overline{M}^m$, let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of $T_p(\overline{M}^m)$, using the compactness of M^n and the assumption that $\phi(M^n)$ is contained in a convex ball B of radius $\pi/4\sqrt{c}$, by a standard argument [4, 5] we can parallel translate the frame $\{e_1, e_2, \ldots, e_m\}$ along every geodesic emanating from p and thereby obtain a differentiable orthonormal frame field $\{E_1, E_2, \ldots, E_m\}$ in a neighborhood of B. We define a vector field near B as

$$Y_q \triangleq \int_M \frac{S_c(s(q, p))}{s(q, p)} \exp_q^{-1}(p)\, dv_p \in T_q(\overline{M}^m),$$

which points towards the interior of B at the boundary ∂B. Thus, by the Brouwer fixed-point theorem and the continuity of $Y_q|_B$, there exists a point $o \in B$, such that $Y_o = 0$; that is

$$\int_M \frac{S_c(s)}{s} x_A\, dv_p = 0, \tag{3.1}$$

where $\{x_A\}$ is the normal coordinates with respect to o.

Since M^n is contained in a convex ball B of radius $\pi/4\sqrt{c}$, this means that M^n lies in a convex ball $\widetilde{B}$ of radius $\pi/2\sqrt{c}$ around o, with $c > 0$.

By

$$s = |X| = \left[\sum_{A=1}^m (x_A)^2\right]^{1/2}, \quad s\overline{\nabla}s = \sum_{A=1}^m x_A \overline{\nabla} x_A \tag{3.2}$$

and $\nabla^M S_c = (\overline{\nabla} S_c)^T = \theta_c \nabla^M s$, we have

$$\nabla^M\left(\frac{S_c}{s}x_A\right) = \frac{x_A}{s}\left(\theta_c - \frac{S_c}{s}\right)\nabla^M s + \frac{S_c}{s}\nabla^M x_A. \tag{3.3}$$

On the other hand,

$$X^\top = (S_c(s)\overline{\nabla}s)^T = S_c(s)\nabla^M s.$$

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Using Lemma 2.2, (3.1) and (3.3) we obtain

$$\begin{aligned}
\lambda_1^T \int_M |X|^2\, dv_g &= \lambda_1^T \int_M \sum_{A=1}^m \left(\frac{S_c}{s} x_A\right)^2 dv_g \\
&\le \sum_{A=1}^m \int_M g_x\left(T\nabla^M\left(\frac{S_c}{s}x_A\right), \nabla^M\left(\frac{S_c}{s}x_A\right)\right) dv_g \\
&= \sum_{A=1}^m \int_M \frac{x_A^2}{s^2}\left(\theta_c - \frac{S_c}{s}\right)^2 g_x(T\nabla^M s, \nabla^M s)\, dv_g \\
&\quad + 2\sum_{A=1}^m \int_M \frac{x_A}{s^2} S_c\left(\theta_c - \frac{S_c}{s}\right) g_x(T\nabla^M s, \nabla^M x_A)\, dv_g \\
&\quad + \sum_{A=1}^m \int_M \frac{S_c^2}{s^2} g_x(T\nabla^M x_A, \nabla^M x_A)\, dv_g \\
&= \int_M \left(\theta_c^2 - \frac{S_c^2}{s^2}\right) g_x(T\nabla^M s, \nabla^M s)\, dv_g \\
&\quad + \sum_{A=1}^m \int_M \frac{S_c^2}{s^2} g_x(T\nabla^M x_A, \nabla^M x_A)\, dv_g. \qquad (3.4)
\end{aligned}$$

Since T is a positive symmetric (1.1)-tensor, we can define a natural positive symmetric (1.1)-tensor $\sqrt{T}$ on M^n, such that $T = \sqrt{T}\sqrt{T}$ (see [5]), we have

$$\begin{aligned}
\frac{S_c^2}{s^2}\sum_{A=1}^m g_x(T\nabla^M x_A, \nabla^M x_A) &= \frac{S_c^2}{s^2}\sum_{A=1}^m g_x(\sqrt{T}\nabla^M x_A, \sqrt{T}\nabla^M x_A) \\
&= \frac{S_c^2}{s^2}\sum_{A=1}^m \sum_{i=1}^n [g_x(\sqrt{T}\nabla^M x_A, e_i)]^2 \\
&= \frac{S_c^2}{s^2}\sum_{A=1}^m \sum_{i=1}^n [\overline{g}_x(\overline{\nabla} x_A, \sqrt{T}e_i)]^2 \\
&\le \sum_{i=1}^n \overline{g}_x(\sqrt{T}e_i, \sqrt{T}e_i) \\
&\qquad + \sum_{i=1}^n \left(\frac{S_c^2}{s^2} - 1\right)[\overline{g}_x(\sqrt{T}e_i, \overline{\nabla} s)]^2 \\
&= \sum_{i=1}^n g_x(Te_i, e_i) + \left(\frac{S_c^2}{s^2} - 1\right)\sum_{i=1}^n [g_x(\sqrt{T}\nabla^M s, e_i)]^2 \\
&= \operatorname{tr} T + \left(\frac{S_c^2}{s^2} - 1\right) g_x(T\nabla^M s, \nabla^M s). \qquad (3.5)
\end{aligned}$$

