SOME INTERPOLATORY PROPERTIES OF LAGUERRE POLYNOMIALS*

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1. Introduction. In 1955, J. Suranyi and P. Turán [8] introduced the nomenclature (0, 2)-interpolation for the problem of finding polynomials of degree $\leq 2n-1$ whose values and second derivatives are prescribed in certain given nodes. In a series of papers ([2], [3], [8]) Professor Turán and his associates discussed the problems of existence, uniqueness, explicit representation and convergence of such interpolatory polynomials when the nodes are the zeros of $(1-x^2)P'_{n-1}(x)$, $P_{n-1}(x)$ being the Legendre polynomial of degree $n-1$.

Later Mathur and Sharma [5] have considered the zeros of the Hermite polynomials as nodes for (0, 2) and (0, 1, 3) interpolation. Kis [4] and Sharma and Varma [7] have considered (0, 2) and (0, M)-interpolation respectively by trigonometric polynomials.

If n is odd and if the nodes are only known to be symmetric about the origin Suranyi and Turán [8] have shown that the interpolatory polynomials either do not exist or are not unique. However in the literature on (0, 2)-interpolation (where the values, first and third derivatives are prescribed) and also in the other extensions of lacunary interpolation, the zeros of Laguerre polynomials do not appear to have been considered as possible nodes for such an interpolation problem.

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For a complete history of the problem and for a detailed bibliography, we refer to a recent paper by J. Balázs [1].

The object of this note is to consider the problems of existence, uniqueness and explicit representation for polynomials $R_n(x)$ of degree $\leq 2n-1$ for $(0,2)$ interpolation when the nodes are the zeros of $x\, L_n'(x)$:

$$(1.1) \qquad 0 = x_0 < x_1 < \ldots < x_{n-1} < \infty$$

and where $L_n(x)$ denotes the Laguerre polynomial of degree n. We shall prove the following:

THEOREM. If α_ν and β_ν $(\nu = 0, 1, \ldots, n-1)$ are any preassigned numbers then there exists a unique polynomial $R_n(x)$ of degree $\leq 2n-1$ satisfying

$$(1.2) \qquad R_n(x_\nu) = \alpha_\nu, \; R_n''(x_\nu) = \beta_\nu \quad (\nu = 0, 1, \ldots, n-1)$$

where x_ν are given by (1.1). For the explicit form of this polynomial we have

$$(1.3) \qquad R_n(x) = \sum_{\nu=0}^{n-1} \alpha_\nu U_\nu(x) + \sum_{\nu=0}^{n-1} \beta_\nu V_\nu(x)$$

where U_ν and $V_\nu(x)$ are given by (5.14), (5.15), (5.3) and 5.4.

If we consider $(0,2)$-interpolation of the above type on the zeros of $L_n^{(\alpha)}(x)$, $\alpha > -1$, then it is possible to prove the existence and uniqueness of the interpolatory polynomials for $-1 < \alpha < 1$, but explicit forms of the interpolatory polynomials become more complicated. So we restrict ourselves to the case treated in the above theorem. It is interesting to observe that the interpolatory polynomials exist for both n odd and even. We shall return to convergence problems later.

2. Preliminaries. In this section we shall state certain well-known formulae which we shall use later on.

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(2.1) $$x L_n''(x) + (1-x) L_n'(x) + n L_n(x) = 0$$

is the differential equation satisfied by $L_n(x)$.

It is easily verified that

(2.2) $$L_n'(0) = -n$$

$$L_n''(0) = \frac{n(n-1)}{2}$$

$$L_n'''(0) = -\frac{n(n-1)(n-2)}{6}$$

and

(2.3) $$x L_n''(x_j) = -n L_n(x_j)$$

$$x_j L_n'''(x_j) = (x_j - 2) L_n''(x_j) = -n(1 - \frac{2}{x_j}) L_n(x_j)$$

$$x_j L_n^{iv}(x_j) = -\frac{n}{x_j} (x_j + \frac{6}{x_j} - n - 3) L_n(x_j).$$

We shall denote by $\ell_\nu(x)$, the fundamental polynomials of Lagrange interpolation based on x_ν 's $(\nu = 0, 1, 2, \ldots, n-1)$, the zeros of $x L_n'(x)$ i.e.

