

PARTITIONS OF THE SET OF NONNEGATIVE INTEGERS WITH THE SAME REPRESENTATION FUNCTIONS

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Abstract

Let \mathbb{N} be the set of all nonnegative integers. For a given set $S \subset \mathbb{N}$ the representation function $R_S(n)$ counts the number of solutions of the equation $n = s + s'$ with $s < s'$ and $s, s' \in S$. We obtain some results on a problem of Chen and Lev [‘Integer sets with identical representation functions’, *Integers* **16** (2016), Article ID A36, 4 pages] about sets A and B such that $A \cup B = \mathbb{N}$, $A \cap B = r + m\mathbb{N}$ and whose representation functions coincide.

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1. Introduction

Let $S \subset \mathbb{N}$ be a set of nonnegative integers. The representation function $R_S(n)$ is the number of solutions of the equation $n = s + s'$ with $s < s'$ and $s, s' \in S$. For a nonnegative integer a , the sumset $a + S = \{a + s : s \in S\}$.

Sárközy asked whether there exist two sets A and B of positive integers with infinite symmetric difference, that is, $|(A \cup B) \setminus (A \cap B)| = \infty$, having $R_A(n) = R_B(n)$ for all sufficiently large n . Dombi [2] used the Thue–Morse sequence to construct such a partition of \mathbb{N} into two sets A and B such that $R_A(n) = R_B(n)$ for all $n \in \mathbb{N}$. Related results can be found in [5–7].

In 2012, Yu and Tang [9] considered the representation functions of two sets A and B which partition the natural numbers and whose intersection is nonempty.

THEOREM A [9, Theorem 1]. *If $\mathbb{N} = A \cup B$ and $A \cap B = \{4k : k \in \mathbb{N}\}$, then $R_A(n) = R_B(n)$ cannot hold for all sufficiently large integers n .*

In 2016, the second author of this paper extended Theorem A as follows.

THEOREM B [8, Theorem 1]. *Let $k \geq 2$ be an integer. If the sets A and B satisfy $\mathbb{N} = A \cup B$ and $A \cap B = \{km : m \in \mathbb{N}\}$, then $R_A(n) = R_B(n)$ cannot hold for all sufficiently large integers n .*

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In 2016, Chen and Lev [1] obtained the following result.

THEOREM C [1, Theorem 1]. *Let l be a positive integer. There exist sets C and D such that $\mathbb{N} = C \cup D$, $C \cap D = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}$ and $R_C(n) = R_D(n)$ for every positive integer n .*

In their paper [1], Chen and Lev posed the following two problems. Let \mathcal{A} denote the Thue–Morse sequence, that is, the set of nonnegative integers whose binary representations contain an even number of 1 binary digits, and set $\mathcal{B} = \mathbb{N} \setminus \mathcal{A}$. Put $A_l = \mathcal{A} \cap [0, 2^l - 1]$ and $B_l = \mathcal{B} \cap [0, 2^l - 1]$.

PROBLEM 1.1. Given that $R_A(n) = R_B(n)$, $A \cup B = \mathbb{N}$ and $A \cap B = r + m\mathbb{N}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$?

PROBLEM 1.2. Given that $R_A(n) = R_B(n)$, $A \cup B = [0, m]$ and $A \cap B = \{r\}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 2$, $A = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $B = B_{2l} \cup (2^{2l} - 1 + A_{2l})$?

Problem 1.2 is the finite version of Problem 1.1. Recently, Kiss and Sándor [3] solved Problem 1.2 affirmatively.

THEOREM D [3, Theorem 7]. *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$, $|C \cap D| = 1$ and $0 \in C$. Then $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a natural number l such that $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$.*

In [4], Kiss and Sándor give further conjectures and extensions of these results and partially describe the structure of the sets which have coinciding representation functions.

In this paper, we focus on Problem 1.1 and obtain the following results.