Furthermore, from (3.5), we have

$$\lambda_1^T \int_M |X|^2 \, dv_g \le \int_M \operatorname{tr} T \, dv_g - c \int_M g_x(TX^\top, X^\top) \, dv_g. \tag{3.6}$$

Let $\overline{\theta}_c = 1/V \int_M \theta_c \, dv_g$, then we obtain

$$\int_M (\theta_c - \overline{\theta}_c) \, dv_g = 0.$$

Using $\nabla^M \theta_c = -cX^\top$, and the Rayleigh quotient with the test function $\theta_c - \overline{\theta}_c$, we obtain

$$\begin{aligned}\lambda_1^T &\int_M (\theta_c - \overline{\theta}_c)^2 dv_g \\ &\le \int_M g_x(T\nabla^M(\theta_c - \overline{\theta}_c), \nabla^M(\theta_c - \overline{\theta}_c)) \, dv_g = c^2 \int_M g_x(TX^\top, X^\top) \, dv_g.\end{aligned}$$

Thus,

$$\lambda_1^T \int_M \theta_c^2 \, dv_g \le c^2 \int_M g_x(TX^\top, X^\top) \, dv_g + \lambda_1^T \frac{1}{V} \left(\int_M \theta_c \, dv_g \right)^2. \tag{3.7}$$

By (3.6) and $\theta_c^2 + cS_c^2 = 1$, we have

$$\lambda_1^T V \le c \int_M \operatorname{tr} T \, dv_g + \frac{\lambda_1^T}{V} \left(\int_M \theta_c \, dv_g \right)^2. \tag{3.8}$$

Let $f(s) = \text{constant} > 0$ in Corollary 2.4, then we obtain

$$\int_M \theta_c \operatorname{tr} T \, dv_g \le \int_M |H_T||X^\perp| \, dv_g.$$

From (3.6), for any $S \in \mathcal{A}$, we have

$$\begin{aligned}\lambda_1^T \inf_M (\operatorname{tr} S)^2 \left(\int_M \theta_c \, dv_g \right)^2 &\le \lambda_1^T \left(\int_M |H_S||X^\perp| \, dv_g \right)^2 \\ &\le \lambda_1^T \int_M |H_S|^2 \, dv_g \int_M |X^\perp|^2 \, dv_g \\ &\le \int_M |H_S|^2 \, dv_g \int_M \operatorname{tr} T \, dv_g.\end{aligned} \tag{3.9}$$

Putting this into (3.8) gives the desired result (1.5), and the equality holds if $(\overline{M}^m, \overline{g})$ is a constant curvature space of sectional curvature c and $X^\top = S_c(s)\nabla^M s = 0$, that is, $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$ centered at o. □

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PROOF OF THEOREM 1.2. Similar to the proof in [5], let $f(s) = \theta_c(s)$ in Corollary 2.4, then we have

$$c \int_M g_x(TX^\top, X^\top)\, dv_g \geq \int_M \theta_c^2(\operatorname{tr} T)\, dv_g - \int_M |H_T||X^\perp|\theta_c\, dv_g.$$

By (3.6), for any $S \in \mathcal{A}$, we immediately obtain

$$\begin{aligned}
\lambda_1^T \int_M S_c^2\, dv_g &\leq \int_M \operatorname{tr} T\, dv_g - \int_M [\theta_c^2(\operatorname{tr} T) - |H_T|\theta_c|X^\perp|]\, dv_g \\
&= c \int_M S_c^2(\operatorname{tr} T)\, dv_g + \int_M \theta_c|H_T||X^\perp|\, dv_g \\
&\leq c \int_M S_c^2(\operatorname{tr} T)\, dv_g + \sup_M\left(\frac{|H_T|}{\operatorname{tr} S}\right) \int_M \theta_c \operatorname{tr} S|X^\perp|\, dv_g. \quad (3.10)
\end{aligned}$$

Taking $f(s) = S_c(s)$ in Corollary 2.4

$$\int_M (\operatorname{tr} T)\theta_c S_c\, dv_g \leq \int_M |H_T|S_c|X^\perp|\, dv_g - \int_M \frac{\theta_c(s)}{S_c(s)} g_x(TX^\top, X^\top)\, dv_g.$$

By the positive definiteness of T and (3.10),

$$\lambda_1^T \int_M S_c^2\, dv_g \leq c \int_M S_c^2(\operatorname{tr} T)\, dv_g + \sup_M\left(\frac{|H_T|}{\operatorname{tr} S}\right) \int_M |H_S|S_c^2\, dv_g,$$

that is,

$$\lambda_1^T \leq \sup_M\left[c \operatorname{tr} T + \sup_M\left(\frac{|H_T|}{\operatorname{tr} S}\right)|H_S|\right], \quad \text{for all } S \in \mathcal{A}.$$