(2.4) $$\ell_\nu(x) = \frac{x L_n'(x)}{x_\nu (x - x_\nu) L_n''(x_\nu)}, \quad \nu \geq 1$$

$$\ell_o(x) = -\frac{L_n'(x)}{n}$$

from which it can be easily seen that

(2.5) $$\ell_\nu(x_\nu) = 1$$

$$\ell_\nu(x_j) = 0$$

(2.6) $$\ell'_\nu(x_\nu) = \frac{1}{2}$$

$$\ell'_\nu(x_j) = \frac{x_j L''_n(x_j)}{x_\nu (x_j - x_\nu) L''_n(x_\nu)}$$

$$\ell''_\nu(x_\nu) = \frac{1}{3} - \frac{n}{3x_\nu}$$

(2.7) $$\ell'_o(0) = -\frac{(n-1)}{2}$$

$$\ell''_o(0) = \frac{(n-1)(n-2)}{2}$$

$$\ell'_\nu(0) = \frac{1}{x_\nu L_n(x_\nu)} .$$

We also require the following well-known relations

(2.8) $$n L_n(x) = (-x+2n-1) L_{n-1}(x) - (n-1) L_{n-2}(x)$$

and

(2.9) $$L_n(x) = \sum_{\nu=o}^{n} \binom{n}{n-\nu} \frac{(-1)^\nu}{\nu!} x^\nu .$$

3. We shall now prove the following lemma which we shall use in the sequel.

LEMMA 3.1. We have

(3.1) $$\int_0^{-\infty} e^{t/2} L'_n(t)dt = 3^n - 1 .$$

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Proof. We have

$$\int_0^{-\infty} e^{t/2} L_n'(t)dt = \int_0^{\infty} e^{-t/2} L_n'(-t)dt$$

$$= \sum_{r=0}^{n-1} \frac{n!}{r!(r+1)!(n-1-r)!} \int_0^{\infty} e^{-t/2} t^r dt$$

(making use of (2.9)). Now substituting $t/2 = u$ we have

$$\int_0^{-\infty} e^{t/2} L_n'(t)dt = \sum_{r=0}^{n-1} \frac{n!\, 2^{r+1}}{r!(r+1)!(n-1-r)!} \int_0^{\infty} e^{-u} u^r du$$

$$= 2 \sum_{r=0}^{n-1} \frac{n!\, 2^r}{(r+1)!(n-1-r)!}$$

$$= \sum_{r=1}^{n} \frac{n!\, 2^r}{r!(n-r)!}$$

$$= \left[\sum_{r=0}^{n} \binom{n}{r} 2^r - 1 \right]$$

$$= (3^n - 1),$$

from which the Lemma follows.

4. Proof of Theorem. The existence and uniqueness assertion will be proved if we show that the polynomial $g_{2n-1}(x)$ of degree $\leq 2n-1$ satisfying the conditions

(4.1) $g_{2n-1}(x_\nu) = 0, \quad g''_{2n-1}(x_\nu) = 0, \quad \nu = 0, 1, \ldots, n-1$

is identically zero. To prove this we write

(4.2) $$g_{2n-1}(x) = x L_n'(x) q_{n-1}(x)$$

which satisfies the first condition in (4.1) with arbitrary polynomial $q_{n-1}(x)$. In order that the second condition in (4.1) is also satisfied, we have on differentiating (4.2) twice and using (2.1):

$$2q'_{n-1}(x_\nu) + q_{n-1}(x_\nu) = 0 , \quad \nu = 1, 2, \ldots, n-1$$

whence

$$(4.3) \qquad 2q'_{n-1}(x) + q_{n-1}(x) = c L'_n(x)$$

where c is a numerical constant. Setting

$$(4.4) \qquad q_{n-1}(x) = \sum_{k=0}^{n-1} a_k L_k(x)$$

and using the recurrence relation ([6] p. 299)

$$(4.5) \qquad L_n(x) = L'_n(x) - L'_{n+1}(x) ,$$

we have from (4.3):

$$\sum_{k=1}^{n} (3a_k - a_{k-1}) L'_k(x) - a_{n-1} L'_n(x) = c L'_n(x) .$$

Equating coefficients of various powers of $L'_k(x)$ in the above we get

$$(4.6) \qquad 3a_k - a_{k-1} = 0 , \quad k = 1, 2, \ldots, n-1$$

$$- a_{n-1} = c .$$

From (4.6) we have

$$(4.7) \qquad a_k = \frac{1}{3^k} a_o , \quad k = 1, 2, \ldots, n-1$$

$$a_{n-1} = - c .$$

Further we should also have $g''_{2n-1}(0) = 0$ which requires

$$2L'_n(0) q'_{n-1}(0) + 2 L''_n(0) q_{n-1}(0) = 0 .$$

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Now using (2.2), (2.3), (4.4) and (4.5) we have

$$\sum_{k=1}^{n-1} [(n-1)\, a_{k-1} - (n-3)\, a_k]\, k + n(n-1)\, a_{n-1} = 0$$

From this on using the second part of (4.6) we get

$$\left[\frac{n-1}{1} + \frac{n+1}{3} + \frac{n+3}{3^2} + \dots + \frac{3n-3}{3^{n-1}}\right] a_o = 0.$$

i.e. $a_o = 0$. Hence from (4.6)

$$a_o = a_1 = a_2 = \dots = a_{n-1} = 0.$$

This shows that $g_{2n-1}(x) \equiv 0$.