THEOREM 1.3. *Given $m > r > 0$, there exists at most one pair of sets (A, B) such that $A \cup B = \mathbb{N}$ and $A \cap B = r + m\mathbb{N}$ and $R_A(n) = R_B(n)$ for every positive integer n .*

THEOREM 1.4. *Let $m > r > 0$ be integers. Let A and B be sets of nonnegative integers such that $A \cup B = \mathbb{N}$ and $A \cap B = \{r + mk : k \in \mathbb{N}\}$. If $R_A(n) = R_B(n)$ for every positive integer n , then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$.*

Throughout the paper, let $f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x]$ and, for $m \leq n$, define

$$(f(x))_m = a_0 + a_1x + \dots + a_mx^m.$$

The characteristic function, $\chi_S(n)$, of the set S is defined by

$$\chi_S(n) = \begin{cases} 1 & \text{if } n \in S, \\ 0 & \text{if } n \notin S. \end{cases}$$

2. Lemmas

LEMMA 2.1 (See [3, Claim 3]). *Let M be a positive integer and $u = \lceil \log_2 M \rceil - 1$. If the integers $M - 1, M - 2, M - 4, M - 8, \dots, M - 2^u$ are all contained in the set \mathcal{A} , then $M = 2^{u+1} - 1$.*

LEMMA 2.2 (See [3, Claim 4]). *Let M be a positive integer and $u = \lceil \log_2 M \rceil - 1$. If the integers $M - 1, M - 2, M - 4, M - 8, \dots, M - 2^u$ are all contained in the set \mathcal{B} , then $M = 2^{u+1} - 1$.*

LEMMA 2.3 (See [3, Claim 1]). *Let $0 < r_1 < \dots < r_s \leq m$ be integers. Then there exists at most one pair of sets (C, D) such that $C \cup D = [0, m]$, $0 \in C$, $C \cap D = \{r_1, \dots, r_s\}$ and $R_C(k) = R_D(k)$ for every $k \leq m$.*

The following lemma is the key of the proof of Theorem 1.4. In fact, the lemma is a revised version of [3, Theorem 7].

LEMMA 2.4. *Let $m \geq 2r > 0$ be integers. Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$, $C \cap D = \{r\}$ and $0 \in C$. If $R_C(n) = R_D(n)$ for every n with $0 \leq n \leq m$, then there exists a positive integer l such that $r = 2^{2l} - 1$.*

PROOF. Let

$$p_C(x) = \sum_{i=0}^m \chi_C(i)x^i, \quad p_D(x) = \sum_{i=0}^m \chi_D(i)x^i. \tag{2.1}$$

Then

$$\frac{1}{2}(p_C(x))^2 - \frac{1}{2}p_C(x^2) = \sum_{n=0}^{\infty} R_C(n)x^n, \tag{2.2}$$

$$\frac{1}{2}(p_D(x))^2 - \frac{1}{2}p_D(x^2) = \sum_{n=0}^{\infty} R_D(n)x^n. \tag{2.3}$$

Since $R_C(n) = R_D(n)$ for $0 \leq n \leq m$,

$$\left(\sum_{n=0}^{\infty} R_C(n)x^n \right)_m = \left(\sum_{n=0}^{\infty} R_D(n)x^n \right)_m. \tag{2.4}$$

Since $C \cup D = [0, m]$ and $C \cap D = \{r\}$,

$$p_D(x) = \frac{1 - x^{m+1}}{1 - x} - p_C(x) + x^r. \tag{2.5}$$

By (2.1)–(2.5),

$$\begin{aligned} \left(\frac{1}{2}(p_C(x))^2 - \frac{1}{2}p_C(x^2) \right)_m &= \left(\sum_{n=0}^{\infty} R_C(n)x^n \right)_m = \left(\sum_{n=0}^{\infty} R_D(n)x^n \right)_m \\ &= \left(\frac{1}{2}(p_D(x))^2 - \frac{1}{2}p_D(x^2) \right)_m, \end{aligned}$$

that is,

$$((p_C(x))^2 - p_C(x^2))_m = \left(\left(\frac{1 - x^{m+1}}{1 - x} - p_C(x) + x^r \right)^2 - \left(\frac{1 - x^{2m+2}}{1 - x^2} - p_C(x^2) + x^{2r} \right) \right)_m.$$

Thus,

$$(2p_C(x^2))_m = \left(\frac{1 - x^{2m+2}}{1 - x^2} + 2p_C(x) \frac{1 - x^{m+1}}{1 - x} - \left(\frac{1 - x^{m+1}}{1 - x} \right)^2 - 2x^r \frac{1 - x^{m+1}}{1 - x} + 2x^r p_C(x) \right)_m. \tag{2.6}$$

Next, we shall prove that there exists a positive integer l such that $r = 2^{2^l} - 1$. The remainder of the proof is the same as the proof of [3, Theorem 7]. We give the details for completeness.