So the equality holds if $(\overline{M}^m, \overline{g})$ is a constant curvature space of sectional curvature c and $X^\top = S_c(s)\nabla^M s = 0$; that is, $\phi(M)$ is contained in a geodesic hypersphere of $\overline{M}^m$ centered at o. □

4. Application to the operator L_r

Let M^n be a connected, orientable and compact manifold without boundary isometrically immersed in space form $\mathbb{R}^{n+1}(c)$, we now introduce the $(1, 1)$-type Newton tensor $T_r^{[1],[2]}$ by

$$\begin{aligned}
T_0 &= I, \\
T_1 &= \sigma_1 I - A, \\
&\vdots \\
T_r &= \sigma_r I - \sigma_{r-1}A + \cdots + (-1)^k \sigma_{r-k}A^k + \cdots + (-1)^r A^r,
\end{aligned}$$

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or inductively by $T_r = \sigma_r I - AI_{r-1}$ $(r = 1, \ldots, n)$, where A is the second fundamental tensor of the isometric immersion. Associated with each T_r, we have on M^n a second-order self-adjoint differential operator L_r defined by

$$L_r f = \operatorname{div}(T_r \nabla^M f),$$

where div_M and ∇^M are the divergence and the gradient of the metric g. On the other hand, by the Codazzi formula, as proved by Rosenberg [7]

$$\operatorname{div}_M T_r = \operatorname{trace}(\nabla^M T_r) = \sum_{i=1}^{n} (\nabla^M_{e_i} T_r(e_i)) = 0. \tag{4.1}$$

So the L_r operator can also be given by

$$L_r f = \operatorname{trace}(T_r \operatorname{Hess}(f)) \tag{4.2}$$

for each $r = 0, 1, \ldots, n$.

In the case $r = 0$, $L_0 = \Delta$ is naturally elliptic operator, but $L_r (r \geq 1)$ is not usually elliptic, the following Lemma 4.1 proves that L_r is elliptic under certain hypotheses.

LEMMA 4.1 (Barbosa and Colares [3]). *Let M^n be a connected, orientable and compact manifold without boundary isometrically immersed by ϕ into space form $\mathbb{R}^{n+1}(c)$, in the case $c > 0$ we assume that $\phi(M)$ is contained in an open hemisphere of the Euclidean sphere $\mathbb{R}^{n+1}(c)$. If $H_{r+1} > 0$, then for each $j (1 \leq j \leq r)$, we have j-mean curvature $H_j > 0$ and L_j is elliptic.*

Therefore, when $H_{r+1} > 0$, $T_r \in \mathcal{A}$, using the relations $\operatorname{tr} T = \operatorname{tr} T_r = c_r H_r$ and

$$|H_T| = \sum_{1 \leq i \leq n} B(Te_i, e_i) = \sum_{1 \leq i \leq n} g(AT(e_i), e_i) = \operatorname{tr}(AT) = c_r H_{r+1}$$

(see [3]). We immediately have the following results by applying Theorems 1.1 and 1.2 to T_r.

COROLLARY 4.2. *Let M^n be a connected, orientable and closed manifold isometrically immersed by ϕ into space form $\mathbb{R}^{n+1}(c)$ $(c > 0)$, and $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$, if there exists a non-negative integer r $(r = 0, 1, \ldots, n-1)$, such that $H_{r+1} > 0$, then*

$$\lambda_1^{L_r} \leq \frac{c_r \int_M H_r \, dv_g}{V} \left[c + \frac{1}{V} \frac{1}{\inf_M H_s^2} \int_M H_{s+1}^2 \, dv_g \right], \quad \text{for all } s = 0, 1, 2, \ldots, r$$

equality holds if and only if $\phi(M)$ is a geodesic hypersphere in $\mathbb{R}^{n+1}(c)$.

COROLLARY 4.3. *Let M^n be a connected, orientable and closed manifold isometrically immersed by ϕ into space form $\mathbb{R}^{n+1}(c)$; in the case $c > 0$ we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{c}$,* *if there exists a nonnegative integer* r $(r = 0, 1, \ldots, n-1)$*, such that* $H_{r+1} > 0$*, then we have*

$$\lambda_1^{L_r} \leq c_r \sup_M \left[cH_r + \sup_M \left(\frac{H_{r+1}}{H_s} \right) H_{s+1} \right] \quad \textit{for all } s = 0, 1, 2, \ldots, r$$

equality holds if and only if $\phi(M)$ *is a geodesic hypersphere in* $\mathbb{R}^{n+1}(c)$.

REMARK 4.4. When $\mathbb{R}^{n+1}(c) = \mathbb{S}^{n+1}(c)$ $(c > 0)$ or $\mathbb{H}^{n+1}(c)$ $(c < 0)$, we improved and obtained the H_r–H_s-type upper bounds of $\lambda_1^{L_r}$ (see [2]) and the corresponding result in [1, 8].

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