5. Explicit form of $U_\nu(x)$ and $V_\nu(x)$:

In this section we determine the explicit form of the polynomials $U_\nu(x)$ and $V_\nu(x)$ in (1.3), which satisfy the following conditions:

$$(5.1) \qquad U_\nu(x_j) = \begin{matrix} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{matrix}, \quad U''_\nu(x_j) = 0$$

$$(5.2) \qquad V_\nu(x_j) = 0, \quad V''_\nu(x_j) = \begin{matrix} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{matrix} \quad j = 0, 1, \dots, n-1$$

The polynomials $V_\nu(x)$.

We have

$$(5.3) \qquad V_o(x) = \frac{-\omega(x)}{n^2(3^n-3)} \left[\int_0^x e^{t/2} L'_n(t)dt - (3^n-1) \right]$$

where $\omega(x) = x L'_n(x)\, e^{-x/2}$ and for $1 \leq \nu \leq n-1$

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$$(5.4) \qquad V_\nu(x) = \omega(x)\,[c_1 \int_0^x L_n'(t)\, e^{t/2}\, dt + \frac{1}{2x_\nu L_n''(x_\nu)} \int_0^x \ell_\nu(t) e^{t/2}\, dt + c_2]$$

where

$$(5.5) \qquad c_1 = -\frac{1}{2(3^n-3)x_\nu L_n''(x_\nu)} \int_0^{-\infty} \ell_\nu(t)\, e^{t/2}\, dt$$

$$(5.6) \qquad c_2 = -2\, c_1 .$$

First we show that (5.3) and (5.4) satisfy (5.2). For this we start with constants c_1 and c_2 to be chosen suitably. Since

$$\omega(x) = e^{-x/2}\, x\, L_n'(x)$$

therefore we have

$$(5.7) \qquad \omega(x_\nu) = \omega''(x_\nu) = 0$$

and

$$(5.8) \qquad \omega'(x_\nu) = -n\, e^{-x_\nu/2}\, L_n(x_\nu) \qquad \nu = 1, 2, \ldots, n-1 .$$

Also

$$\omega(0) = 0$$

$$\omega'(0) = -n$$

$$\omega''(0) = n^2$$

Due to (5.7) and (5.9) the first condition of (5.2) is satisfied. In order to satisfy the second condition of (5.2), we have on account of (5.7) $V_\nu'(x_j) = 0$, $j \neq \nu$ at any choice of c_1 and c_2.

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Similarly

$$(5.10) \qquad V''_\nu(x_\nu) = \frac{\omega'(x_\nu)\, e^{x_\nu/2}}{x_\nu L''_n(x_\nu)} .$$

Hence from (5.10), using (2.3) and (5.8) we have

$$V''_\nu(x_\nu) = \frac{-n L_n(x_\nu)}{x_\nu L''_n(x_\nu)} = 1 .$$

On account of (2.2) and (5.9) we have

$$(5.11) \qquad V''_\nu(0) = c_2\, \omega''(0) + 2c_1\, \omega'(0)\, L'_n(0) .$$

Using (2.2) and (5.9) we have for $V''_\nu(0) = 0$

$$(5.12) \qquad c_2 + 2\, c_1 = 0 .$$

In order that $V_\nu(x)$ is a polynomial we must have with the help of Lemma 3.1

$$(5.13) \quad (3^n-1)\, c_1 + \frac{1}{2x_\nu L''_n(x_\nu)} \int_0^{-\infty} \ell_\nu(t)\, e^{t/2}\, dt + c_2 = 0 .$$

Thus (5.12) and (5.9) are simultaneously satisfied if we choose c_1 and c_2 as given by (5.5) and (5.6).

Similarly we can verify the conditions (5.2) for $V_0(x)$ given by (5.3). We omit the details.