First, we shall prove r is odd. Suppose that r is even and let $r \leq k \leq 2r \leq m$ be also even. It is easy to see from (2.1) and the coefficient of x^k in (2.6) that

$$2\chi_C\left(\frac{k}{2}\right) = 1 + 2 \sum_{0 \leq i \leq k} \chi_C(i) - (k + 1) - 2 + 2\chi_C(k - r). \tag{2.7}$$

If $k + 1 \leq 2r \leq m$, then from the coefficient of x^{k+1} in (2.6),

$$0 = 2 \sum_{0 \leq i \leq k+1} \chi_C(i) - (k + 2) - 2 + 2\chi_C(k + 1 - r). \tag{2.8}$$

By (2.7) and (2.8),

$$\chi_C\left(\frac{k}{2}\right) = 1 - \chi_C(k + 1) + \chi_C(k - r) - \chi_C(k + 1 - r). \tag{2.9}$$

Noting that $k + 1 - r < r$ and that $k - r$ is even, by Lemma 2.3,

$$C \cap [0, r - 1] = \mathcal{A} \cap [0, r - 1].$$

By the definition of \mathcal{A} ,

$$\chi_C(k + 1 - r) + \chi_C(k - r) = 1.$$

If $\chi_C(k - r) = 1$ and $\chi_C(k + 1 - r) = 0$, then by (2.9), $\chi_C(k + 1) = \chi_C(\frac{1}{2}k) = 1$. If $\chi_C(k - r) = 0$ and $\chi_C(k + 1 - r) = 1$, then by (2.9), $\chi_C(k + 1) = \chi_C(\frac{1}{2}k) = 0$. Thus $\chi_C(k - r) = \chi_C(\frac{1}{2}k)$. Putting $k = 2r - 2^{i+1}$, where $i + 1 \leq \lceil \log_2 r \rceil$, we have $\chi_C(r - 2^{i+1}) = \chi_C(r - 2^i)$. It follows that

$$\chi_C(r - 1) = \chi_C(r - 2) = \chi_C(r - 4) = \dots = \chi_C(r - 2^t),$$

that is,

$$\chi_{\mathcal{A}}(r - 1) = \chi_{\mathcal{A}}(r - 2) = \chi_{\mathcal{A}}(r - 4) = \dots = \chi_{\mathcal{A}}(r - 2^t)$$

for $t = \lceil \log_2 r \rceil - 1$. Now, Lemmas 2.1 and 2.2 imply that $r = 2^l - 1$, which contradicts the assumption that r is even.

Second, we prove that $r = 2^{2l} - 1$. If $r \leq k < 2r \leq m$ and k is even, then from the coefficient of x^k in (2.6),

$$2\chi_C\left(\frac{k}{2}\right) = 1 + 2 \sum_{i \leq k} \chi_C(i) - (k + 1) - 2 + 2\chi_C(k - r). \tag{2.10}$$

In this case $k - 1$ is odd and $k - 1 \geq r$; therefore, from the coefficient of x^{k-1} in (2.6),

$$0 = 2 \sum_{i \leq k-1} \chi_C(i) - k - 2 + 2\chi_C(k - 1 - r). \tag{2.11}$$

By (2.10) and (2.11),

$$\chi_C\left(\frac{k}{2}\right) = 1 - \chi_C(k) + \chi_C(k - r) - \chi_C(k - 1 - r).$$

If r is odd, then $k - 1 - r$ is even and Lemma 2.3 gives

$$C \cap [0, r - 1] = \mathcal{A} \cap [0, r - 1].$$

By the definition of \mathcal{A} ,

$$\chi_C(k - 1 - r) + \chi_C(k - r) = 1.$$

If $\chi_C(k - r) = 1$ and $\chi_C(k - 1 - r) = 0$, then $\chi_C(\frac{1}{2}k) = \chi_C(k - r) = 1$. If $\chi_C(k - r) = 0$ and $\chi_C(k - 1 - r) = 1$, then $\chi_C(\frac{1}{2}k) = \chi_C(k - r) = 0$. Thus, $\chi_C(r - 2^{i+1}) = \chi_C(r - 2^i)$. It follows that

$$\chi_C(r - 1) = \chi_C(r - 2) = \chi_C(r - 4) = \dots = \chi_C(r - 2^t),$$

for $t = \lceil \log_2 r \rceil - 1$, that is

$$\chi_{\mathcal{A}}(r - 1) = \chi_{\mathcal{A}}(r - 2) = \chi_{\mathcal{A}}(r - 4) = \dots = \chi_{\mathcal{A}}(r - 2^t),$$

