LATTICES WITH A GIVEN ABSTRACT GROUP OF AUTOMORPHISMS

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THE problem of finding a lattice[1] with a given abstract group of automorphisms has been solved by Garrett Birkhoff[2] who proved that for any group of order g there exists a distributive lattice with at most 2^{g^2+g} elements. That this number can be somewhat reduced by modifications of Birkhoff's original procedure has already been shown by the author[3]; it turns out, however, that it remains rather high for finite groups of relatively low order.

The purpose of the present paper is to show that a lattice with fewer elements can be found by a completely different method; in general, however, this lattice will not be distributive. Indeed we shall prove (see Theorem 2 below) that for any group of finite order g which can be generated by n of its elements a lattice can be found with at most $5(n + 2)\, g + 2$ elements. (To obtain an upper bound independent of n it suffices to recall that always $n \leqslant \dfrac{\log g}{\log 2}$.)

Since our method of finding a lattice with a given group of automorphisms is rather closely related to some theorems on graphs and their groups, we begin by recalling the definitions of these two notions.

By a *graph* we mean a finite set of elements called vertices some of which are joined by edges (or arcs), but so that two vertices are never joined by more than one edge; also the case of isolated vertices (which are not endpoints of any edge) will be excluded. If in a graph with q vertices $P_1, P_2, \ldots, P_q$ we define incidence-numbers I_{P_i,P_k} $(i \neq k)$ by

$$I_{P_i,P_k} = I_{P_k,P_i} = \begin{cases} 0, & \text{if } P_i \text{ and } P_k \text{ are not joined by an edge,} \\ 1, & \text{if } P_i \text{ and } P_k \text{ are joined by an edge,} \end{cases}$$

then the graph itself may also be characterized by the following quadratic form in q indeterminates $x_1, x_2, \ldots, x_q$:

$$F(x_1, x_2, \ldots, x_q) = \sum_{i<k} I_{P_i,P_k} x_i x_k.$$

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[1]For the definition of lattice and other basic notions of lattice theory, see Garrett Birkhoff's *Lattice Theory* (1st ed., New York, 1940).

[2]Garrett Birkhoff, *Sobre los grupos de automorfismos.* Revista de la Union Matematica Argentina, vol. 11 (1946), pp. 155-157.

[3]R. Frucht, *Sobre la construccion de sistemas parcialmente ordenados con grupo de automorfismos dado.* Revista de la Union Matematica Argentina, vol. 13 (1948), pp. 12-18. See also: *On the construction of partially ordered systems with a given group of automorphisms.* Amer. J. Math., vol. 72 (1950), pp. 195-199.

The *group* (of automorphisms) of the graph then consists of those permutations of $x_1, x_2, \ldots, x_q$ which leave the quadratic form $F(x_1, x_2, \ldots, x_q)$ unaltered; it is obvious that the corresponding permutations of the vertices $P_1, P_2, \ldots, P_q$ of the graph represent all the possible mappings of the graph into itself which preserve incidence-relations.

The connexion between lattices and graphs is given by the following general theorem.

THEOREM 1. *Given any graph (in the sense defined above) with q vertices and p edges, there is always a lattice with $p + q + 2$ elements such that the group of automorphisms of the lattice is simply isomorphic to that of the graph.*

Proof. Let $P_1, P_2, \ldots, P_q$ be the vertices of the given graph G, and let $a_1, a_2, \ldots, a_p$, be its edges. A partially ordered system S with $p + q + 2$ elements $I, A_1, A_2, \ldots, A_p, B_1, B_2, \ldots, B_q, O$ may then be defined by the following order-relations:

(1) $\quad I > A_i > O$ (for $i = 1, 2, \ldots, p$),

(2) $\quad I > B_j > O$ (for $j = 1, 2, \ldots, q$),

(3) $\quad A_i > B_j$ if, and only if, the vertex P_j is in G one of the endpoints of the edge[4] a_i.

This system S is a lattice, as it is evident that any two of its elements have always a greatest lower bound or meet (symbol: $\cap$) and a lowest upper bound or join (symbol: $\cup$); e.g. it is obvious that

$$A_i \cup A_k = I \text{ for any } i \neq k,$$

and that

$$A_i \cap A_k = \begin{cases} O, \text{ if in } G \text{ the edges } a_i \text{ and } a_k \text{ have no common endpoint,} \\ B_j, \text{ if in } G \text{ the edges } a_i \text{ and } a_k \text{ have the common endpoint } P_j. \end{cases}$$

(By our rather restricted definition of "graph" we have excluded the possibility of two edges with both endpoints in common.)

Finally it is easy to recognize that the groups of automorphisms of G and S are simply isomorphic, since any automorphism of G obviously induces one of S, and conversely.

That the lattice S is in general not distributive (nor even modular) may be shown by the following example. As graph G take that characterized by the quadratic form $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$, i.e., a quadrilateral with the edges

$$a_1 = P_1P_2, \ a_2 = P_2P_3, \ a_3 = P_3P_4, \ a_4 = P_1P_4.$$

The group of automorphisms of G is of course simply isomorphic to the octic group (= dihedral group of order 8). In the corresponding lattice S we have[5]

[4]In other words, S is the "cell-space" $P(G)$ of G (see *Lattice Theory*, 1st ed., p. 15) to which an O has been added in order to obtain a lattice.

[5]The "Hasse diagram" of this lattice may be obtained from the right-hand half of Fig 2, p. 15, of *Lattice Theory* by adding an O and joining it with B_1, B_2, B_3 and B_4.

$$I > A_i > B_i > O \qquad (i = 1, 2, 3, 4),$$

and also

$$A_1 > B_2,\ A_2 > B_3,\ A_3 > B_4,\ A_4 > B_1.$$

It is easily seen that this lattice S is not modular (hence not distributive). Indeed any modular lattice must satisfy the following condition (called (ξ') by Birkhoff[6]): "In a modular lattice, if X and Y cover[7] A, and $X \neq Y$, then $X \cup Y$ covers X and Y"; but the elements $X = B_1$ and $Y = B_3$ of S do not fulfil this condition.

We are now going to prove the following

THEOREM 2. *If $\mathfrak{G}$ is any abstract group of finite order g which can be generated by n of its elements, it is possible to find a lattice with at most $5(n + 2)\, g + 2$ elements whose group of automorphisms is simply isomorphic to $\mathfrak{G}$.*

Proof. It has been shown elsewhere[8] how to obtain a graph with at most $q = 2(n + 2)g$ vertices whose group of automorphisms is simply isomorphic to a given abstract group $\mathfrak{G}$; and since each of the vertices of that graph is of degree 3 (i.e., an endpoint of 3 edges), we have

$$p = 3q/2 = 3(n + 2)g.$$

With these values of p and q, Theorem 2 follows immediately from Theorem 1.

Of course it should be remarked that for special groups where a graph with fewer vertices and edges than the one used here is known, Theorem 1 will furnish a lattice with fewer elements than Theorem 2. E.g., for the octic group ($g = 8$, $n = 2$) Theorem 2 would give a lattice with 162 elements, but we know already that there is one with only 10 elements (see the example after the proof of Theorem 1).

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[6] *Lattice Theory*, 1st ed., p. 34, Corollary 3 to Theorem 3.1.

[7] By "X covers A" it is meant that $X > A$, while no Z of S satisfies $X > Z > A$.

[8] R. Frucht, *Graphs of degree 3 with a given abstract group.* Can. J. Math., vol. 1 (1949), pp. 365-378.