The polynomials $U_\nu(x)$.

$$(5.14) \quad U_0(x) = \frac{1}{n^2} L'^2(x) + \omega(x)\, [c_3 \int_0^x e^{t/2} L'_n(t) dt$$

$$+ \frac{e^{x/2}}{4n^2} \{2L''_n(x) - 3L'_n(x)\} + c_4]$$

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where

$$\omega(x) = x L_n'(x) e^{-x/2}$$

$$c_3 = \frac{5n^2 - 6n + 1}{4n^2(3^n - 3)}$$

$$c_4 = -\frac{3^n - 1}{3^n - 3} \cdot \frac{5n^2 - 6n + 1}{4n^2}$$

and for $1 \leq \nu \leq n-1$

(5.15) $$U_\nu(x) = \ell_\nu^2(x) + \omega(x) \Big[c_5 \int_0^x L_n'(t) e^{t/2} dt + c_6 \int_0^x \frac{(1 + x_\nu - t)\ell_\nu(t) - 2\ell_\nu'(t)}{t - x_\nu} e^{t/2} dt + c_7 \Big]$$

where

(5.16) $$c_5 = \frac{1}{2x_\nu(3^n - 3) L_n''(x_\nu)} \int_0^{-\infty} \frac{(t - x_\nu - 1)\ell_\nu(t) + 2\ell_\nu'(t)}{t - x_\nu} e^{t/2} dt$$

(5.17) $$c_6 = \frac{1}{2x_\nu L_n''(x_\nu)}$$

and

(5.18) $$c_7 = -2 c_5 .$$

We shall only give the details of the verification that $U_\nu(x)$ given by (5.15) satisfies (5.1). For $U_o(x)$ the verification can be done similarly.

We see from (5.15) that on account of (2.5), (5.7) and (5.9) the first condition of (5.1) is satisfied. Now from (5.15)

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using (2.6), (5.7) and (5.8) we have

$$U''_\nu(x_j) = 2\ell'^2_\nu(x_j) - \frac{2x_j^2 \, L''^2_n(x_j)}{x_\nu^2(x_j - x_\nu)^2 \, L''_n(x_\nu)} = 0$$

if c_6 is chosen as in (5.17).

Now from (2.6), (5.8), (5.17) and (2.1) we have

$$U''_\nu(x_\nu) = \left(\frac{7}{6} - \frac{2n}{3x_\nu}\right) + 2\, c_6\, \omega'(x_\nu)\, [\ell'_\nu(x_\nu) - \ell_\nu(x_\nu) - 2\ell''_\nu(x_\nu)]\, e^{x_\nu/2} = 0$$

if c_6 is given by (5.17). It remains to show that $U''_\nu(0) = 0$ and the expression of $U_\nu(x)$ is a polynomial of degree $2n-1$. To show this we get from (5.15), (5.9), (2.7) and (5.17)

$$U''_\nu(0) = \frac{2}{x_\nu^2 L_n^2(x_\nu)} + n^2 c_7 + 2\omega'(0)\left[c_5 L'_n(0) + \frac{2c_6\, \ell'_\nu(0)}{x_\nu}\right]$$

$$= \frac{2}{x_\nu^2 L_n^2(x_\nu)} + n^2 c_7 + 2n^2 c_5 - \frac{2}{x_\nu^2 L_n^2(x_\nu)}$$

$$= 2n^2 c_5 + n^2 c_7$$

$$= 0$$

if

$$(5.19) \qquad 2\, c_5 + c_7 = 0\, .$$

In order that $U_\nu(x)$ is a polynomial of degree $2n-1$ we must have from (5.15) and Lemma 3.1,

$$c_5(3^n - 3) - \frac{1}{2x_\nu L''_n(x_\nu)} \int_0^{-\infty} \frac{(t - x_\nu - 1)\ell_\nu(t) + 2\ell'_\nu(t)}{t - x_\nu}\, e^{t/2}\, dt = 0$$

or

$$(5.20)\quad (3^n-3)\,c_5 = \frac{1}{2x_\nu L_n''(x_\nu)} \int_0^{-\infty} \frac{(t-x_\nu-1)\ell_\nu(t)+2\ell_\nu'(t)}{t-x_\nu}\, e^{t/2}\, dt$$

(5.19) and (5.20) are simultaneously satisfied by the values of c_5 and c_7 given in (5.16) and (5.18) respectively.

Results analogous to those of Balázs [1] under the conditions used by him can be obtained in this case also. In other words we could replace (1.2) with

$$(1.2a)\qquad R_n(x_\nu) = \alpha_\nu, \quad \left(e^{-\frac{x}{2}} x^{\frac{1+\alpha}{2}} R_n(x)\right)''_{x_\nu} = \beta_\nu$$

However we shall not pursue the subject here.

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