and, by Lemmas 2.1 and 2.2, $r = 2^u - 1$. If $k = r$, then from the coefficient of x^k in (2.6),

$$0 = 2 \sum_{i \leq r} \chi_C(i) - (r + 1) - 2 + 2. \tag{2.12}$$

On the other hand, if $k = r - 1$, then from the coefficient of x^{k-1} in (2.6),

$$2\chi_C\left(\frac{r-1}{2}\right) = 1 + 2 \sum_{i \leq r-1} \chi_C(i) - r. \tag{2.13}$$

Since $r \in C$, by (2.12) and (2.13),

$$-2\chi_C\left(\frac{r-1}{2}\right) = -2 + 2\chi_C(r) = 0.$$

Consequently, $0 = \chi_C(\frac{1}{2}(r - 1)) = \chi_{\mathcal{A}}(\frac{1}{2}(r - 1))$ and so $\frac{1}{2}(r - 1) = 2^{u-1} - 1$, where $u - 1$ is odd, that is $r = 2^{2l} - 1$. This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.3

Assume there exist at least two pairs of sets (A, B) and (A', B') which satisfy the conditions

$$A \cup B = \mathbb{N}, \quad A \cap B = r + m\mathbb{N}, \quad R_A(n) = R_B(n) \quad \text{for every } n \geq 1,$$

$$A' \cup B' = \mathbb{N}, \quad A' \cap B' = r + m\mathbb{N}, \quad R_{A'}(n) = R_{B'}(n) \quad \text{for every } n \geq 1.$$

We may assume that $0 \in A \cap A'$. Let k be the smallest positive integer such that $\chi_A(k) \neq \chi_{A'}(k)$. Write

$$\{r + m\mathbb{N}\} \cap [0, k] = \{r_1, \dots, r_s\},$$

$$A_1 = A \cap [0, k], \quad B_1 = B \cap [0, k],$$

$$A_2 = A' \cap [0, k], \quad B_2 = B' \cap [0, k].$$

Then

$$A_1 \cup B_1 = A_2 \cup B_2 = [0, k], \tag{3.1}$$

$$A_1 \cap B_1 = A_2 \cap B_2 = \{r_1, \dots, r_s\}, \tag{3.2}$$

$$\chi_{A_1}(k) \neq \chi_{A_2}(k), \quad 0 \in A_1 \cap B_1. \tag{3.3}$$

For any integer n with $0 \leq n \leq k$, by the hypothesis,

$$R_{A_1}(n) = \#\{(a, a') : a < a' \leq n, a, a' \in A_1, a + a' = n\}$$

$$= R_A(n) = R_B(n) = R_{B_1}(n). \tag{3.4}$$

$$R_{A_2}(n) = \#\{(a, a') : a < a' \leq n, a, a' \in A_2, a + a' = n\}$$

$$= R_{A'}(n) = R_{B'}(n) = R_{B_2}(n). \tag{3.5}$$

Hence, there exist two pairs of sets (A_1, B_1) and (A_2, B_2) satisfying (3.1)–(3.5). By Lemma 2.3, this is impossible. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

Let

$$C = A \cap [0, m - 1 + r],$$

$$D = B \cap [0, m - 1 + r].$$

Then

$$C \cup D = [0, m - 1 + r], \quad C \cap D = \{r\}.$$

Moreover, for any n with $0 \leq n \leq m - 1 + r$,

$$R_C(n) = \#\{(c, c') : c < c' \leq n, c, c' \in C, c + c' = n\}$$

$$= \#\{(c, c') : c < c' \leq n, c, c' \in A, c + c' = n\}$$

$$= R_A(n),$$

$$R_D(n) = \#\{(c, c') : c < c' \leq n, c, c' \in D, c + c' = n\}$$

$$= \#\{(c, c') : c < c' \leq n, c, c' \in B, c + c' = n\}$$

$$= R_B(n).$$

Thus, for any n with $0 \leq n \leq m - 1 + r$,

$$R_C(n) = R_A(n) = R_B(n) = R_D(n).$$

Noting that $r \leq m - 1$, we see that $m - 1 + r \geq 2r > 0$ and, by Lemma 2.4, there exists a positive integer l such that $r = 2^{2^l} - 1$. This completes the proof of Theorem 1.4.

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